The Binomial Theorem

In these notes we prove the binomial theorem, which says that for any integer \( n \geq 1 \),

\[
(x + y)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell y^{n-\ell} = \sum_{\ell, m \geq 0, \ell + m = n} \binom{n}{\ell} m x^\ell y^m \quad \text{where} \quad \binom{n}{\ell} = \frac{n!}{\ell!(n-\ell)!} \quad (B_n)
\]

Here \( n! \) (read “\( n \) factorial”) means \( 1 \times 2 \times 3 \times \cdots \times n \) so that, for example,

\[
\binom{n}{1} = \binom{n}{n-1} = \frac{n!}{1!(n-1)!} = \frac{1 \times 2 \times 3 \times \cdots \times (n-1) \times n}{1 \times 2 \times 3 \times \cdots \times (n-1)} = n
\]

\[
\binom{n}{2} = \binom{n}{n-2} = \frac{n!}{2!(n-2)!} = \frac{1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n}{(1 \times 2) \times (1 \times 2 \times 3 \times \cdots \times (n-2))} = \frac{n(n-1)}{2}
\]

By convention \( 0! = 1 \) so that \( \binom{n}{0} = \binom{n}{n} = \frac{n!}{0!n!} = 1 \). As special cases of the Binomial Theorem, we have

\[
\begin{align*}
n = 1 & \quad (x + y)^1 = x + y \\
n = 2 & \quad (x + y)^2 = x^2 + 2xy + y^2 \\
n = 3 & \quad (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3
\end{align*}
\]

Proof of the Binomial Theorem: The proof is by induction on \( n \). First we check that, when \( n = 1 \),

\[
\sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell y^{n-\ell} = \left. \binom{n}{\ell} x^\ell y^{n-\ell} \right|_{\ell=0} + \left. \binom{n}{\ell} x^\ell y^{n-\ell} \right|_{\ell=1} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0
\]

so that (\( B_n \)) is correct for \( n = 1 \). To complete the proof we have to show that, for any integer \( n \geq 2 \), (\( B_n \)) is a consequence of (\( B_{n-1} \)). So pick any integer \( n \geq 2 \) and assume that

\[
(x + y)^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-1-\ell}
\]

is true. Now compute

\[
(x + y)^n = (x + y)^{n-1}(x + y) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell+1} y^{n-1-\ell} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-\ell}
\]

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The second sum has the same powers of \( x \) and \( y \), namely \( x^\ell y^{n-\ell} \), as appear in \((B_n)\). The make the powers of \( x \) and \( y \) in the first sum, namely \( x^{\ell+1} y^{n-1-\ell} \) look more like those of \((B_n)\), we make the change of summation variable from \( \ell \) to \( \tilde{\ell} = \ell + 1 \). The first sum

\[
\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell+1} y^{n-1-\ell} = \sum_{\ell=1}^{n} \binom{n-1}{\ell-1} x^{\ell} y^{n-\ell}
\]

As \( \tilde{\ell} \) is just a dummy summation variable, we may call it anything we like. In particular, we may rename \( \tilde{\ell} \) back to \( \ell \). So we now have

\[
(x + y)^n = \sum_{\ell=1}^{n} \binom{n-1}{\ell-1} x^\ell y^{n-\ell} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-\ell}
\]

Recalling that \( n! = n \ (n-1)! \) we have

\[
\binom{n}{\ell} = \frac{n!}{\ell! (n-\ell)!} = \frac{n(n-1)!}{\ell! (\ell-1)! (n-\ell)!} = \frac{n}{\ell} \binom{n-1}{\ell-1}
\]

So

\[
(x + y)^n = \binom{n-1}{0} x^n + \binom{n-1}{n-1} y^n + \sum_{\ell=1}^{n-1} \left[ \binom{n-1}{\ell-1} + \binom{n-1}{\ell} \right] x^\ell y^{n-\ell}
\]

\[
= x^n + y^n + \sum_{\ell=1}^{n-1} \left[ \frac{n}{\ell} + \frac{n-\ell}{n} \right] x^\ell y^{n-\ell}
\]

\[
= x^n + y^n + \sum_{\ell=1}^{n-1} \binom{n}{\ell} x^\ell y^{n-\ell}
\]

\[
= \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell y^{n-\ell}
\]

as desired.