MATHEMATICS 121, Problem Set D

Not to be handed in

1) For each of the following numbers: (i) sketch its position in the complex plane, (ii) find its real and imaginary parts, (iii) find its complex conjugate and (iv) find its modulus $r = |z|$ and argument $\theta = A(z)$.

a) $-1+i$ b) $3-4i$ c) $-\pi i$ d) -4

2) Simplify

a) $i-(3-2i)+(7-3i)$ b) $(4-i)(4+i)$ c) $\frac{1+3i}{2-i}$ d) $\frac{(1+2i)(2-3i)}{(2-i)(3+2i)}$

3) Express each of the numbers $z = 3 + i\sqrt{3}$ and $w = -1 + i\sqrt{3}$ in the form $re^{i\theta}$. Use these to compute zw and z/w.

4) Use $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ to find identities for $\sin(3\theta)$ and $\cos(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

5) For each natural number n and each nonzero complex number a, the equation $z^n = a$ has precisely n distinct solutions, called the n^{th} roots of a.

a) Find the three cube roots of -1.

b) Show that the sum of the n n^{th} roots of 1 is zero.

6) Evaluate the given integrals, using complex numbers.

a) $\int_0^1 \frac{2x^2-2}{(x^2+1)^2}\,dx$ b) $\int_0^{\pi/2} \sin^4\theta\,d\theta$

7) Solve.

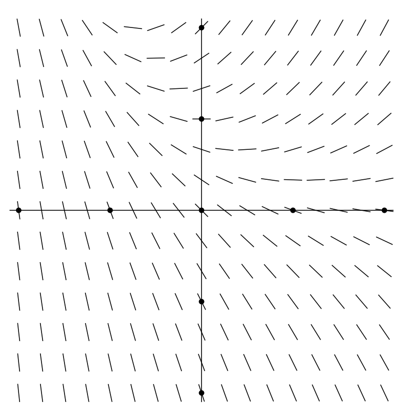
a) $\frac{dy}{dx} = y^2$ b) $y' = xy$ c) $y' = \frac{\ln x}{xy+xy^3}$

d) $\frac{dx}{dt} = 1 + t - x - tx$ e) $\frac{dy}{dt} = \frac{ty+3t}{t^2+1}$, $y(2) = 2$

8) Find an equation for the curve that passes through the point $(1,1)$ and whose slope at (x,y) is $\frac{y^2}{x^3}$.

9) A direction field for the differential equation $y' = y - e^{-x}$ is shown below. The spacing between the dots on the axes is one unit. Sketch the graphs of the solutions that satisfy the initial conditions

a) $y(0) = 0$ b) $y(0) = 1$ c) $y(0) = -1$



10) In 1986, the population of the world was 5 billion and was increasing at a rate of 2% per year. Using the exponential model for population growth,

a) find an expression for the population of the world in the year t,

b) predict the population of the world in the years 2000, 2100 and 2500 and

see over

c) determine how many square feet of land there will be per person in the year 2500. The total land surface area of the earth is about 1.8×10^{15} ft^2.

11) In 1986, the population of the world was 5 billion and was increasing at a rate of 2% per year. Assuming the logistic growth model with an assumed maximum population of 100 billion, predict the population of the world in the years 2000, 2100 and 2500.

12) When a raindrop falls it increases in size, so that its mass at time t is a function of t, $m(t)$. The rate of growth of the mass is proportional to the mass itself. Newton's law of motion, when applied to an object whose mass is varying, is $(mv)' = F$, where F is the applied force, which for the raindrop is mg, v is the velocity of the raindrop and g is the acceleration due to gravity. Find the terminal velocity, $\lim\limits_{t\to\infty} v(t)$, for a raindrop.

13) According to Newton's Law of Gravitation, the gravitational force on an object of mass m that is a distance x from the surface of the earth is

$$F = -\tfrac{mgR^2}{(x+R)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity at the earth's surface. Hence by Newton's Second Law of Motion

$$m\tfrac{dv}{dt} = -\tfrac{mgR^2}{(x+R)^2}$$

a) Suppose a projectile is fired vertically with an initial velocity v_0 and achieves a maximum height of h. Show that

$$v_0 = \sqrt{\tfrac{2gRh}{R+h}}$$

Hint: By the Chain Rule $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$.

b) Calculate $v_e = \lim\limits_{h\to\infty} v_0$. This is called the escape velocity.

c) Calculate v_e in miles per second for $R = 3960$ miles and $g = 32$ ft/sec^2.

14) A population of rare South American monkeys has been found to have a population growth rate which decreases with time. Specifically, the population size, $y(t)$, satisfies the differential equation

$$y'(t) = \tfrac{y}{(t+1)^2}$$

a) Given that at time $t = 0$ the population is 100, solve for the population size $y(t)$ after time t.

b) In the limit $t \to \infty$, what is the size of the population?

15) Suppose that $y(t)$ (measured in thousands of dollars – "kilobucks") represents the balance in a savings account at time t years after the account was opened with an initial balance of 2 kilobucks. Interest is being paid into the account continuously at a nominal rate of 5% per annum; that is to say, interest is flowing into the account at a rate of $\frac{5}{100}y(t)$ kilobucks per year at time t. At the same time, the account is being continuously depleted by taxes and other charges at a rate $\frac{y(t)^2}{1000}$ kilobucks per year. No other deposits or withdrawals are made on the account.

a) What is balance in the account at time $t = 20$ years?

b) How large can the balance in the account grow over time?

Reminder: Quiz III is on Tuesday, March 4. It will cover up to the end of the class of Friday, February 28.