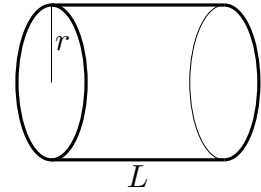


The Fuel Tank

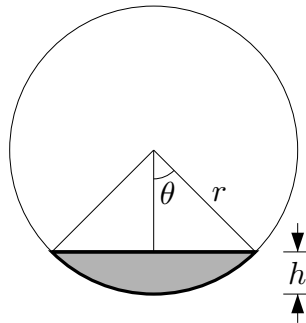
Question

Consider a cylindrical fuel tank of radius r and length L , that is lying on its side. Suppose that fuel is being pumped into the tank at a rate q . At what rate is the fuel level rising?



Solution

Here is an end view of the tank. The shaded part of the circle is filled with fuel.



Let us denote by $V(t)$ the volume of fuel in the tank at time t and by $h(t)$ the fuel level at time t . We have been told that $V'(t) = q$ and have been asked to determine $h'(t)$. It is possible to do so by finding a formula relating $V(t)$ and $h(t)$. But it is computationally easier to first find a formula relating V and the angle θ shown in the end view. Once we know $\theta'(t)$, we can easily obtain $h'(t)$ by differentiating $h = r - r \cos \theta$.

The volume of fuel is L times the cross-sectional area filled by the fuel. That is,

$$V = L \times \text{Area}\left(\text{shaded segment}\right)$$

While we do not have a canned formula for the area of a chord of a circle like this, it is easy to express the area of the chord in terms of two areas that we can compute.

$$V = L \times \text{Area}\left(\text{shaded segment}\right) = L \times \left[\text{Area}\left(\text{sector with angle } 2\theta\right) - \text{Area}\left(\text{triangle with angle } \theta\right) \right] \quad (1)$$

The piece of pie on the right hand side of formula (1) is the fraction $\frac{2\theta}{2\pi}$ of the full circle, so its area is $\frac{2\theta}{2\pi} \pi r^2 = \theta r^2$. The triangle on the right hand side of formula (1) has height $r \cos \theta$ and base $2r \sin \theta$ and hence has area $\frac{1}{2}(r \cos \theta)(2r \sin \theta) = r^2 \sin \theta \cos \theta$. Subbing these two

areas into formula (1) gives

$$V(t) = L \times [\theta r^2 - r^2 \sin \theta \cos \theta] = Lr^2[\theta(t) - \sin \theta(t) \cos \theta(t)] \quad (2)$$

Since

$$\begin{aligned} \frac{d}{dt} \sin \theta(t) \cos \theta(t) &= \cos \theta(t) \frac{d}{dt} \sin \theta(t) + \sin \theta(t) \frac{d}{dt} \cos \theta(t) \\ &= \cos \theta(t) [\theta'(t) \cos \theta(t)] + \sin \theta(t) [-\theta'(t) \sin \theta(t)] \\ &= \theta'(t) [\cos^2 \theta(t) - \sin^2 \theta(t)] \end{aligned}$$

differentiating both sides of formula (2) with respect to t gives

$$\begin{aligned} V'(t) &= Lr^2[\theta'(t) - \theta'(t) \cos^2 \theta(t) + \theta'(t) \sin^2 \theta(t)] = Lr^2[1 - \cos^2 \theta(t) + \sin^2 \theta(t)]\theta'(t) \\ &= 2\theta'(t)Lr^2 \sin^2 \theta(t) \end{aligned}$$

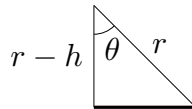
In the last step, I just subbed in $1 = \sin^2 \theta + \cos^2 \theta$. We can now easily solve for $\theta'(t)$.

$$\theta'(t) = \frac{V'(t)}{2Lr^2 \sin^2 \theta(t)} = \frac{q}{2Lr^2 \sin^2 \theta(t)}$$

Since $h(t) = r - r \cos \theta(t)$

$$h'(t) = r\theta'(t) \sin \theta(t) = \frac{qr \sin \theta(t)}{2Lr^2 \sin^2 \theta(t)} = \frac{q}{2Lr \sin \theta(t)}$$

We have now found $h'(t)$. It is more useful to express the answer in terms of h , rather than θ . From the triangle



and Pythagorouss we have

$$\sin \theta = \frac{\sqrt{r^2 - (r-h)^2}}{r} = \frac{\sqrt{2rh - h^2}}{r}$$

and hence

$$h'(t) = \boxed{\frac{q}{2L\sqrt{2rh(t) - h(t)^2}}}$$