Evaluating Limits Using Taylor Expansions

Taylor polynomials provide a good way to understand the behaviour of a function near a specified point and so are useful for evaluating complicated limits. We’ll see examples of this later in these notes.

We’ll just start by recalling that if, for some natural number \( n \), the function \( f(x) \) has \( n + 1 \) derivatives near the point \( x_0 \), then

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + E_n(x)
\]

where

\[
P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n
\]

is the Taylor polynomial of degree \( n \) for the function \( f(x) \) and expansion point \( x_0 \) and

\[
E_n(x) = f(x) - P_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x - x_0)^{n+1}
\]

is the error introduced when we approximate \( f(x) \) by the polynomial \( P_n(x) \). Here \( c \) is some unknown number between \( x_0 \) and \( x \). As \( c \) is not known, we do not know exactly what the error \( E_n(x) \) is. But that is usually not a problem. In taking the limit \( x \to x_0 \), we are only interested in \( x \)’s that are very close to \( x_0 \), and when \( x \) is very close \( x_0 \), \( c \) must also be very close to \( x_0 \). As long as \( f^{(n+1)}(x) \) is continuous at \( x_0 \), \( f^{(n+1)}(c) \) must approach \( f^{(n)}(x_0) \) as \( x \to x_0 \). In particular there must be constants \( M, D > 0 \) such that \( |f^{(n+1)}(c)| \leq M \) for all \( c \)’s within a distance \( D \) of \( x_0 \). If so, there is another constant \( C \) (namely \( \frac{M}{(n+1)!} \)) such that

\[
|E_n(x)| \leq C|x - x_0|^{n+1} \quad \text{whenever } |x - x_0| \leq D
\]

There is some notation for this behaviour.

**Definition 1 (Big O)** We say “\( F(x) \) is of order \( |x - x_0|^m \) near \( x_0 \)” and we write \( F(x) = O(|x - x_0|^m) \) if there exist constants \( C, D > 0 \) such that

\[
|F(x)| \leq C|x - x_0|^m \quad \text{whenever } |x - x_0| \leq D
\]

Whenever \( O(|x - x_0|^m) \) appears within an algebraic expression, it just stands for some (unknown) function \( F(x) \) that obeys (1). This is called “big O” notation. Here are some examples.

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Example 2 Let \( f(x) = \sin x \) and \( x_0 = 0 \). Then
\[
\begin{align*}
f(x) &= \sin x, & f'(x) &= \cos x, & f''(x) &= -\sin x, & f^{(3)}(x) &= -\cos x, & f^{(4)}(x) &= \sin x, & \cdots \\
f(0) &= 0, & f'(0) &= 1, & f''(0) &= 0, & f^{(3)}(0) &= -1, & f^{(4)}(0) &= 0, & \cdots 
\end{align*}
\]
and the pattern repeats. Thus \( |f^{(n+1)}(c)| \leq 1 \) for all \( c \). So the Taylor polynomial of, for example, degree 4 and its error term are
\[
\begin{align*}
sin x &= x - \frac{1}{3!}x^3 + \frac{\cos c}{5!}x^5 \\
&= x - \frac{1}{3!}x^3 + O(|x|^5)
\end{align*}
\]
under Definition 1, with \( C = \frac{1}{5!} \) and any \( D > 0 \). Similarly, for any natural number \( n \),
\[
\begin{align*}
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + O(|x|^{2n+3}) \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} + O(|x|^{2n+2})
\end{align*}
\]

Example 3 Let \( n \) be any natural number. We have seen that, since \( \frac{d^m}{dx^m} e^x = e^x \) for every integer \( m \geq 0 \),
\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}
\]
for some \( c \) between 0 and \( x \). If, for example, \( |x| \leq 1 \), then \( |e^c| \leq e \), so that the error term
\[
\left| \frac{e^c}{(n+1)!}x^{n+1} \right| \leq C|x|^{n+1}
\]
with \( C = \frac{e}{(n+1)!} \) whenever \( |x| \leq 1 \).

So, under Definition 1, with \( C = \frac{e}{(n+1)!} \) and \( D = 1 \),
\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + O(|x|^{n+1})
\]

Example 4 Let \( f(x) = \ln(1 + x) \) and \( x_0 = 0 \). Then
\[
\begin{align*}
f'(x) &= \frac{1}{1+x}, & f''(x) &= -\frac{1}{(1+x)^2}, & f^{(3)}(x) &= \frac{2}{(1+x)^3}, & f^{(4)}(x) &= -\frac{2x^3}{(1+x)^4}, & f^{(5)}(x) &= \frac{2x^3}{(1+x)^5} \\
f'(0) &= 1, & f''(0) &= -1, & f^{(3)}(0) &= 2, & f^{(4)}(0) &= -3!, & f^{(5)}(0) &= 4!
\end{align*}
\]
For any natural number \( n \),
\[
f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}, \quad \frac{1}{n!} f^{(n)}(0) x^n = (-1)^{n-1} \frac{(n-1)!}{n!} x^n = (-1)^{n-1} \frac{x^n}{n}
\]
so
\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + E_n(x)
\]

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Remark 6

The big O notation has a few properties that are useful in computations and taking limits. All follow immediately from Definition 1.

1. If $p > 0$, then $\lim_{x \to 0} O(|x|^p) = 0$.

2. For any real numbers $p$ and $q$, $O(|x|^p)O(|x|^q) = O(|x|^{p+q})$.

   (This is just because $C|x|^p \times C'|x|^q = (CC')|x|^{p+q}$.)

   In particular, $ax^m O(|x|^p) = O(|x|^{p+m})$, for any constant $a$ and any integer $m$.

3. For any real numbers $p$ and $q$, $O(|x|^p) + O(|x|^q) = O(|x|^\min\{p,q\})$.

   (For example, if $p = 2$ and $q = 5$, then $C|x|^2 + C'|x|^5 = (C+C'|x|^3)|x|^2 \leq (C+C')|x|^2$ whenever $|x| \leq 1$.)

4. For any real numbers $p$ and $q$ with $p > q$, any function which is $O(|x|^p)$ is also $O(|x|^q)$ because $C|x|^p = C|x|^{p-q}|x|^q \leq C|x|^q$ whenever $|x| \leq 1$.

The Taylor expansion (2) with $p > q$ and $c = 1$ is the same as $O(|x|^q)$.
Example 7  In this example we’ll evaluate the harder limit

\[
\lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2} x \sin x}{[\ln(1 + x)]^4}
\]

Using Examples 2 and 4,

\[
\lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2} x \sin x}{[\ln(1 + x)]^4} = \lim_{x \to 0} \frac{[1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 + O(x^6)] - 1 + \frac{1}{2} x [x - \frac{1}{3!} x^3 + O(|x|^5)]}{[x + O(x^2)]^4}
\]

\[
= \lim_{x \to 0} \frac{[\frac{1}{4!} - \frac{1}{2 \times 3!}]x^4 + O(x^6) + \frac{x}{2} O(|x|^5)}{[x + O(x^2)]^4}
\]

\[
= \lim_{x \to 0} \frac{[\frac{1}{4!} - \frac{1}{2 \times 3!}]x^4 + O(x^6) + O(x^6)}{[x + O(x^2)]^4}
\]

by Remark 6, part (2)

\[
= \lim_{x \to 0} \frac{[\frac{1}{4!} - \frac{1}{2 \times 3!}]x^4 + O(x^6)}{[x + x O(|x|)]^4}
\]

by Remark 6, parts (2), (3)

\[
= \lim_{x \to 0} \frac{[\frac{1}{4!} - \frac{1}{2 \times 3!}]x^4 + x^4 O(x^2)}{x^4 [1 + O(|x|)]^4}
\]

by Remark 6, part (2)

\[
= \lim_{x \to 0} \frac{[\frac{1}{4!} - \frac{1}{2 \times 3!}] + O(x^2)}{[1 + O(|x|)]^4}
\]

\[
= \frac{1}{4!} - \frac{1}{2 \times 3!}
\]

by Remark 6, part (1)

\[
= \frac{1}{3!} \left( \frac{1}{4} - \frac{1}{2} \right) = - \frac{1}{4!}
\]