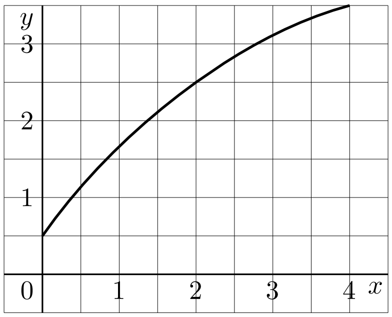
MATHEMATICS 121, Problem Set IV
Due Tuesday, March 11

1) Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is



Denote by L_n, R_n, M_n, T_n the approximate value for I generated when the "left endpoint rule", "right endpoint rule", "midpoint rule" and "trapezoidal rule", respectively, is applied using n subintervals.

   a) Find L_2, R_2 and M_2.
   b) Are these underestimates or over estimates for I?.
   c) For any value of n, list L_n, R_n, M_n, T_n and I in increasing order.

2) Use (a) the Trapezoidal Rule, (b) the Midpoint Rule and (c) Simpson's Rule with $n = 10$ subintervals to approximate $\int_2^3 \frac{1}{\ln x}\,dx$. (Round your answer to six decimal places.)

3 a) Find the approximations T_4, T_8, M_4 and M_8 for $\int_0^1 \cos(x^2)\,dx$.
b) Estimate the errors involved in these approximations.

4) How large do we have to choose n so that the approximations T_n and M_n to the integral $\int_0^1 e^{-x^2}\,dx$ are accurate to within 0.00001?

5) How large do we have to choose n so that the approximation S_n to the integral $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?

6) The speedometer reading (v) on a car was observed at one minute intervals and recorded in the following chart. Use Simpson's Rule to estimate the distance traveled by the car.

t (min)	0	1	2	3	4	5	6	7	8	9	10
v (mi/hr)	40	42	45	49	52	54	56	57	57	55	56

7 a) Read the generalization of the Mean Value Theorem given on the next page.
b) Suppose that $f(x)$ has a continuous second derivative that obeys $|f''(x)| \le K$ for all $a \le x \le b$. Let T_n be the result of applying the Trapezoidal Rule to $\int_a^b f(x)\,dx$ with n subintervals. Use the generalization of the Mean Value Theorem given on the next page to prove that

$$\left| \int_a^b f(x)\,dx - T_n \right| \le \tfrac{K(b-a)^3}{12n^2}$$

8) Let $I = \int_{\pi/6}^{\pi/2} \ln(\sin x)\,dx$.

   a) Determine the maximum K of $|f''(x)|$, for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$, where $f(x) = \ln(\sin x)$.
   b) How large should n be in order that the approximation $I \approx M_n$ be accurate to within 10^{-4}?

see over

Recall that

Theorem (Mean Value Theorem) *Suppose that the function $f(x)$ is continuous on $a \le x \le b$ and differentiable of $a < x < b$. Then there exists a c with $a < c < b$ such that*

$$f(b) - f(a) = (b-a)f'(c)$$

The following is a generalization of the Mean Value Theorem:

Theorem (Mean Value Theorem for Interpolation)
Let $a \le y_0 < y_1 < y_2 < \cdots < y_n \le b$. Let $f(y)$ and its first n derivatives be continuous on $a \le y \le b$ and let $f^{(n+1)}(y)$ exist on $a < y < b$. Suppose that $p(y)$ is a polynomial of degree at most n that coincides with $f(y)$ at $y_0, \cdots, y_n$. That is, assume that

$$f(y_0) = p(y_0),\ f(y_1) = p(y_1),\ \cdots,\ f(y_n) = p(y_n)$$

Then, for each $a \le y \le b$, there is a number $a < c_y < b$, depending on y, such that

$$f(y) - p(y) = (y-y_0)(y-y_1)\cdots(y-y_n)\tfrac{f^{(n+1)}(c_y)}{(n+1)!}$$

Proof: If y is one of $y_0,\ \cdots,\ y_n$ then both of $f(y)-p(y)$ and $(y-y_0)(y-y_1)\cdots(y-y_n)\frac{f^{(n+1)}(c_y)}{(n+1)!}$ are zero, for any c_y, and hence are equal. So it suffices to consider a y that is different from all of $y_0,\ \cdots,\ y_n$. Fix any such y and define

$$g(z) = f(z) - p(z) - \frac{(z-y_0)\cdots(z-y_n)}{(y-y_0)\cdots(y-y_n)}[f(y)-p(y)]$$

Note that $g(z)$ is zero for at least $n+2$ different values of z, namely, $z = y$ and $z = y_0,\ \cdots,\ y_n$. The Mean Value Theorem implies that $\frac{dg}{dz}$ has a zero between any two successive zeroes of $g(z)$. So $\frac{dg}{dz}(z)$ is zero for at least $n+1$ different values of z. By the Mean Value Theorem, $\frac{d^2g}{dz^2}$ has a zero between any two successive zeroes of $\frac{dg}{dz}$. So $\frac{d^2g}{dz^2}(z)$ is zero for at least n different values of z. Continuing in this way, we see that $\frac{d^{n+1}g}{dz^{n+1}}$ has at least one zero. Call it c_y.

As $p(z)$ is a polynomial in z of degree at most n, $\frac{d^{n+1}p}{dz^{n+1}} = 0$. Recall that y is being held fixed in this computation. So $\frac{(z-y_0)\cdots(z-y_n)}{(y-y_0)\cdots(y-y_n)}[f(y)-p(y)]$ is a polynomial in z of degree $n+1$ whose degree $n+1$ term is $\frac{z^{n+1}}{(y-y_0)\cdots(y-y_n)}[f(y)-p(y)]$. Hence

$$\tfrac{d^{n+1}}{dz^{n+1}}\tfrac{(z-y_0)\cdots(z-y_n)}{(y-y_0)\cdots(y-y_n)}[f(y)-p(y)] = \tfrac{(n+1)!}{(y-y_0)\cdots(y-y_n)}[f(y)-p(y)]$$

and

$$0 = \tfrac{d^{n+1}g}{dz^{n+1}}(c_y) = f^{(n+1)}(c_y) - \tfrac{(n+1)!}{(y-y_0)\cdots(y-y_n)}[f(y)-p(y)]$$

Moving $f^{(n+1)}(c_y)$ to the other side of the equation and cross multiplying gives

$$f(y) - p(y) = \tfrac{(y-y_0)\cdots(y-y_n)}{(n+1)!}f^{(n+1)}(c_y)$$

as desired ■

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