Trig Identities – Cosine Law and Addition Formulae

The cosine law

If a triangle has sides of length $A$, $B$ and $C$ and the angle opposite the side of length $C$ is $\theta$, then

$$C^2 = A^2 + B^2 - 2AB \cos \theta$$

Proof: Applying Pythagorous to the right hand triangle of the right hand figure of

```
A
  \theta
/    \
\    \\
/      \
B

C
```

```
A
  \theta
/    \
\    \\
/      \
\cos \theta

A sin \theta
```

gives

$$C^2 = (B - A \cos \theta)^2 + (A \sin \theta)^2$$

$$= B^2 - 2AB \cos \theta + A^2 \cos^2 \theta + A^2 \sin^2 \theta$$

$$= B^2 - 2AB \cos \theta + A^2$$

Addition and subtraction formulae

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
**Proof:** We first prove $\cos(x - y) = \cos x \cos y + \sin x \sin y$. The angle, of the upper triangle, that is opposite the side of length $C$ is $x - y$. So, by the cosine law,

$$C^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(x - y) = 2 - 2 \cos(x - y)$$

But the side of length $C$ joins the points $(\cos y, \sin y)$ and $(\cos x, \sin x)$ and so we also have, by Pythagorous,

$$C^2 = (\cos y - \cos x)^2 + (\sin y - \sin x)^2$$

$$= \cos^2 y - 2 \cos x \cos y + \cos^2 x + \sin^2 y - 2 \sin x \sin y + \sin^2 x$$

$$= 2 - 2 \cos x \cos y - 2 \sin x \sin y$$

Setting the two formulae for $C^2$ equal to each other gives

$$2 - 2 \cos(x - y) = 2 - 2 \cos x \cos y - 2 \sin x \sin y$$

$$\implies -2 \cos(x - y) = -2 \cos x \cos y - 2 \sin x \sin y$$

$$\implies \cos(x - y) = \cos x \cos y + \sin x \sin y$$

which is the fourth addition formula. Replacing $y$ by $-y$ gives

$$\cos(x + y) = \cos x \cos(-y) + \sin x \sin(-y) = \cos x \cos y - \sin x \sin y$$

which is the third addition formula. Now, replacing $x$ by $\frac{\pi}{2} - x$ gives

$$\cos \left(\frac{\pi}{2} - x + y\right) = \cos \left(\frac{\pi}{2} - x\right) \cos y - \sin \left(\frac{\pi}{2} - x\right) \sin y$$

Recalling that $\sin \left(\frac{\pi}{2} - z\right) = \cos z$ and $\cos \left(\frac{\pi}{2} - z\right) = \sin z$,

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$

which is the second addition formula. Finally, replacing $y$ by $-y$ gives the first addition formula.

$\blacksquare$