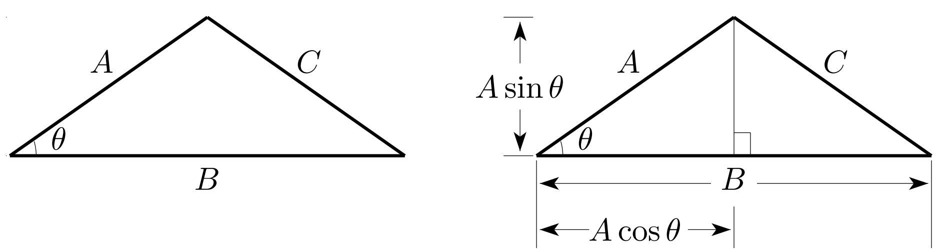
Trig Identities – Cosine Law and Addition Formulae

The cosine law

If a triangle has sides of length A, B and C and the angle opposite the side of length C is θ, then

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

Proof: Applying Pythagorous to the right hand triangle of the right hand figure of



gives

$$\begin{aligned} C^2 &= \big(B - A\cos\theta\big)^2 + \big(A\sin\theta\big)^2 \\ &= B^2 - 2A\,B\,\cos\theta + A^2\cos^2\theta + A^2\sin^2\theta \\ &= B^2 - 2A\,B\,\cos\theta + A^2 \end{aligned}$$

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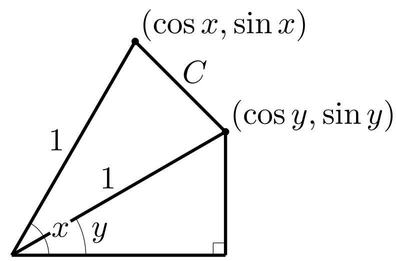
Addition and subtraction formulae

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x\sin y \\ \sin(x-y) &= \sin x\cos y - \cos x\sin y \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y \\ \cos(x-y) &= \cos x\cos y + \sin x\sin y \end{aligned}$$

Proof: We first prove $\cos(x-y) = \cos x \cos y + \sin x \sin y$. The angle, of the upper triangle,



that is opposite the side of length C is $x - y$. So, by the cosine law,

$$C^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(x-y) = 2 - 2\cos(x-y)$$

But the side of length C joins the points $(\cos y, \sin y)$ and $(\cos x, \sin x)$ and so we also have, by Pythagorous,

$$\begin{aligned} C^2 &= (\cos y - \cos x)^2 + (\sin y - \sin x)^2 \\ &= \cos^2 y - 2\cos x \cos y + \cos^2 x + \sin^2 y - 2 \sin x \sin y + \sin^2 x \\ &= 2 - 2\cos x \cos y - 2 \sin x \sin y \end{aligned}$$

Setting the two formulae for C^2 equal to each other gives

$$\begin{aligned} 2 - 2\cos(x-y) &= 2 - 2\cos x \cos y - 2 \sin x \sin y \\ \implies \quad -2\cos(x-y) &= -2\cos x \cos y - 2 \sin x \sin y \\ \implies \quad \cos(x-y) &= \cos x \cos y + \sin x \sin y \end{aligned}$$

which is the fourth addition formula. Replacing y by $-y$ gives

$$\cos(x+y) = \cos x \cos(-y) + \sin x \sin(-y) = \cos x \cos y - \sin x \sin y$$

which is the third addition formula. Now, replacing x by $\frac{\pi}{2} - x$ gives

$$\cos\left(\tfrac{\pi}{2} - x + y\right) = \cos\left(\tfrac{\pi}{2} - x\right)\cos y - \sin\left(\tfrac{\pi}{2} - x\right)\sin y$$

Recalling that $\sin\left(\frac{\pi}{2} - z\right) = \cos z$ and $\cos\left(\frac{\pi}{2} - z\right) = \sin z$,

$$\sin\left(x - y\right) = \sin x \cos y - \cos x \sin y$$

which is the second addition formula. Finally, replacing y by $-y$ gives the first addition formula. ■