Integration of $\sec\theta$, $\csc\theta$, $\sec^3\theta$ and $\csc^3\theta$

$\int \sec\theta \; d\theta$ – by trickery

The standard trick used to integrate $\sec\theta$ is to multiply the integrand by $1 = \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$ and then substitute $y = \sec\theta + \tan\theta$, $dy = (\sec\theta\tan\theta + \sec^2\theta)\, d\theta$.

$$\begin{aligned}\int \sec\theta \; d\theta &= \int \sec\theta \; \tfrac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta} \; d\theta = \int \tfrac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta} \; d\theta = \int \tfrac{dy}{y} = \ln|y| + C \\ &= \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

$\int \sec\theta \; d\theta$ – by partial fractions

Another method for integrating $\int \sec\theta \; d\theta$, that is more tedious, but less dependent on a memorized trick, is to convert $\int \sec\theta \, d\theta$ into the integral of a rational function using the substitution $y = \sin\theta$, $dy = \cos\theta\, d\theta$ and then use partial fractions.

$$\begin{aligned}\int \sec\theta \; d\theta &= \int \tfrac{1}{\cos\theta} \; d\theta = \int \tfrac{\cos\theta}{\cos^2\theta} \; d\theta = \int \tfrac{\cos\theta}{1-\sin^2\theta} \; d\theta = \int \tfrac{1}{1-y^2} \; dy \\ &= \int \tfrac{1}{2}\Big[\tfrac{1}{1+y} + \tfrac{1}{1-y}\Big] \; dy = \tfrac{1}{2}\int \Big[\tfrac{1}{y+1} - \tfrac{1}{y-1}\Big] \; dy = \tfrac{1}{2}\Big[\ln|y+1| - \ln|y-1|\Big] + C \\ &= \tfrac{1}{2}\ln\Big|\tfrac{y+1}{y-1}\Big| + C = \tfrac{1}{2}\ln\Big|\tfrac{\sin\theta+1}{\sin\theta-1}\Big| + C\end{aligned}$$

To see that this answer is really the same as the one above, note that

$$\begin{aligned}&\tfrac{\sin\theta+1}{\sin\theta-1} = \tfrac{(\sin\theta+1)^2}{\sin^2\theta-1} = \tfrac{(\sin\theta+1)^2}{-\cos^2\theta} \\ \Rightarrow &\tfrac{1}{2}\ln\Big|\tfrac{\sin\theta+1}{\sin\theta-1}\Big| = \tfrac{1}{2}\ln\Big|\tfrac{(\sin\theta+1)^2}{-\cos^2\theta}\Big| = \tfrac{1}{2}\ln\Big|\tfrac{(\sin\theta+1)^2}{\cos^2\theta}\Big| = \ln\Big|\tfrac{\sin\theta+1}{\cos\theta}\Big| = \ln|\tan\theta + \sec\theta|\end{aligned}$$

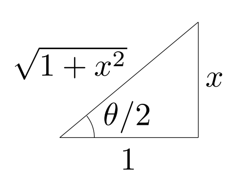
$\int \csc\theta \; d\theta$ – by the $x = \tan\frac{\theta}{2}$ substitution

The integral $\int \csc\theta \; d\theta$ may also be evaluated by both the methods above. The answers are

$$\int \csc\theta \; d\theta = \ln|\cot\theta - \csc\theta| + C = -\tfrac{1}{2}\ln\Big|\tfrac{\cos\theta+1}{\cos\theta-1}\Big| + C$$

Since $\csc\theta$ is a rational function of $\sin\theta$ and $\cos\theta$, the substitution

$$\begin{aligned}x &= \tan\tfrac{\theta}{2} \qquad \theta = 2\tan^{-1}x \\ d\theta &= \tfrac{2}{1+x^2}dx \\ \sin\theta &= 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = 2\tfrac{x}{\sqrt{1+x^2}}\tfrac{1}{\sqrt{1+x^2}} = \tfrac{2x}{1+x^2} \\ \cos\theta &= \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = \tfrac{1}{1+x^2} - \tfrac{x^2}{1+x^2} = \tfrac{1-x^2}{1+x^2}\end{aligned}$$



converts $\int \csc\theta\, d\theta$ into the integral of a rational function.

$$\int \csc\theta\ d\theta = \int \tfrac{1}{\sin\theta}\ d\theta = \int \tfrac{1+x^2}{2x}\ \tfrac{2}{1+x^2}\,dx = \int \tfrac{1}{x}\ dx = \ln|x| + C = \ln\left|\tan\tfrac{\theta}{2}\right| + C$$

To see that this answer is really the same as the one above, note that

$$\cot\theta - \csc\theta = \tfrac{\cos\theta-1}{\sin\theta} = \tfrac{-2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\tan\tfrac{\theta}{2}$$

$\int \sec^3 x\ dx$ – by trickery

The standard trick used to evaluate $\int \sec^3 x\ dx$ is integration by parts with $u = \sec x$, $dv = \sec^2 x\ dx$, $du = \sec x\tan x\ dx$, $v = \tan x$.

$$\int \sec^3 x\ dx = \int \sec x\ \sec^2 x\ dx = \sec x\tan x - \int \tan x\ \sec x\tan x\ dx$$

Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\int \sec^3 x\ dx = \sec x\tan x - \int [\sec^3 x - \sec x]\ dx = \sec x\tan x + \ln|\sec x+\tan x| + C - \int \sec^3 x\ dx$$

where we used $\int \sec x\ dx = \ln|\sec x + \tan x| + C$. Now moving the $\int \sec^3 x\ dx$ from the right hand side to the left hand side

$$2\int \sec^3 x\ dx = \sec x\tan x + \ln|\sec x + \tan x| + C$$

$$\Rightarrow \int \sec^3 x\ dx = \tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C$$

for a new arbitrary constant C.

$\int \sec^3 x\ dx$ – by partial fractions

Another method for integrating $\int \sec^3 x\ dx$, that is more tedious, but less dependent on trickery, is to convert $\int \sec^3 x\, dx$ into the integral of a rational function using the substitution $y = \sin x$, $dy = \cos x\, dx$ and then use partial fractions.

$$\begin{aligned}
\int \sec^3 x\ dx &= \int \tfrac{1}{\cos^3 x}\ dx = \int \tfrac{\cos x}{\cos^4 x}\ dx = \int \tfrac{\cos x}{[1-\sin^2 x]^2}\ dx = \int \tfrac{1}{[1-y^2]^2}\ dy = \int \tfrac{1}{[y^2-1]^2}\ dy \\
&= \int \left[\tfrac{1}{2}\left(\tfrac{1}{y-1} - \tfrac{1}{y+1}\right)\right]^2 dy = \tfrac{1}{4}\int \left[\tfrac{1}{(y-1)^2} - \tfrac{2}{(y-1)(y+1)} + \tfrac{1}{(y+1)^2}\right] dy \\
&= \tfrac{1}{4}\int \left[\tfrac{1}{(y-1)^2} - \tfrac{1}{y-1} + \tfrac{1}{y+1} + \tfrac{1}{(y+1)^2}\right] dy \\
&= \tfrac{1}{4}\left[-\tfrac{1}{y-1} - \ln|y-1| + \ln|y+1| - \tfrac{1}{y+1}\right] + C = -\tfrac{1}{4}\tfrac{2y}{y^2-1} + \tfrac{1}{4}\ln\left|\tfrac{y+1}{y-1}\right| + C \\
&= \tfrac{1}{2}\tfrac{y}{1-y^2} + \tfrac{1}{4}\ln\left|\tfrac{y+1}{y-1}\right| + C = \tfrac{1}{2}\tfrac{\sin x}{\cos^2 x} + \tfrac{1}{2}\ln\left|\tfrac{\sin x+1}{\sin x-1}\right| + C
\end{aligned}$$

$\int \csc^3\theta\ d\theta$ – by the $x = \tan\frac{\theta}{2}$ substitution

As another example of the $x = \tan\frac{\theta}{2}$, $d\theta = \frac{2}{1+x^2}dx$, $\sin\theta = \frac{2x}{1+x^2}$, $\cos\theta = \frac{1-x^2}{1+x^2}$ substitution, we evaluate

$$\begin{aligned}\int \csc^3\theta\ d\theta &= \int \tfrac{1}{\sin^3\theta}\ d\theta = \int \big(\tfrac{1+x^2}{2x}\big)^3\ \tfrac{2}{1+x^2}dx = \tfrac{1}{4}\int \tfrac{1+2x^2+x^4}{x^3}\ dx \\ &= \tfrac{1}{4}\big[\tfrac{x^{-2}}{-2} + 2\ln|x| + \tfrac{x^2}{2}\big] + C = \tfrac{1}{8}\big[-\cot^2\tfrac{\theta}{2} + 4\ln\big|\tan\tfrac{\theta}{2}\big| + \tan^2\tfrac{\theta}{2}\big] + C\end{aligned}$$

By the usual double angle formulae

$$\begin{aligned}\tan^2\tfrac{\theta}{2} - \cot^2\tfrac{\theta}{2} &= \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} - \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{\sin^4\frac{\theta}{2} - \cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} \\ &= \frac{\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} \qquad \text{since } \sin^2\tfrac{\theta}{2} + \cos^2\tfrac{\theta}{2} = 1 \\ &= \frac{-\cos\theta}{\frac{1}{4}\sin^2\theta}\end{aligned}$$

so we may also write

$$\int \csc^3\theta\ d\theta = -\tfrac{1}{2}\cot\theta\csc\theta + \tfrac{1}{2}\ln|\cot\theta - \csc\theta| + C$$