Derivatives of Exponentials

Fix any $a > 0$. The definition of the derivative of $a^x$ is

$$\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = C(a) a^x$$

where we are using $C(a)$ to denote the coefficient $\lim_{h \to 0} \frac{a^h - 1}{h}$ that appears in the derivative. This coefficient does not depend on $x$. So, at this stage, we know that $\frac{d}{dx} a^x$ is $a^x$ times some fixed constant $C(a)$. We can learn more about $C(a)$ by just writing $a^h = (10^{\log_{10} a})^h = 10^{h \log_{10} a}$:

$$C(a) = \lim_{h \to 0} \frac{a^h - 1}{h} = \lim_{h \to 0} \frac{10^{h \log_{10} a} - 1}{h} = h \to 0 \frac{10^{h' - 1} h'}{\log_{10} a} = \log_{10} a \lim_{h' \to 0} \frac{10^{h' - 1}}{h'} = C(10) \log_{10} a$$

So we now know

$$\frac{d}{dx} a^x = C(10) (\log_{10} a) a^x$$

We will get a formula for $C(10)$ later in these notes. For now, we just try to get an idea of what $C(10)$ looks like by computing $\frac{10^h - 1}{h}$ for various values of $a$ and various small values of $h$. Here is a table of such values.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\frac{10^h - 1}{h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.5893</td>
</tr>
<tr>
<td>0.01</td>
<td>2.3293</td>
</tr>
<tr>
<td>0.001</td>
<td>2.3052</td>
</tr>
<tr>
<td>0.0001</td>
<td>2.3028</td>
</tr>
<tr>
<td>0.00001</td>
<td>2.3026</td>
</tr>
<tr>
<td>0.000001</td>
<td>2.3026</td>
</tr>
</tbody>
</table>

So it looks like $C(10) = 2.3026$, to four decimal places. In any event, recall that

- $\log_{10} a |_{a=1} = 0$ so that $C(a)|_{a=1} = 0$ (This is to be expected — when $a = 1$, $\frac{d}{dx} a^x = \frac{d}{dx} 1 = 0$.)
- $\log_{10} a$ increases as $a$ increases, and hence $C(a)$ increases as $a$ increases
- $\log_{10} a$ tends to $+\infty$ as $a \to \infty$, and hence $C(a)$ tends to $+\infty$ as $a \to \infty$

Consequently there is exactly one value of $a$ for which $C(a) = 1$. See the figure below. The value of $a$ for which $C(a) = C(10) \log_{10} a = 1$ is given the name $e$. That is, $e$ is defined by the condition...
\[ C(e) = C(10) \log_{10} e = 1, \text{ or equivalently, by the condition that } \frac{d}{dx} e^x = e^x. \] From our previous numerical experiment, it looks like

\[ 2.3026 \log_{10} e \approx 1 \implies \log_{10} e \approx \frac{1}{2.3026} \implies e \approx 10^{1/2.3026} \approx 2.7183 \]

We shall find a much better way to determine \( e \), to any desired degree of accuracy, shortly.

**The Taylor Expansion of \( e^x \)**

Let \( f(x) = e^x \). Then

\[
\begin{align*}
  f(x) &= e^x \\
  f'(x) &= e^x \\
  f''(x) &= e^x \\
  f'''(x) &= e^x \\
  \vdots
\end{align*}
\]

Recall that, for any positive integer \( n \),

\[
\begin{align*}
  f(x) &= f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + \frac{1}{(n+1)!} f^{(n+1)}(c)(x - x_0)^{n+1}
\end{align*}
\]

for some \( c \) between \( x_0 \) and \( x \). Applying this with \( f(x) = e^x \) and \( x_0 = 0 \), and using that \( f^{(m)}(x_0) = e^{x_0} = e^0 = 1 \) for all \( m \),

\[
e^x = f(x) = 1 + x + \cdots + \frac{x^n}{n!} + \frac{1}{(n+1)!} e^c x^{n+1}
\]

for some \( c \) between 0 and \( x \).

I claim that, for any fixed \( x \), the error term \( \frac{1}{(n+1)!} e^c x^{n+1} \) always goes to zero as \( n \) goes to infinity. To see this, first observe that \( e^c \) increases as \( c \) increases, so \( e^c \) is necessarily between \( e^0 = 1 \) and \( e^x \), for all \( n \). So to show that the error term \( \frac{1}{(n+1)!} e^c x^{n+1} \) always goes to zero as \( n \) goes to infinity, I just have to show that \( \varepsilon_n = \frac{|x|^{n+1}}{(n+1)!} \) always goes to zero as \( n \) goes to infinity. Now note that

\[
\varepsilon_{n+1} = \frac{|x|^{n+2}}{(n+2)!} = \frac{|x|^{n+1}}{(n+1)!} \frac{|x|}{(n+2)} = \frac{|x|}{(n+2)} \varepsilon_n
\]

Once \( n \) gets bigger than, for example, \( 2|x| \), we have \( \varepsilon_{n+1} = \frac{|x|}{(n+2)} \varepsilon_n < \frac{1}{2} \varepsilon_n \). That is, increasing \( n \) by decreases \( \varepsilon_n \) by a factor of at least 2. So \( \varepsilon_n \) must tend to zero as \( n \) tends to infinity.
Because, for any fixed $x$, the error term $\frac{1}{(n+1)!}e^x x^{n+1}$ always goes to zero as $n$ goes to infinity, we have, exactly,

$$e^x = \lim_{n \to \infty} \left[ 1 + x + \cdots + \frac{x^n}{n!} \right]$$

This limit is generally written

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + \cdots$$

or

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$$

In fact one may take $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ as the definition of $e^x$. If we set $x = 1$ we get

$$e = e^x \bigg|_{x=1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \bigg|_{x=1}$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \cdots$$

$$= 1 + 1 + 0.5 + 0.16 + 0.0416 + 0.0083 + 0.00138 + 0.00019841 + 0.00002480 + 0.00000276 + \cdots$$

$$= 2.71828182846$$

and, since $e$ was defined by $1 = C(e) = C(10) \log_{10} e$,

$$C(10) = \frac{1}{\log_{10} e} = \frac{\log_{10} 10}{\log_{10} e} = \ln 10 = 2.30258509299$$

and $C(a) = C(10) \log_{10} a = \frac{\log_{10} a}{\log_{10} e} = \ln a$ and

$$\frac{d}{dx} a^x = C(a) a^x = (\ln a) a^x$$

I do not have this derivative memorised. Every time I need it, I use

$$a^x = (e^{\ln a})^x = e^{x \ln a} \implies \frac{d}{dx} a^x = (\ln a) e^{x \ln a} = (\ln a) a^x$$

by the chain rule.