Inequalities

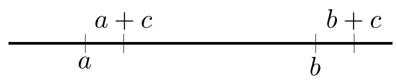
Every real number a is designated as being either positive ($a > 0$) or zero ($a = 0$) or negative ($a < 0$). By definition $a > b$ if and only if $a - b > 0$ and $a < b$ if and only if $a - b < 0$. Hence, given any two real numbers a and b,

$$\text{either} \qquad a > b \qquad \text{or} \qquad a = b \qquad \text{or} \qquad a < b$$

Properties of Inequalities

Let a, b and c be real numbers.

1) If $a < b$, then $a + c < b + c$.

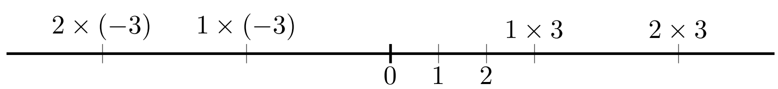


2) If $a < b$ and $c > 0$ then $ac < bc$. But if $a < b$ and $c < 0$ then $ac > bc$.

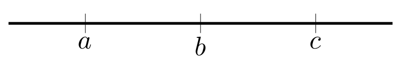
Proof: If $a < b$ and $c > 0$, then, by definition, $b - a > 0$ and $c > 0$. But the product of any two positive numbers is positive, so $bc - ac = c(b - a) > 0$, which implies $ac < bc$. On the other hand, the product of a positive number and a negative number is a negative number. So if $a < b$ and $c < 0$, we have $b - a > 0$ and $c < 0$ implying $bc - ac = c(b - a) < 0$ so that $ac > bc$.

Example: Here is an example with $a = 1$ and $b = 2$.

- For $c = 3$, $ac = (1)(3) = 3 < bc = (2)(3) = 6$.
- For $c = -3$, $ac = (1)(-3) = -3 > bc = (2)(-3) = -6$.
- For $c = 0$, $ac = (1)(0) = 0 = bc = (2)(0) = 0$.



3) If $a < b$ and $b < c$, then $a < c$.

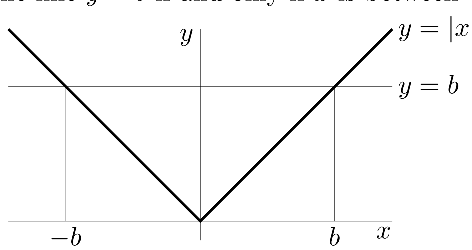


4) If $a > 0$, then $\frac{1}{a} > 0$. If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.

5) If $0 \le a < b$, then $a^2 < b^2$.

Proof: This is an easy consequence of Properties (2) and (3). By Property (2) with $c = a$, $a^2 \le ab$. By Property (2) with $c = b$, $ab < b^2$. By Property (3), $a^2 \le ab < b^2$.

6) $|a| < b$ if and only if $-b < a < b$. This is most easily seen by looking at the graph $y = |x|$. To determine which values of x obey, $|x| < b$, we have to determine which points on $y = |x|$ have $y < b$. That is, we have to determine which points on $y = |x|$ lie below the line $y = b$. The point $(x, |x|)$ on the graph $y = |x|$ lies below the line $y = b$ if and only if x is between $-b$ and b.



Example 1 Find all real numbers x obeying $|\sqrt{x} - 3| < 10^{-6}$.

Solution.

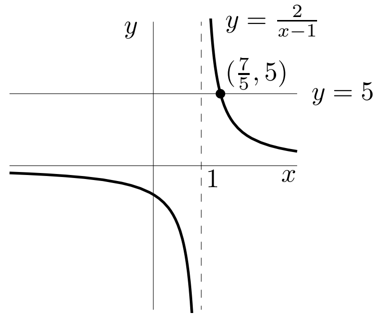
$$\begin{aligned} |\sqrt{x} - 3| < 10^{-6} &\iff -10^{-6} < \sqrt{x} - 3 < 10^{-6} && \text{by Property (6)} \\ &\iff 3 - 10^{-6} < \sqrt{x} < 3 + 10^{-6} && \text{by Property (1) with } c = 3 \\ &\iff (3 - 10^{-6})^2 < x < (3 + 10^{-6})^2 && \text{by Property (5)} \end{aligned}$$

Example 2 Find all real numbers x obeying $\frac{2}{x-1} \geq 5$.

Solution. If $x - 1 < 0$, then $\frac{2}{x-1} < 0$ while $5 > 0$. So no x's with $x - 1 < 0$, that is $x < 1$, are allowed. If $x - 1 > 0$, that is $x > 1$,

$$\begin{aligned} \tfrac{2}{x-1} \geq 5 &\iff 2 \geq 5(x-1) && \text{by Property (2) with } c = x - 1 > 0 \\ &\iff \tfrac{2}{5} \geq x - 1 && \text{by Property (2) with } c = \tfrac{1}{5} \\ &\iff \tfrac{7}{5} \geq x && \text{by Property (1) with } c = 1 \end{aligned}$$

So the allowed x's are $\boxed{1 < x \leq \frac{7}{5}}$. This is consistent with the graph below.

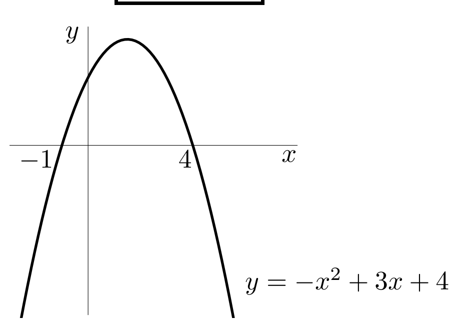


Example 3 Find all real numbers x obeying $-x^2 + 3x + 4 > 0$.

Solution. Factoring

$$-x^2 + 3x + 4 = -(x^2 - 3x - 4) = -(x - 4)(x + 1)$$

If the two factors $(x-4)$ and $(x+1)$ have the same sign (either both positive or both negative) then $(x-4)(x+1)$ will be positive and $-x^2 + 3x + 4 = -(x-4)(x+1)$ will be negative. Hence we need the two factors $(x - 4)$ and $(x + 1)$ to have opposite sign. That is, we need $\boxed{-1 < x < 4}$. This is consistent with the graph below.



Example 3 Let $\varepsilon > 0$. Find $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon$ for all $|x-5| < \delta$.

Solution. I shall pick a δ that is smaller than 1. Then, if $|x-5| < \delta$, we have $|x-5| < 1$ so that $4 < x < 6$ and

$$\left|\tfrac{1}{x} - \tfrac{1}{5}\right| = \left|\tfrac{5-x}{5x}\right| = \tfrac{|5-x|}{5|x|} < \tfrac{|5-x|}{5\times 4}$$

since $x > 4$. As

$$\tfrac{|5-x|}{5\times 4} < \varepsilon \qquad \text{if} \qquad |5-x| < 20\varepsilon$$

we have that

$$|5-x| < \min\{1, 20\varepsilon\} \quad \Rightarrow \quad \left|\tfrac{1}{x} - \tfrac{1}{5}\right| < \varepsilon$$

Hence $\boxed{\delta = \min\{1, 20\varepsilon\}}$ does the trick.

Example 4 Suppose that we had guessed, incorrectly, that $\lim\limits_{x\to 5} \frac{1}{x} = \frac{1}{6}$. If we were to try and verify this, by applying the precise definition of the limit, we would let $\varepsilon > 0$ and try to find a $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{6}\right| < \varepsilon$ for all $|x-5| < \delta$. The following shows that this cannot be done.

Solution. We shall pick $\varepsilon = \frac{1}{100}$ and show that there is no $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{6}\right| < \frac{1}{100}$ for all $|x-5| < \delta$. For all x obeying $|x-5| < 0.1$ we have $4.9 < x < 5.1$ and hence

$$\left|\tfrac{1}{x} - \tfrac{1}{6}\right| = \left|\tfrac{6-x}{6x}\right| \ge \tfrac{6-5.1}{6\times 4.9} = \tfrac{0.9}{29.4} > \tfrac{1}{100} = \varepsilon$$

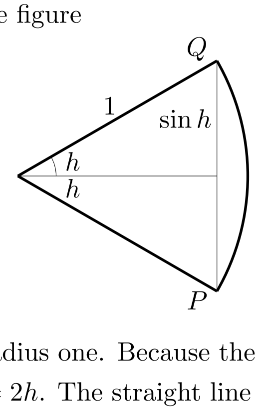
No matter what $\delta > 0$ we pick, there will be some x's obeying $|x-5| < \delta$ that also obey $|x-5| < 0.1$ and hence that also obey $\left|\frac{1}{x} - \frac{1}{6}\right| > \frac{1}{100} = \varepsilon$.

Example 5 Let $\varepsilon > 0$. Find $\delta > 0$ such that $\left|\sin(\theta + h) - \sin\theta\right| < \varepsilon$ for all $|h| < \delta$.

Solution. We shall use the fact that

$$|\sin h| \le |h| \tag{1}$$

for all h, provided the angle h is given in radians. First, we verify this fact. If $|h| \ge 1$, (1) is obvious because $|\sin h| \le 1 \le |h|$. For $0 \le h \le 1$, consider the figure



The arc from P to Q is part of a circle of radius one. Because the arc subtends the angle $2h$, it is the fraction $\frac{2h}{2\pi}$ of the circle and so has length $\frac{2h}{2\pi} \times 2\pi = 2h$. The straight line from P to Q has length $2\sin h$. Because the straight line from P to Q is shorter than the arc from P to Q, we have $2\sin h \le 2h$. For $-1 \le h < 0$, $\sin h$ is negative so that

$$|\sin h| = -\sin h = \sin(-h)$$

But $-h$ is between 0 and 1, so we already know that $\sin(-h) \le -h = |h|$. This completes the verification of (1).
Now back to the main problem. By the trig identities $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos 2a = 1 - 2\sin^2 a$,

$$\begin{aligned}\sin(\theta + h) - \sin\theta &= \sin\theta\cos h + \cos\theta\sin h - \sin\theta \\ &= \big(\cos h - 1\big)\sin\theta + \cos\theta\sin h \\ &= -2\sin^2\tfrac{h}{2}\sin\theta + \cos\theta\sin h\end{aligned}$$

Since $|\sin h| \le |h|$, $\big|\sin\frac{h}{2}\big| \le \big|\frac{h}{2}\big|$, $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$

$$\begin{aligned}\big|\sin(\theta + h) - \sin\theta\big| &\le 2\big|\tfrac{h}{2}\big|^2 \times 1 + 1 \times |h| \\ &= |h| + \tfrac{1}{2}|h|^2\end{aligned}$$

If we pick $\delta < 1$, then $|h| < \delta < 1$ implies $|h|^2 = |h|\,|h| < |h|$ and

$$\big|\sin(\theta + h) - \sin\theta\big| \le |h| + \tfrac{1}{2}|h| = \tfrac{3}{2}|h| < \varepsilon \text{ if } |h| < \tfrac{2}{3}\varepsilon$$

Hence $\boxed{\delta = \min\big\{1, \frac{2}{3}\varepsilon\big\}}$ does the trick.