Error Behaviour of Newton’s Method

Newton’s method is a procedure for finding approximate solutions to equations of the form \( f(x) = 0 \). The procedure is to

1) Make a preliminary guess \( x_1 \).
2) Define \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \).
3) Iterate. That is, once you have computed \( x_n \), define \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \).

Newton’s method usually works spectacularly well, provided your initial guess is reasonably close to a solution of \( f(x) = 0 \). A good way to select this initial guess is to sketch the graph of \( y = f(x) \). In these notes we shall see why “Newton’s method usually works spectacularly well, provided your initial guess is reasonably close to a solution of \( f(x) = 0 \).” We shall assume that there are two numbers \( L, M > 0 \) such that \( f \) obeys:

\[
\begin{align*}
\text{H1)} & \quad |f'(x)| \geq L \text{ for all } x \\
\text{H2)} & \quad |f''(x)| \leq M \text{ for all } x
\end{align*}
\]

Let \( r \) be any solution of \( f(x) = 0 \). Then \( f(r) = 0 \). Suppose that we have already computed \( x_n \). The error in \( x_n \) is \(|x_n - r|\). We now derive a formula that relates the error after the next step, \(|x_{n+1} - r|\), to \(|x_n - r|\). We have seen in class that

\[
f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(c)(x - x_n)^2
\]

for some \( c \) between \( x_n \) and \( x \). In particular, choosing \( x = r \),

\[
0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{1}{2} f''(c)(r - x_n)^2
\]

(1)

By the definition of \( x_{n+1} \),

\[
0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)
\]

(2)

(In fact, we defined \( x_{n+1} \) as the solution of \( 0 = f(x_n) + f'(x_n)(x - x_n) \).) Subtracting (2) from (1).

\[
0 = f'(x_n)(r - x_{n+1}) + \frac{1}{2} f''(c)(r - x_n)^2 \Rightarrow x_{n+1} - r = \frac{f''(c)}{2f'(x_n)}(x_n - r)^2
\]

\[
\Rightarrow |x_{n+1} - r| = \frac{|f''(c)|}{2|f'(x_n)|}|x_n - r|^2
\]

If the guess \( x_n \) is close to \( r \), then \( c \), which must be between \( x_n \) and \( r \), is also close to \( r \) and \(|x_{n+1} - r| \approx \frac{|f''(c)|}{2|f'(r)|}|x_n - r|^2 \). Even if \( x_n \) is not close to \( r \), by the hypotheses (H1) and (H2) on the behaviour of \( f \)

\[
|x_{n+1} - r| \leq \frac{M}{L^2}|x_n - r|^2
\]

(3)
Let’s denote by $\varepsilon_1$ the error $|x_1 - r|$ of our initial guess. In fact, let’s denote by $\varepsilon_n$ the error $|x_n - r|$ in $x_n$. Then (3) says

$$\varepsilon_{n+1} \leq \frac{M}{2L} \varepsilon_n^2$$

In particular

$$\begin{align*}
\varepsilon_2 &\leq \frac{M}{2L} \varepsilon_1^2 \\
\varepsilon_3 &\leq \frac{M}{2L} \varepsilon_2^2 \leq \frac{M}{2L} \left( \frac{M}{2L} \varepsilon_1^2 \right)^2 = \left( \frac{M}{2L} \right)^3 \varepsilon_1^4 \\
\varepsilon_4 &\leq \frac{M}{2L} \varepsilon_3^2 \leq \frac{M}{2L} \left( \left( \frac{M}{2L} \right)^3 \varepsilon_1^4 \right)^2 = \left( \frac{M}{2L} \right)^7 \varepsilon_1^8 \\
\varepsilon_5 &\leq \frac{M}{2L} \varepsilon_4^2 \leq \frac{M}{2L} \left( \left( \frac{M}{2L} \right)^7 \varepsilon_1^8 \right)^2 = \left( \frac{M}{2L} \right)^{15} \varepsilon_1^{16}
\end{align*}$$

By now we can see a pattern forming, that is easily verified by induction

$$\varepsilon_n \leq \left( \frac{M}{2L} \right)^{2^{n-1}-1} \varepsilon_1^{2^{n-1}} = \frac{2L}{M} \left( \frac{M}{2L} \varepsilon_1 \right)^{2^{n-1}}$$

As long as $\frac{M}{2L} \varepsilon_1 < 1$ (which tells us quantitatively how good our first guess has to be in order for Newton’s method to converge), this goes to zero extremely quickly as $n$ increases. For example, suppose that $\frac{M}{2L} \varepsilon_1 \leq \frac{1}{2}$. Then

$$\varepsilon_n \leq 2L \left( \frac{1}{2} \right)^{2^{n-1}} \leq \frac{2L}{M} \begin{cases} 0.25 & \text{if } n = 2 \\ 0.0625 & \text{if } n = 3 \\ 0.0039 & \text{if } n = 4 \\ 0.000015 & \text{if } n = 5 \\ 0.0000000023 & \text{if } n = 6 \\ 0.000000000000000054 & \text{if } n = 7 \end{cases}$$

Each time you increase $n$ by one, the number of zeroes after the decimal place roughly doubles.