Long Division of Polynomials

Suppose that $P(x)$ is a polynomial of degree p and suppose that you know that r is a root of that polynomial. In other words, suppose you know that $P(r) = 0$. Then it is always possible to factor $(x - r)$ out of $P(x)$. More precisely, it is alway possible to find a polynomial $Q(x)$ of degree $p - 1$ such that

$$P(x) = (x - r)Q(x)$$

In sufficiently simple cases, you can probably do this factoring by inspection. For example, $P(x) = x^2 - 4$ has $r = 2$ as a root because $P(2) = 2^2 - 4 = 0$. In this case, $P(x) = (x - 2)(x + 2)$ so that $Q(x) = (x + 2)$. As another example, $P(x) = x^2 - 2x - 3$ has $r = -1$ as a root because $P(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$. In this case, $P(x) = (x + 1)(x - 3)$ so that $Q(x) = (x - 3)$.

Once you have found a root r of a polynomial, even if you cannot factor $(x - r)$ out of the polynomial by inspection, you can find $Q(x)$ by dividing $P(x)$ by $x - r$, using the long division algorithm you learned in public school, but with 10 replaced by x.

Example. $P(x) = x^3 - x^2 + 2$.

Because $P(-1) = (-1)^3 - (-1)^2 + 2 = -1 - 1 + 2 = 0$, $r = -1$ is a root of this polynomial. So we divide $\frac{x^3 - x^2 + 2}{x+1}$. The first term, x^2, in the quotient is chosen so that when you multiply it by the denominator, $x^2(x + 1) = x^3 + x^2$, the leading term, x^3, matches the leading term in the numerator, $x^3 - x^2 + 2$, exactly.

$$\begin{array}{r|l} & x^2 \\ \hline x+1 & x^3 - x^2 + \qquad 2 \\ & x^3 + x^2 \end{array}$$

When you subtract $x^2(x + 1) = x^3 + x^2$ from the numerator $x^3 - x^2 + 2$ you get the remainder $-2x^2 + 2$. Just like in public school, the 2 is not normally "brought down" until it is actually needed.

$$\begin{array}{r|l} & x^2 \\ \hline x+1 & x^3 - x^2 + \qquad 2 \\ & \underline{x^3 + x^2} \\ & \quad -2x^2 \end{array}$$

The next term, $-2x$, in the quotient is chosen so that when you multiply it by the denominator, $-2x(x + 1) = -2x^2 - 2x$, the leading term $-2x^2$ matches the leading term in the remainder exactly.

$$\begin{array}{r|l} & x^2 - 2x \\ \hline x+1 & x^3 - x^2 + \qquad 2 \\ & \underline{x^3 + x^2} \\ & \quad -2x^2 \\ & \quad -2x^2 - 2x \end{array}$$

And so on.

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ \hline x+1 & x^3 - x^2 + \qquad 2 \\ & \underline{x^3 + x^2} \\ & \quad -2x^2 \\ & \quad \underline{-2x^2 - 2x} \\ & \qquad\qquad 2x + 2 \\ & \qquad\qquad \underline{2x + 2} \\ & \qquad\qquad\qquad 0 \end{array}$$

Note that we finally end up with a remainder 0. Since -1 is a root of the numerator, $x^3 - x^2 + 2$, the denominator $x - (-1)$ must divide the numerator exactly.

There is an alternative to long division that involves more writing. In the previous example, we know that $\frac{x^3-x^2+2}{x+1}$ must be a polynomial (since -1 is a root of the numerator) of degree 2. So

$$\frac{x^3 - x^2 + 2}{x + 1} = ax^2 + bx + c$$

for some, as yet unknown, coefficients a, b and c. Cross multiplying and simplifying

$$\begin{aligned} x^3 - x^2 + 2 &= (ax^2 + bx + c)(x + 1) \\ &= ax^3 + (a + b)x^2 + (b + c)x + c \end{aligned}$$

Matching coefficients of the various powers of x on the left and right hand sides

$$\begin{aligned} &\text{coefficient of } x^3: & a &= 1 \\ &\text{coefficient of } x^2: & a + b &= -1 \\ &\text{coefficient of } x^1: & b + c &= 0 \\ &\text{coefficient of } x^0: & c &= 2 \end{aligned}$$

tells us directly that $a = 1$ and $c = 2$. Subbing $a = 1$ into $a+b = -1$ tells us that $1+b = -1$ and hence $b = -2$.