Units and the Derivative of $\sin x$

Any derivative measures the rate of change of one quantity per unit change of a second quantity. If you change the units of either quantity, the numerical value of the derivative changes. For example, the speed of a car is given by the derivative $\frac{dx}{dt}$, where x is the position of the car and t is time. If x is measured in kilometers and t is measured in hours, one might have, for example, $\frac{dx}{dt} = 60$ km/hr. But if you decide to measure t in seconds rather than hours, the same vehicle moving at the same speed will have $\frac{dx}{dt} = 60 \times \frac{1}{3600}$ km/sec.

As a second example, consider the average rate of change of $\sin x$ as x moves from 0 to a right angle. If x is measured in radians,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1-0}{\frac{\pi}{2}-0} = \frac{2}{\pi}$$

because $\sin x$ changes from 0 to 1 as x changes from 0 radians to $\frac{\pi}{2}$ radians. But, if x is measured in degrees,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1-0}{90-0} = \frac{1}{90}$$

The two answers are different even though x moved from 0 to a right angle in both cases. The usual formula that you have memorized for the derivative of $\sin x$, namely $\frac{d}{dx}\sin x = \cos x$, **is valid only if x is measured in radians**.

Problem. Find $\frac{d}{dx}\sin x$, if x is in degrees, rather than radians.

Solution 1. I think that the safest and clearest way to approach this problem is to give different names to the sin of x degrees and the sin of x radians. Of course, in the real world, nobody does this. Use $\mathrm{Sin}\,(x)$ to denote the sin of x degrees and $\sin(x)$ to denote the sin of x radians. Similarly, use $\mathrm{Cos}\,(x)$ to denote the cos of x degrees and $\cos(x)$ to denote the cos of x radians. For example, $\mathrm{Sin}\,(90) = \sin\left(\frac{\pi}{2}\right) = 1$ and $\mathrm{Cos}\,(90) = \cos\left(\frac{\pi}{2}\right) = 0$. In general, $\mathrm{Sin}\,(x) = \sin\left(\frac{\pi}{180}x\right)$. By definition,

$$\begin{aligned}\frac{d}{dx}\mathrm{Sin}\,(x) &= \lim_{h\to0}\frac{\mathrm{Sin}\,(x+h)-\mathrm{Sin}\,(x)}{h} = \lim_{h\to0}\frac{\sin\left(\frac{\pi}{180}(x+h)\right)-\sin(\frac{\pi}{180}x)}{h}\\ &= \lim_{h\to0}\frac{\sin\left(\frac{\pi}{180}x+\frac{\pi}{180}h\right)-\sin(\frac{\pi}{180}x)}{h}\end{aligned}$$

Set $H = \frac{\pi}{180}h$. Then

$$\begin{aligned}\frac{d}{dx}\mathrm{Sin}\,(x) &= \lim_{H\to0}\frac{\sin\left(\frac{\pi}{180}x+H\right)-\sin(\frac{\pi}{180}x)}{\frac{180}{\pi}H} = \frac{\pi}{180}\lim_{H\to0}\frac{\sin\left(\frac{\pi}{180}x+H\right)-\sin(\frac{\pi}{180}x)}{H}\\ &= \frac{\pi}{180}\frac{d}{dt}\sin(t)\Big|_{t=\frac{\pi}{180}x} = \frac{\pi}{180}\cos(\tfrac{\pi}{180}x) = \frac{\pi}{180}\mathrm{Cos}\,(x)\end{aligned}$$

Thus $\boxed{\frac{d}{dx}\mathrm{Sin}\,(x) = \frac{\pi}{180}\mathrm{Cos}\,(x)}$. Of course, people do not usually give separate names to $\mathrm{Sin}\,(x)$ and $\sin(x)$. So, it is also correct to say that

$$\boxed{\frac{d}{dx}\sin(x) = \frac{\pi}{180}\cos(x), \text{ when } \sin(x) \text{ and } \cos(x) \text{ mean sin and cos of } x \text{ degrees}}$$

Solution 2. Using the same notation as in Solution 1, $\mathrm{Sin}\,(x) = \sin\left(\frac{\pi}{180}x\right)$. So, by the chain rule, with $f(y) = \sin y$ and $g(x) = \frac{\pi}{180}x$,

$$\frac{d}{dx}\mathrm{Sin}\,(x) = \frac{d}{dx}\sin\left(\frac{\pi}{180}x\right) = \frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x) = \cos\left(\frac{\pi}{180}x\right)\frac{\pi}{180}$$

This is the same answer as we got in Solution 1.