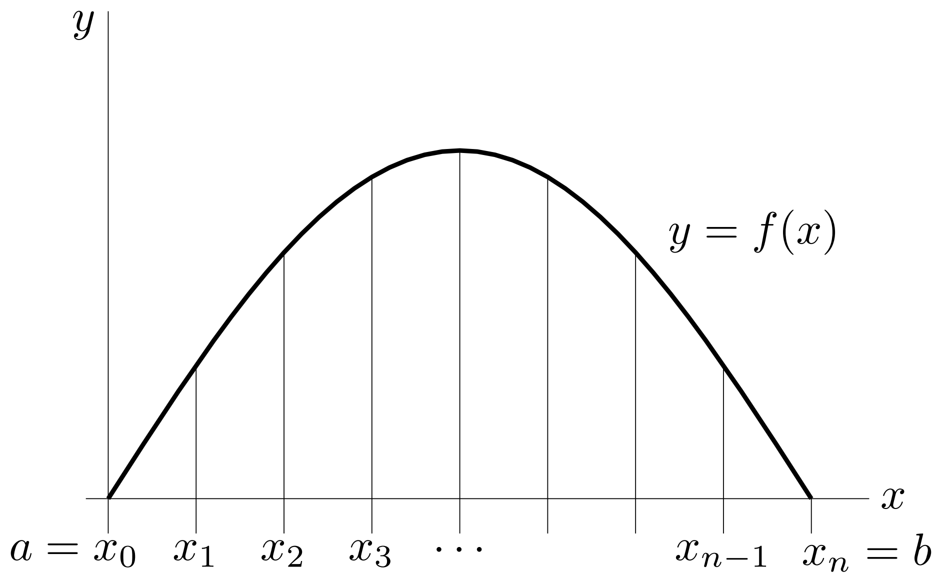
Numerical Integration

Select an integer n, called the "number of steps". Divide the interval of integration, $a \le x \le b$ into n equal subintervals, each of size $\Delta x = \frac{b-a}{n}$. Decompose

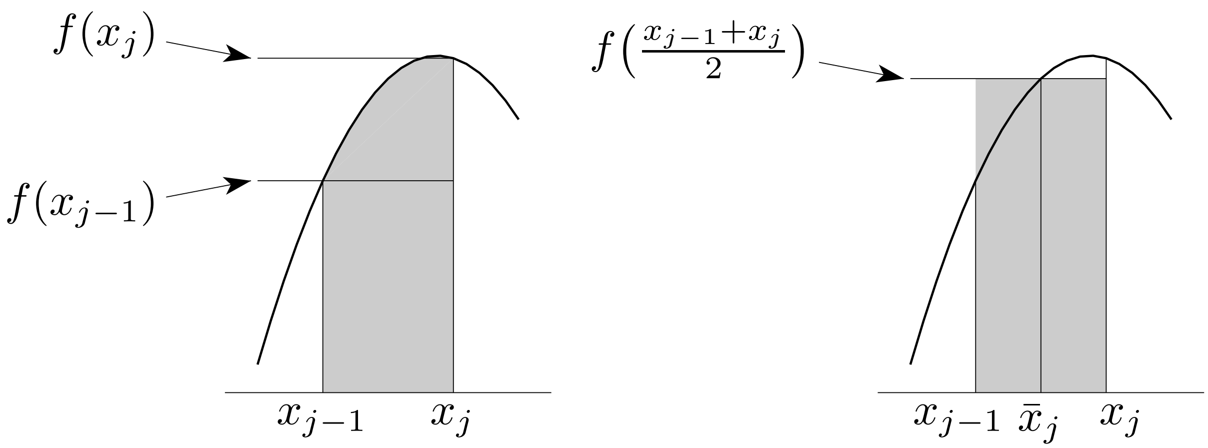


$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Each subintegral $\int_{x_{j-1}}^{x_j} f(x)\,dx$ is approximated by the area of a simple geometric figure.

Midpoint Rule

The Midpoint Rule approximates $\int_{x_{j-1}}^{x_j} f(x)\,dx$ by the area of a rectangle of width $x_j - x_{j-1} = \Delta x$ and height $f\left(\frac{x_{j-1}+x_j}{2}\right)$, which is the exact height at the midpoint of the range of x. So the



Midpoint Rule approximates each subintegral by

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx f(\bar{x}_j)\Delta x \text{ where } \bar{x}_j = \tfrac{x_{j-1}+x_j}{2}$$

and the full integral by

$$\int_a^b f(x)\,dx \approx \Big[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\Big]\Delta x$$

For example, here is the approximation for $\int_0^\pi \sin x\,dx$ with $n = 8$. First note that $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \quad x_1 = \tfrac{\pi}{8} \quad x_2 = \tfrac{2\pi}{8} \quad \cdots \quad x_7 = \tfrac{7\pi}{8} \quad x_8 = \tfrac{8\pi}{8} = \pi$$

Consequently,

$$\bar{x}_1 = \tfrac{\pi}{16} \quad \bar{x}_2 = \tfrac{3\pi}{16} \quad \cdots \quad \bar{x}_7 = \tfrac{13\pi}{16} \quad x_8 = \tfrac{15\pi}{16}$$

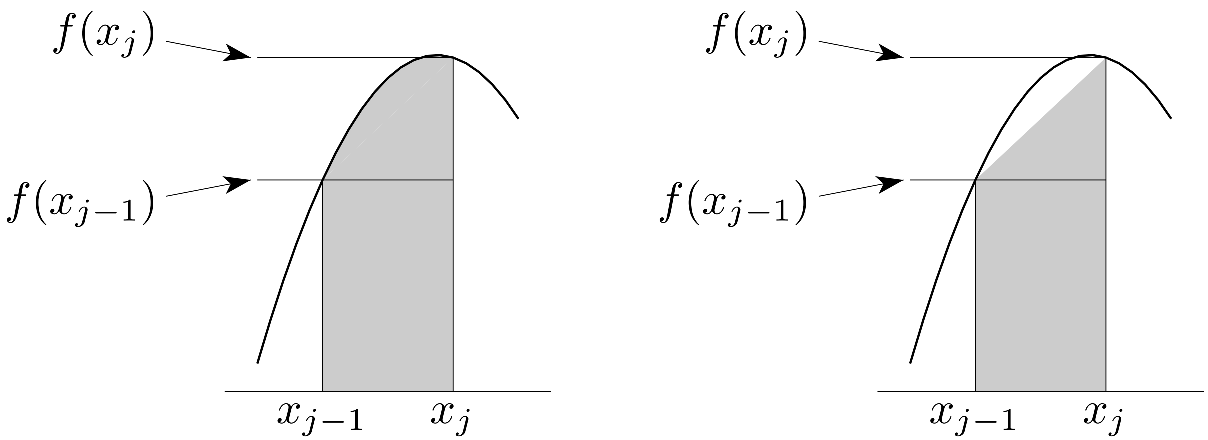
and

$$\begin{aligned}
\int_0^\pi \sin x\,dx &\approx \Big[\sin(\bar{x}_1) + \sin(\bar{x}_2) + \cdots + \sin(\bar{x}_8)\Big]\Delta x \\
&= \Big[\sin(\tfrac{\pi}{16}) + \sin(\tfrac{3\pi}{16}) + \sin(\tfrac{5\pi}{16}) + \sin(\tfrac{7\pi}{16}) \\
&\qquad + \sin(\tfrac{9\pi}{16}) + \sin(\tfrac{11\pi}{16}) + \sin(\tfrac{13\pi}{16}) + \sin(\tfrac{15\pi}{16})\Big]\tfrac{\pi}{8} \\
&= \Big[0.1951 + 0.5556 + 0.8315 + 0.9808 \\
&\qquad + 0.9808 + 0.8315 + 0.5556 + 0.1951\Big] \times 0.3927 \\
&= 5.1260 \times 0.3927 \\
&= 2.013
\end{aligned}$$

The exact answer is $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$. So with eight steps of the Midpoint Rule we achieved $100\frac{2.013-2}{2} = 0.6\%$ accuracy.

Trapezoidal Rule

The Trapezoidal Rule approximates $\int_{x_{j-1}}^{x_j} f(x)\,dx$ by the area of the trapezoid with width $x_j - x_{j-1} = \Delta x$ left side height $f(x_{j-1})$ and right hand side height $f(x_j)$. So the Trapezoidal Rule ap-



proximates

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx \tfrac{f(x_{j-1})+f(x_j)}{2}\Delta x$$

and the full integral by

$$\int_a^b f(x)\,dx = \Big[\tfrac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\tfrac{1}{2}f(x_n)\Big]\Delta x$$

For example, here is the approximation for $\int_0^\pi \sin x\,dx$ with $n = 8$. First note that $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \quad x_1 = \tfrac{\pi}{8} \quad x_2 = \tfrac{2\pi}{8} \quad \cdots \quad x_7 = \tfrac{7\pi}{8} \quad x_8 = \tfrac{8\pi}{8} = \pi$$

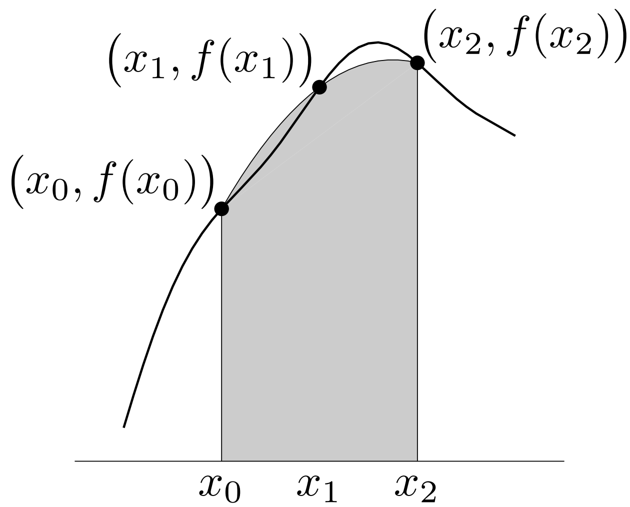
Consequently,

$$\begin{aligned}
\int_0^\pi \sin x\,dx &\approx \Big[\tfrac{1}{2}\sin(x_0) + \sin(x_1) + \cdots + \sin(x_7) + \tfrac{1}{2}\sin(x_8)\Big]\Delta x \\
&= \Big[\tfrac{1}{2}\sin(0) + \sin(\tfrac{\pi}{8}) + \sin(\tfrac{2\pi}{8}) + \sin(\tfrac{3\pi}{8}) \\
&\qquad + \sin(\tfrac{4\pi}{8}) + \sin(\tfrac{5\pi}{8}) + \sin(\tfrac{6\pi}{8}) + \sin(\tfrac{7\pi}{8}) + \tfrac{1}{2}\sin(\tfrac{8\pi}{8})\Big]\tfrac{\pi}{8} \\
&= \Big[\tfrac{1}{2} \times 0 + 0.3827 + 0.7071 + 0.9239 \\
&\qquad + 1.0000 + 0.9239 + 0.7071 + 0.3827 + \tfrac{1}{2} \times 0\Big] \times 0.3927 \\
&= 5.0274 \times 0.3927 \\
&= 1.974
\end{aligned}$$

The exact answer is $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$. So with eight steps of the Trapezoidal Rule we achieved $100\frac{|1.974-2|}{2} = 1.3\%$ accuracy.

Simpson's Rule

Simpson's Rule approximates $\int_{x_0}^{x_2} f(x)\,dx$ by the area under the part of a parabola with x running from x_0 to x_2. The parabola used passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$.



So Simpson's rule approximates

$$\int_{x_0}^{x_2} f(x)\,dx \approx \tfrac{1}{3}\Delta x\big[f(x_0) + 4f(x_1) + f(x_2)\big]$$

and

$$\int_a^b f(x)\,dx = \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots$$
$$+2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]\tfrac{\Delta x}{3}$$

As an example we approximate $\int_0^\pi \sin x \, dx$ with $n = 8$, yet again. Under Simpson's rule

$$\begin{aligned}
&\int_0^\pi \sin x \, dx \\
&\quad = \Big[\sin(0) + 4\sin(\tfrac{\pi}{8}) + 2\sin(\tfrac{2\pi}{8}) + 4\sin(\tfrac{3\pi}{8}) + 2\sin(\tfrac{4\pi}{8}) \\
&\qquad\qquad + 4\sin(\tfrac{5\pi}{8}) + 2\sin(\tfrac{6\pi}{8}) + 4\sin(\tfrac{7\pi}{8}) + \sin(\tfrac{8\pi}{8})\Big]\tfrac{\pi}{8\times 3} \\
&\quad = \Big[0 + 4 \times 0.382683 + 2 \times 0.707107 + 4 \times 0.923880 + 2 \times 1.0 \\
&\qquad + 4 \times 0.923880 + 2 \times 0.707107 + 4 \times 0.382683 + 0\Big]\tfrac{\pi}{8\times 3} \\
&\quad = 15.280932 \times 0.130900 \\
&\quad = 2.00027
\end{aligned}$$

The exact answer is $\int_0^\pi \sin x \, dx = -\cos x\Big|_0^\pi = 2$. With only eight steps of Simpson's rule we achieved $100\frac{2.00027-2}{2} = 0.013\%$ accuracy.

Error Behaviour

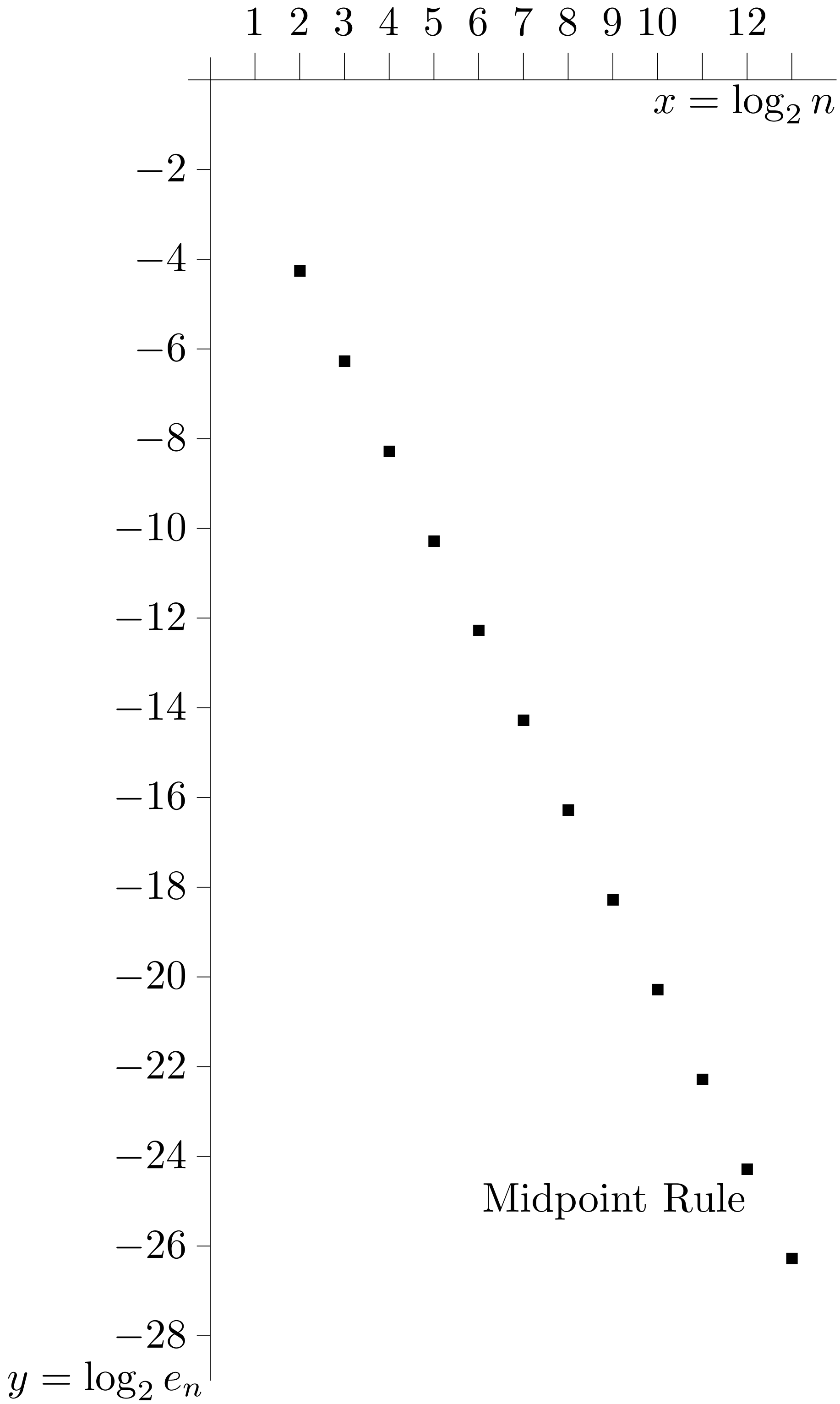
The exact value of the integral $\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi$ is 2. The following table lists the error in the approximate value for this number generated by our three rules applied with three different choices of n. It also lists the number of evaluations of f required to compute the approximation.

n	**Midpoint** error	**Midpoint** #evals	**Trapezoidal** error	**Trapezoidal** #evals	**Simpson's** error	**Simpson's** #evals
10	4.1×10^{-1}	10	8.2×10^{-1}	11	5.5×10^{-3}	11
100	4.1×10^{-3}	100	8.2×10^{-3}	101	5.4×10^{-7}	101
1000	4.1×10^{-5}	1000	8.2×10^{-5}	1001	5.5×10^{-11}	1001

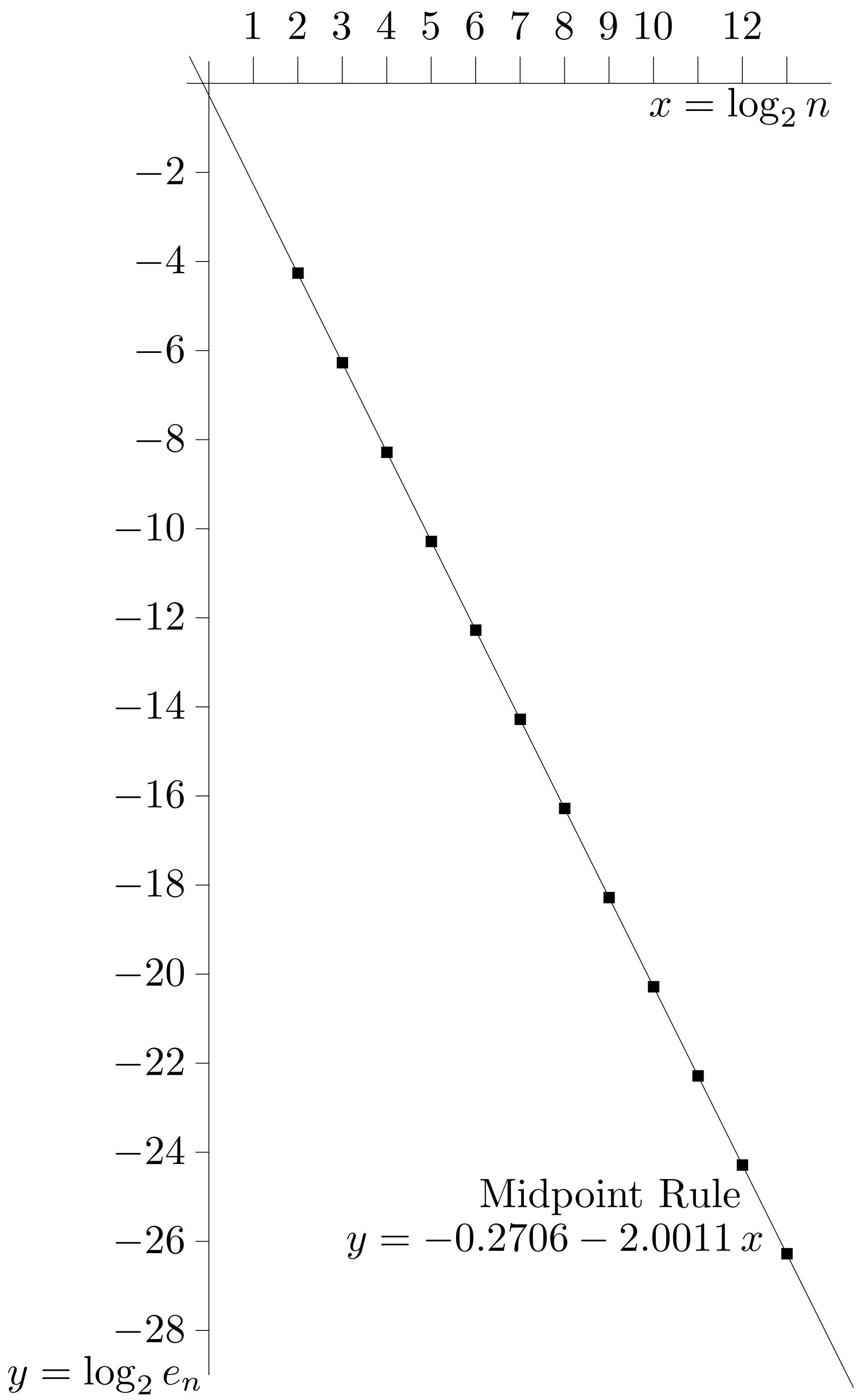
Observe that

- Using 101 evaluations of f worth of Simpson's rule gives an error 80 times smaller than 1000 evaluations of f worth of the Midpoint Rule.
- The Trapezoidal Rule error with n steps is about twice the Midpoint Rule error with n steps.
- With the Midpoint Rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $100 = 10^2 = n^2$.
- With the Trapezoidal Rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $10^2 = n^2$.
- With Simpson's Rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $10^4 = n^4$.

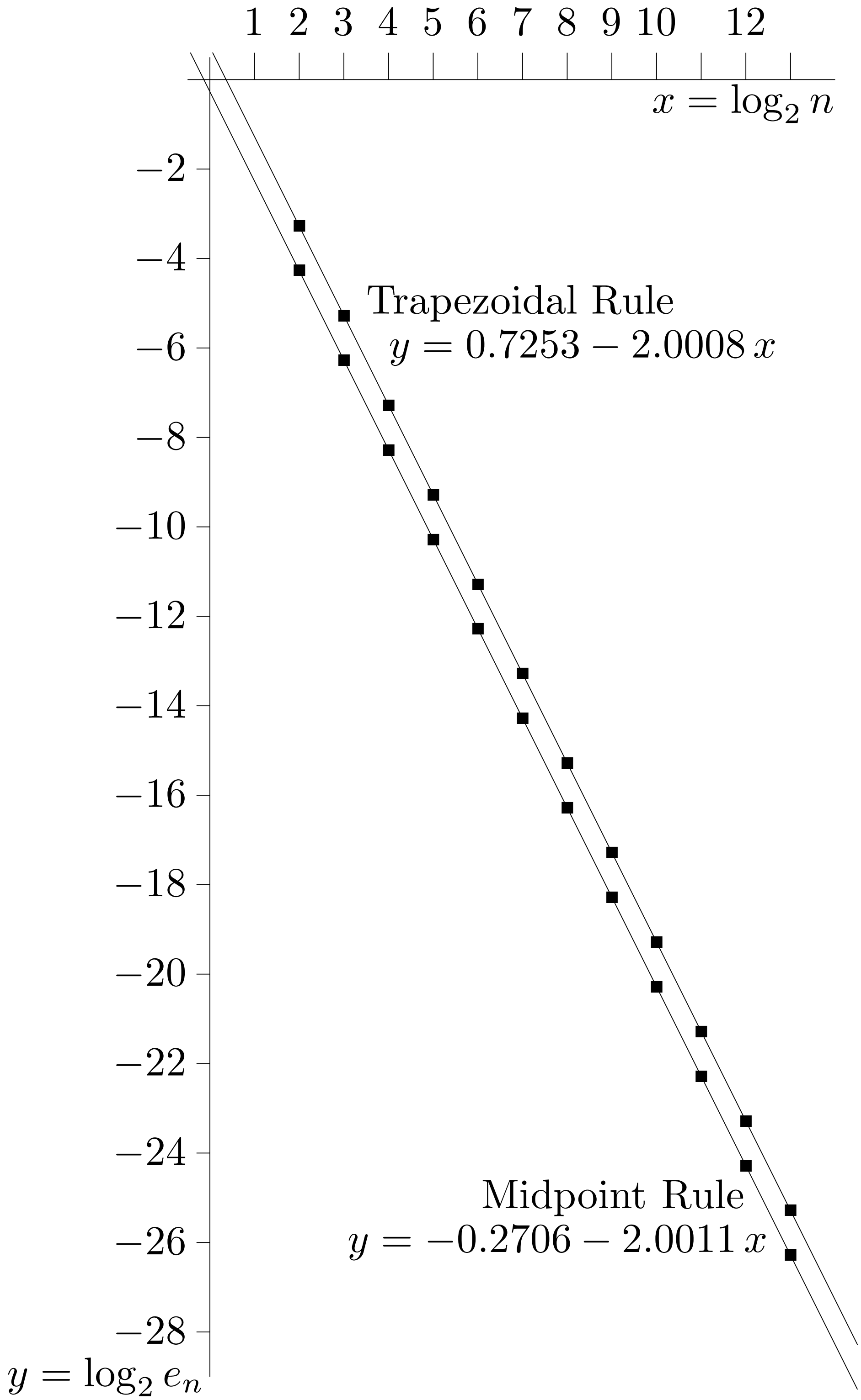
Error in the approximation, with n steps, to $\int_0^\pi \sin x\,dx$



Error in the approximation, with n steps, to $\int_0^\pi \sin x\, dx$



Error in the approximation, with n steps, to $\int_0^{\pi} \sin x \, dx$



Error in the approximation, with n steps, to $\int_0^\pi \sin x\, dx$