Computation of π

We use the formula

$$\tfrac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}$$

which was discovered by the English mathematician John Machin. He used it to compute π to 100 decimal places in 1706.

Proof of Machin's Formula

Let $\beta = \tan^{-1}\frac{1}{5}$. Then $\tan\beta = \frac{1}{5}$. Using the double angle formula

$$\tan(2\theta) = \tfrac{\sin(2\theta)}{\cos(2\theta)} = \tfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tfrac{2\tan\theta}{1-\tan^2\theta}$$

twice

$$\begin{aligned} \tan(2\beta) &= \tfrac{2\tan\beta}{1-\tan^2\beta} &&= \tfrac{2/5}{1-1/25} &&= \tfrac{10}{24} = \tfrac{5}{12} \\ \tan(4\beta) &= \tfrac{2\tan(2\beta)}{1-\tan^2(2\beta)} &&= \tfrac{5/6}{1-25/144} &&= \tfrac{120}{119} \end{aligned}$$

Then using the addition formula

$$\tan(x-y) = \tfrac{\sin(x-y)}{\cos(x-y)} = \tfrac{\sin x\cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y} = \tfrac{\tan x - \tan y}{1 + \tan x \tan y}$$

and $\tan\frac{\pi}{4} = 1$

$$\tan\left(4\beta - \tfrac{\pi}{4}\right) = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{120-119}{120+119} = \frac{1}{239}$$

Taking the arctan of both sides gives

$$4\tan^{-1}\tfrac{1}{5} - \tfrac{\pi}{4} = \tan^{-1}\tfrac{1}{239}$$

which is what we want.

A Series Expansion for $\tan^{-1} x$

We already know that for $|s| < 1$

$$\tfrac{1}{1-s} = 1 + s + s^2 + s^3 + \cdots$$

Subbing in $s = -t^2$ gives, for $|t| < 1$

$$\tfrac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

Integrating gives, for $|x| < 1$,

$$\tan^{-1} x = \int_0^x \tfrac{1}{1+t^2}\,dt = \int_0^x \left[1 - t^2 + t^4 - t^6 + \cdots\right]\,dt = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \tfrac{x^7}{7} + \cdots$$

Furthermore, for $0 \le x < 1$, the requirements of the alternating series test apply and truncating the series introduces an error between 0 and the first term omitted.

The Computation of π

$$
\begin{aligned}
\tan^{-1}\tfrac{1}{5} &= \tfrac{1}{5} - \tfrac{1}{3\times 5^3} + \tfrac{1}{5\times 5^5} \\
&\quad - \tfrac{1}{7\times 5^7} + \tfrac{1}{9\times 5^9} - \tfrac{1}{11\times 5^{11}} \\
&= 0.2000000000 - 0.0026666667 + 0.0000640000 \\
&\quad - 0.0000018286 + 0.0000000569 - 0.0000000019 \\
&= 0.197395562 - e \text{ with } 0 \le e \le 1.9\times 10^{-9} + 5\times 0.5\times 10^{-10} \\
\tan^{-1}\tfrac{1}{239} &= \tfrac{1}{239} - \tfrac{1}{3\times 239^3} + \tfrac{1}{5\times 239^5} \\
&= 0.0041841004 - 0.0000000244 + 2.6\times 10^{-13} \\
&= 0.0041840760 + e' \text{ with } 0 \le e' \le 2.6\times 10^{-13} + 3\times 0.5\times 10^{-10}
\end{aligned}
$$

The extra 10^{-10} terms take roundoff error into account. So

$$4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} = 0.785398172 - 4e - e'$$

and

$$\pi = 3.141592688 - 16e - 4e'$$
$$\pi = 3.14159269 - e'' \text{ with } e'' \text{ between } 0 \text{ and } 4\times 10^{-8}$$