A Delta–Epsilon Example.

**Problem:** Let \( \varepsilon > 0 \). Find a \( \delta > 0 \) such that \( |\cos(2\pi x - \sin(x - 1)) - 1| < \varepsilon \) for all \( |x - 1| < \delta \).

**Solution:** Define \( f(x) = \cos(2\pi x - \sin(x - 1)) \). We are given some number \( \varepsilon > 0 \). We have to find a \( \delta > 0 \) such that \( |f(x) - f(1)| < \varepsilon \) for all \( |x - 1| < \delta \). We wish, in the end, to write an argument of the form

\[
\text{Set } \delta = \cdots \text{. If } |x - 1| < \delta \text{ then }
\]

\[
|f(x) - f(1)| \leq \cdots
\]

\[
\vdots
\]

\[
< \varepsilon
\]

However at this stage, we still do not know what \( \delta \) to pick. So I like to start by writing out an argument of the above form, but leaving the choice of \( \delta \) blank.

Set \( \delta = \cdots \). If \( |x - 1| < \delta \) then

\[
|f(x) - f(1)| = |f'(z) (x - 1)|
\]

for some \( z \) between \( x \) and 1,

by the Mean–Value Theorem

\[
= | - \sin(2\pi z - \sin(z - 1)) \{2\pi - \cos(x - 1)\} (x - 1)|
\]

\[
\leq |\{2\pi - \cos(x - 1)\} (x - 1)| \text{ since } |\sin(2\pi z - \sin(z - 1))| \leq 1
\]

\[
= |2\pi - \cos(x - 1)| |x - 1|
\]

\[
\leq (2\pi + 1) |x - 1| \text{ since } -1 \leq \cos(x - 1) \leq 1
\]

We would now like to terminate the string of inequalities with \( < \varepsilon \). But for that to be true we need \( (2\pi + 1) |x - 1| < \varepsilon \). That is, we need \( |x - 1| < \frac{\varepsilon}{2\pi + 1} \). This tells us to choose \( \delta = \frac{\varepsilon}{2\pi + 1} \). We may now \( \delta \) and give the full argument.

Set \( \delta = \frac{\varepsilon}{2\pi + 1} \). If \( |x - 1| < \delta \) then

\[
|f(x) - f(1)| = |f'(z) (x - 1)|
\]

for some \( z \) between \( x \) and 1,

\[
= | - \sin(2\pi z - \sin(z - 1)) \{2\pi - \cos(x - 1)\} (x - 1)|
\]

\[
\leq |\{2\pi - \cos(x - 1)\} (x - 1)| \text{ since } |\sin(2\pi z - \sin(z - 1))| \leq 1
\]

\[
= |2\pi - \cos(x - 1)| |x - 1|
\]

\[
\leq (2\pi + 1) |x - 1| \text{ since } -1 \leq \cos(x - 1) \leq 1
\]

\[
< \varepsilon
\]

\[
\text{since } |x - 1| < \delta = \frac{\varepsilon}{2\pi + 1}
\]