Substitution Examples

Example 1

$$\int_0^{\pi/2} \cos(3x)\,dx = \int_0^{3\pi/2} \cos(y)\,\frac{dy}{3} \qquad \begin{cases} y = 3x \\ dy = 3\,dx \\ y(0) = 0 \\ y(1) = \frac{3}{2}\pi \end{cases}$$

$$= \frac{\sin y}{3}\Big|_0^{3\pi/2} = \frac{-1}{3} - \frac{0}{3} = \boxed{-\frac{1}{3}}$$

Example 2

$$\int_0^1 \frac{1}{(2x+1)^3}\,dx = \int_1^3 \frac{1}{y^3}\,\frac{dy}{2} \qquad \begin{cases} y = 2x+1 \\ dy = 2\,dx \\ y(0) = 1 \\ y(1) = 3 \end{cases}$$

$$= \frac{1}{2}\int_1^3 y^{-3}\,dy = \frac{1}{2}\frac{y^{-2}}{-2}\Big|_1^3 = \frac{3^{-2}}{-4} - \frac{1^{-2}}{-4} = \frac{1}{4}\left[1 - \frac{1}{9}\right] = \boxed{\frac{2}{9}}$$

Example 3

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{1}{y}\,\frac{dy}{2} \qquad \begin{cases} y = 1+x^2 \\ dy = 2x\,dx \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$

$$= \frac{1}{2}\ln|y|\Big|_1^2 = \frac{\ln 2}{2} - \frac{0}{2} = \boxed{\frac{1}{2}\ln 2}$$

Example 4

$$\int \tan x\,dx = \int \frac{1}{\cos x}\sin x\,dx = \int \frac{1}{y}\,\frac{dy}{-1} \qquad \begin{cases} y = \cos x \\ dy = -\sin x\,dx \end{cases}$$

$$= -\ln|y| + C = -\ln|\cos x| + C = \ln|\cos x|^{-1} + C$$

$$= \boxed{\ln|\sec x| + C}$$

$$\int \sec x \tan x\,dx = \int \frac{1}{\cos^2 x}\sin x\,dx = \int \frac{1}{y^2}\,\frac{dy}{-1} \qquad \begin{cases} y = \cos x \\ dy = -\sin x\,dx \end{cases}$$

$$= -\frac{y^{-1}}{-1} + C = \frac{1}{y} + C = \boxed{\sec x + C}$$

Example 5

$$\int x^3 \cos\left(x^4+2\right) dx = \int \cos(y)\,\tfrac{dy}{4} \qquad \begin{cases} y = x^4+2 \\ dy = 4x^3\,dx \end{cases}$$

$$= \tfrac{1}{4}\sin y + C = \boxed{\tfrac{1}{4}\sin\left(x^4+2\right)+C}$$

Example 6

$$\int \sqrt{1+x^2}\,x^3\,dx = \int \sqrt{1+x^2}\,x^2\,x\,dx$$

$$= \int \sqrt{y}\,(y-1)\,\tfrac{dy}{2} \qquad \begin{cases} y = 1+x^2 \\ dy = 2x\,dx \end{cases}$$

$$= \tfrac{1}{2}\int \left(y^{3/2}-y^{1/2}\right) dy = \tfrac{1}{2}\left[\tfrac{y^{5/2}}{5/2} - \tfrac{y^{3/2}}{3/2}\right] + C$$

$$= \boxed{\tfrac{1}{5}(1+x^2)^{5/2} - \tfrac{1}{3}(1+x^2)^{3/2} + C}$$

Example 7

$$\int_0^1 \frac{1}{x^2+2x+1}\,dx = \int_0^1 \frac{1}{(x+1)^2}\,dx = \int_1^2 \frac{1}{y^2}\,dy \qquad \begin{cases} y = x+1 \\ dy = dx \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$

$$= \tfrac{y^{-1}}{-1}\Big|_1^2 = -\tfrac{1}{2} - (-1) = \boxed{\tfrac{1}{2}}$$

$$\int_0^1 \frac{1}{x^2+2x+2}\,dx = \int_0^1 \frac{1}{(x+1)^2+1}\,dx = \int_1^2 \frac{1}{y^2+1}\,dy \qquad \begin{cases} y = x+1 \\ dy = dx \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$

$$= \arctan y\Big|_1^2 = \boxed{\arctan 2 - \arctan 1}$$

$$\int_0^1 \frac{1}{x^2+2x+5}\,dx = \int_0^1 \frac{1}{(x+1)^2+4}\,dx = \int_{\frac{1}{2}}^1 \frac{2dy}{4y^2+4} \qquad \begin{cases} 2y = x+1 \\ 2dy = dx \\ y(0) = 1/2 \\ y(1) = 1 \end{cases}$$

$$= \tfrac{1}{2}\arctan y\Big|_{1/2}^1 = \boxed{\tfrac{1}{2}\left[\arctan 1 - \arctan \tfrac{1}{2}\right]}$$

Example 8

$$\int_0^a x\sqrt{a^2-x^2}\,dx = \int_{a^2}^0 \sqrt{y}\,\tfrac{dy}{-2} \qquad \begin{cases} y = a^2 - x^2 \\ dy = -2x\,dx \\ y(0) = a^2 \\ y(a) = 0 \end{cases}$$

$$= -\tfrac{1}{2}\,\tfrac{y^{3/2}}{3/2}\Big|_{a^2}^{0} = \tfrac{1}{3}\left(a^2\right)^{3/2} = \boxed{\tfrac{1}{3}a^3}$$

Example 9

$$\int \sqrt[3]{1-x}\,dx = \int \sqrt[3]{y}\,\tfrac{dy}{-1} \qquad \begin{cases} y = 1-x \\ dy = -dx \end{cases}$$

$$= -\tfrac{y^{4/3}}{4/3} + C = \boxed{-\tfrac{3}{4}(1-x)^{4/3} + C}$$

Example 10

$$\int \sin\left(e^x\right)\,e^x\,dx = \int \sin y\,dy \qquad \begin{cases} y = e^x \\ dy = e^x\,dx \end{cases}$$

$$= -\cos y + C = \boxed{-\cos(e^x) + C}$$

Example 11

$$\int x^3(1-x^2)^{3/2}\,dx = \int (1-x^2)^{3/2}\,x^2\,x\,dx$$

$$= \int y^{3/2}\,(1-y)\,\tfrac{dy}{-2} \qquad \begin{cases} y = 1-x^2 \\ dy = -2x\,dx \end{cases}$$

$$= -\tfrac{1}{2}\int \left(y^{3/2} - y^{5/2}\right)dy = -\tfrac{1}{2}\left[\tfrac{y^{5/2}}{5/2} - \tfrac{y^{7/2}}{7/2}\right] + C$$

$$= \boxed{\tfrac{1}{7}(1-x^2)^{7/2} - \tfrac{1}{5}(1-x^2)^{5/2} + C}$$

Example 12

$$\int \frac{\cos x}{4+\sin^2 x}\,dx = \int \frac{2dy}{4+4y^2} \qquad \begin{cases} 2y = \sin x \\ 2dy = \cos x\,dx \end{cases}$$

$$= \tfrac{1}{2}\int \tfrac{dy}{1+y^2} = \tfrac{1}{2}\arctan y + C$$

$$= \boxed{\tfrac{1}{2}\arctan\left(\tfrac{1}{2}\sin x\right) + C}$$

Example 13

$$\int \frac{\sin^3(\ln x)\ \cos^3(\ln x)}{x}\,dx = \int \sin^3 y\ \cos^3 y\ dy \qquad \begin{cases} y = \ln x \\ dy = \frac{dx}{x} \end{cases}$$

$$= \int \sin^3 y\ (1 - \sin^2 y)\ \cos y\ dy$$

$$= \int u^3\ (1 - u^2)\ du \qquad \begin{cases} u = \sin y \\ du = \cos y\,dy \end{cases}$$

$$= \int \left[u^3 - u^5\right]\ du = \left[\tfrac{u^4}{4} - \tfrac{u^6}{6} + C\right]\Big|_{u=\sin y}\Big|_{y=\ln x}$$

$$= \boxed{\tfrac{\sin^4(\ln x)}{4} - \tfrac{\sin^6(\ln x)}{6} + C}$$

It is always a good idea to check that an indefinite integral is correct by checking that its derivative is the integrand.

$$\begin{aligned} \tfrac{d}{dx}\left[\tfrac{\sin^4(\ln x)}{4} - \tfrac{\sin^6(\ln x)}{6}\right] &= \sin^3(\ln x)\cos(\ln x)\tfrac{1}{x} - \sin^5(\ln x)\cos(\ln x)\tfrac{1}{x} \\ &= \sin^3(\ln x)\cos(\ln x)\tfrac{1}{x}\left[1 - \sin^2(\ln x)\right] \\ &= \sin^3(\ln x)\cos^3(\ln x)\tfrac{1}{x} \end{aligned}$$

There are often many different correct ways to evaluate an integral. Here is a second evaluation of the same indefinite integral that uses $\sin(2y) = 2\sin y\cos y$.

$$\int \frac{\sin^3(\ln x)\ \cos^3(\ln x)}{x}\,dx = \int \sin^3 y\ \cos^3 y\ dy \qquad \begin{cases} y = \ln x \\ dy = \frac{dx}{x} \end{cases}$$

$$= \tfrac{1}{8}\int \sin^3(2y)\ dy = \tfrac{1}{8}\int \big(1 - \cos^2(2y)\big)\sin(2y)\ dy$$

$$= \tfrac{1}{8}\int \left[1 - z^2\right]\ \tfrac{dz}{-2} \qquad \begin{cases} z = \cos(2y) \\ dz = -2\sin(2y)\,dy \end{cases}$$

$$= -\tfrac{1}{16}\left[z - \tfrac{z^3}{3} + C\right]\Big|_{z=\cos(2y)}\Big|_{y=\ln x}$$

$$= \boxed{-\tfrac{1}{16}\left[\cos(2\ln x) - \tfrac{1}{3}\cos^3(2\ln x)\right] + C}$$

We can also check that this answer is correct by differentiating:

$$\begin{aligned} &-\tfrac{1}{16}\tfrac{d}{dx}\left[\cos(2\ln x) - \tfrac{1}{3}\cos^3(2\ln x)\right] \\ &\quad = -\tfrac{1}{16}\left[-\sin(2\ln x)\tfrac{2}{x} + \cos^2(2\ln x)\sin(2\ln x)\tfrac{2}{x}\right] \\ &\quad = \tfrac{1}{16}\left[1 - \cos^2(2\ln x)\right]\sin(2\ln x)\tfrac{2}{x} \\ &\quad = \tfrac{1}{8}\sin^3(2\ln x)\tfrac{1}{x} \\ &\quad = \sin^3(\ln x)\cos^3(\ln x)\tfrac{1}{x} \end{aligned}$$

We can also use the identities

$$\cos(2y) = 1 - 2\sin^2 y \qquad \text{and} \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

to check that the answers are the same

$$\begin{aligned}
&-\tfrac{1}{16}\left[\cos(2\ln x) - \tfrac{1}{3}\cos^3(2\ln x)\right] \\
&\qquad = -\tfrac{1}{48}\left[3(1-2\sin^2 y) - (1-2\sin^2 y)^3\right] \\
&\qquad = -\tfrac{1}{48}\left[3 - 6\sin^2 y - \big(1 - 6\sin^2 y + 12\sin^4 y - 8\sin^6 y\big)\right] \\
&\qquad = -\tfrac{1}{48}\left[2 - 12\sin^4 y + 8\sin^6 y\right] \\
&\qquad = \tfrac{\sin^4(\ln x)}{4} - \tfrac{\sin^6(\ln x)}{6} - \tfrac{1}{24}
\end{aligned}$$

The extra term $\frac{1}{24}$ just means that the symbol C that appears in the two answers

$$\tfrac{\sin^4(\ln x)}{4} - \tfrac{\sin^6(\ln x)}{6} + C \quad \text{and} \quad -\tfrac{1}{16}\left[\cos(2\ln x) - \tfrac{1}{3}\cos^3(2\ln x)\right] + C$$

actually stands two different constants.