Powers and Roots

The symbol $x^3$ means $x \cdot x \cdot x$. More generally, if $n$ is any strictly positive integer, then $x^n$ means the product $x \cdot x \cdot \ldots \cdot x$.

If $x$ is a positive number, then $\sqrt[n]{x} = x^{\frac{1}{n}}$ is used to denote the positive number that obeys $\sqrt[n]{x} \cdot \sqrt[n]{x} = x$. For example $2 \times 2 = 4$, so $\sqrt{4} = 2$. The equation $x^2 = 137$ has two solutions. The positive one is denoted $\sqrt{137}$ and the negative one $-\sqrt{137}$. So the general solution to $x^2 = 137$ is $x = \pm \sqrt{137}$. It is possible to define the square root of a negative number. But this involves enlarging the real number system to the complex number system and will not be covered in this course.

If $x$ is a positive number and $n$ is a strictly positive integer, then $\sqrt[n]{x} = x^{\frac{1}{n}}$ is used to denote the positive number that obeys $(\sqrt[n]{x})^n = x$. For example $2 \cdot 2 \cdot 2 \cdot 2 = 16$ so $\sqrt[4]{16} = 2$.

Call

$$p(x) = x^n \quad r(x) = x^{\frac{1}{n}}$$

So $p$ is the symbol for a machine (let’s call it a powifier) that outputs $x^n$ in response to the input $x$ and $r$ is the symbol for a machine (let’s call it a rootifier) that outputs $x^{\frac{1}{n}}$ in response to the input $x$. If you put $x$ into the input hopper of the rootifier you get the output $r(x) = \sqrt[n]{x}$. If you take this output of the rootifier and put it into the input hopper of the powifier, the output of the powifier will be

$$p(r(x)) = (r(x))^n = (\sqrt[n]{x})^n = x$$

The last equality was a consequence of the definition of $\sqrt[n]{x}$. Similarly, if you feed $x > 0$ into the powifier first and then feed the resulting output into the rootifier, the output of the rootifier is

$$r(p(x)) = \sqrt[n]{p(x)} = \sqrt[n]{x^n} = x$$

The equations $r(p(x)) = p(r(x)) = x$ say that the rootifier undoes whatever the powifier does and vice versa. Consequently, $p$ and $r$ are called inverse functions.

Because $p$ and $r$ are inverse functions, it is very easy to convert the graph of $p(x)$ into the graph of $r(x)$. Here is how. Graph $y = x^n$. Observe that if $(x_0, y_0)$ is any point on the graph, then $x_0$ and $y_0$ are related by $y_0 = x_0^n$. Consequently, by the definition of the $n^{th}$ root, $x_0 = \sqrt[n]{y_0}$.
Now redraw the same graph, but this time flip it over so that the $y$–axis runs horizontally and the $x$–axis runs vertically. This is a little unorthodox, but perfectly legal.

Then replace every $x$ with a $Y$ and every $y$ with an $X$.

Any point on the curve has its $X$ and $Y$ coordinates related by $Y = \sqrt[2n]{X}$. So the curve is the graph of $\sqrt[2n]{X}$ against $X$. 