Centroid Example

Find the centroid of the region bounded by \( y = \sin x \), \( y = \cos x \), \( x = 0 \) and \( x = \frac{\pi}{2} \).

**Solution.** We apply the formulae that the coordinates of the centroid (at centre of mass assuming constant density) of the region with top \( y = f(x) \), bottom \( y = g(x) \), left hand side \( x = a \) and right hand side \( x = b \) are

\[
\bar{x} = \frac{\int_a^b x (f(x) - g(x)) \, dx}{\int_a^b (f(x) - g(x)) \, dx} \quad \bar{y} = \frac{\int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx}{\int_a^b [f(x) - g(x)] \, dx}
\]

Before we apply these formulae, we recall where they came from. Assume that the region has density one. Consider a thin vertical slice, of width \( dx \), running from \((x, g(x))\) to \((x, f(x))\). It has area, and hence mass,

\[
[f(x) - g(x)] \, dx.
\]

On this slice \( x \) is essentially constant. So the formula for \( \bar{x} \) is just the formula for the (weighted) average of \( x \) over the whole region. On the slice \( y \) runs from \( g(x) \) to \( f(x) \). The average value of \( y \) on the slice is \( \frac{1}{2} [f(x) + g(x)] \). Because \( \frac{1}{2} [f(x) + g(x)] [f(x) - g(x)] = \frac{1}{2} [f(x)^2 - g(x)^2] \), the formula for \( \bar{y} \) is the formula for the average of \( y \) over the whole region.

In the given problem, \( a = 0 \), \( b = \frac{\pi}{4} \), \( f(x) = \cos x \) and \( g(x) = \sin x \). Subbing these in to the numerator of the formula for \( \bar{x} \) and \( \bar{y} \) gives

\[
\int_a^b x (f(x) - g(x)) \, dx = \int_0^{\pi/4} (\sin x - \cos x) \, dx = \left[ \sin x + \cos x \right]_0^{\pi/4} = \left( \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} \right) - [0 + 1] = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1
\]

Subbing into the numerator of the formula for \( \bar{x} \) gives

\[
\int_a^b x (f(x) - g(x)) \, dx = \int_0^{\pi/4} x (\cos x - \sin x) \, dx
\]

To integrate this, use integration by parts with \( u = x \) and \( dv = (\cos x - \sin x) \, dx \). So \( du = dx \), \( v = \sin x + \cos x \) and

\[
\int_0^{\pi/4} x (\cos x - \sin x) \, dx = x [\sin x + \cos x]_0^{\pi/4} - \int_0^{\pi/4} [\sin x + \cos x] \, dx
\]

\[
= \frac{\pi}{4} \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] - \left[ \sin x + \cos x \right]_0^{\pi/4}
\]

\[
= \frac{\pi}{4\sqrt{2}} - \left( \left( -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 + 1) \right) = \frac{\pi}{2\sqrt{2}} - 1
\]

Finally, subbing into the numerator of the formula for \( \bar{y} \) gives

\[
\int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx = \int_0^{\pi/4} \frac{1}{2} [\cos^2 x - \sin^2 x] \, dx = \int_0^{\pi/4} \frac{1}{2} \cos (2x) \, dx = \left[ \frac{1}{4} \sin (2x) \right]_0^{\pi/4} = \frac{\pi}{4}
\]

Putting the formulae together

\[
\bar{x} = \frac{\int_a^b x (f(x) - g(x)) \, dx}{\int_a^b (f(x) - g(x)) \, dx} = \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = 1
\]

\[
\bar{y} = \frac{\int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] \, dx}{\int_a^b (f(x) - g(x)) \, dx} = \frac{1}{2\sqrt{2} - 1}
\]