

The Form of Partial Fractions Decompositions

In the following it is assumed that

- N and D are polynomials with the degree of N strictly smaller than the degree of D .
- K is a constant.
- a_1, a_2, \dots, a_j are all different.
- $m_1, m_2, \dots, m_j, n_1, n_2, \dots, n_k$ are all strictly positive integers.
- $x^2 + b_1x + c_1, x^2 + b_2x + c_2, \dots, x^2 + b_kx + c_k$ are all different.

Simple Linear Factor Case

If $D(x) = K(x - a_1)(x - a_2) \cdots (x - a_j)$ then

$$\frac{N(x)}{D(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_j}{x - a_j}$$

The proof that such a decomposition always exists is given in notes that I have posted on our course home page.

General Linear Factor Case

If $D(x) = K(x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_j)^{m_j}$ then

$$\begin{aligned} \frac{N(x)}{D(x)} = & \frac{B_{1,m_1-1}x^{m_1-1} + \cdots + B_{1,1}x + B_{1,0}}{(x - a_1)^{m_1}} \\ & + \frac{B_{2,m_2-1}x^{m_2-1} + \cdots + B_{2,1}x + B_{2,0}}{(x - a_2)^{m_2}} + \cdots \\ & + \frac{B_{j,m_j-1}x^{m_j-1} + \cdots + B_{j,1}x + B_{j,0}}{(x - a_j)^{m_j}} \end{aligned}$$

Each numerator is the most general polynomial with degree strictly smaller than that of the denominator. The proof that such a decomposition always exists is given in notes that I have posted on our course home page.

It is generally mechanically advantageous to rewrite

$$\frac{B_{m-1}x^{m-1} + \dots + B_1x + B_0}{(x-a)^m} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_m}{(x-a)^m}$$

To see that these forms are equivalent, rename $x - a = y$. Then

$$\begin{aligned} \frac{B_{m-1}x^{m-1} + \dots + B_1x + B_0}{(x-a)^m} &= \frac{B_{m-1}(y+a)^{m-1} + \dots + B_1(y+a) + B_0}{y^m} \\ &= \frac{A_1y^{m-1} + \dots + A_{m-1}y + A_m}{y^m} \\ &= \frac{A_1}{x-a} + \dots + \frac{A_{m-1}}{(x-a)^{m-1}} + \frac{A_m}{(x-a)^m} \end{aligned}$$

The rewritten decomposition is

$$\begin{aligned} \frac{N(x)}{D(x)} &= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \dots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} \\ &+ \frac{A_{2,1}}{x-a_2} + \frac{A_{2,2}}{(x-a_2)^2} + \dots + \frac{A_{2,m_2}}{(x-a_2)^{m_2}} + \dots \\ &+ \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \dots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}} \end{aligned}$$

Simple Linear and Quadratic Factor Case

If $D(x) = K(x - a_1) \cdots (x - a_j)(x^2 + b_1x + c_1) \cdots (x^2 + b_kx + c_k)$ then

$$\frac{N(x)}{D(x)} = \frac{A_1}{x - a_1} + \cdots + \frac{A_j}{x - a_j} + \frac{B_1x + C_1}{x^2 + b_1x + c_1} + \cdots + \frac{B_kx + C_k}{x^2 + b_kx + c_k}$$

General Linear and Quadratic Factor Case

If $D(x) = K(x - a_1)^{m_1} \cdots (x - a_j)^{m_j}(x^2 + b_1x + c_1)^{n_1} \cdots (x^2 + b_kx + c_k)^{n_k}$ then

$$\begin{aligned} \frac{N(x)}{D(x)} = & \frac{A_{1,1}}{x - a_1} + \frac{A_{1,2}}{(x - a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x - a_1)^{m_1}} + \cdots \\ & + \frac{A_{j,1}}{x - a_j} + \frac{A_{j,2}}{(x - a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x - a_j)^{m_j}} \\ & + \frac{B_{1,1}x + C_{1,1}}{x^2 + b_1x + c_1} + \frac{B_{1,2}x + C_{1,2}}{(x^2 + b_1x + c_1)^2} + \cdots + \frac{B_{1,n_1}x + C_{1,n_1}}{(x^2 + b_1x + c_1)^{n_1}} + \cdots \\ & + \frac{B_{k,1}x + C_{k,1}}{x^2 + b_kx + c_k} + \frac{B_{k,2}x + C_{k,2}}{(x^2 + b_kx + c_k)^2} + \cdots + \frac{B_{k,n_k}x + C_{k,n_k}}{(x^2 + b_kx + c_k)^{n_k}} \end{aligned}$$