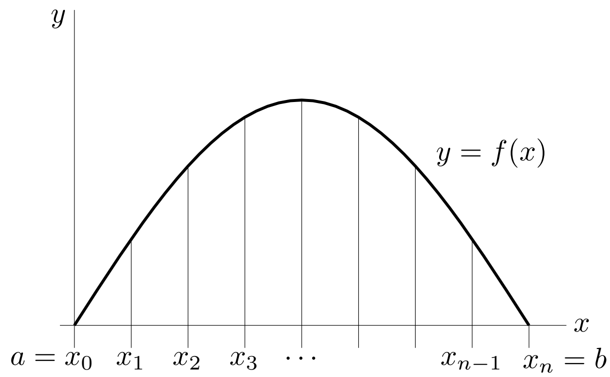
Simple Numerical Integrators – Derivation

These notes provide derivations of three simple algorithms for generating, numerically, approximate values for the definite integral $\int_a^b f(x)\,dx$. In each algorithm, we first select an integer n, called the "number of steps". We then divide the interval of integration, $a \le x \le b$ into n equal subintervals, each of size $\Delta x = \frac{b-a}{n}$. The end points of these intervals are $x_0 = a,\ x_1 = a + \Delta x,\ x_2 =$



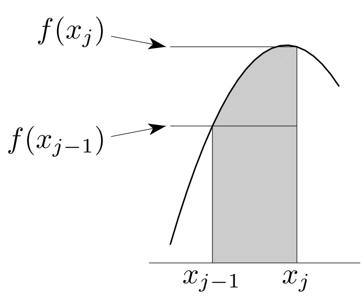
$a + 2\Delta x,\ \cdots,\ x_{n-1} = b - \Delta x,\ x_n = b$. The corresponding decomposition of the integral is

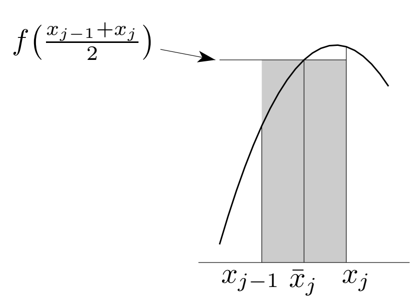
$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Each subintegral $\int_{x_{j-1}}^{x_j} f(x)\,dx$ is approximated by the area of a simple geometric figure. The three different algorithms use three different figures.

The Midpoint Rule

The integral $\int_{x_{j-1}}^{x_j} f(x)\,dx$ represents the area under the curve $y = f(x)$ with x running from x_{j-1} to x_j. The width of this region is $x_j - x_{j-1}$. The height varies over the different values that $f(x)$ takes as x runs from x_{j-1} to x_j. The Midpoint Rule approximates this area by the area of a rectangle of width $x_j - x_{j-1} = \Delta x$ and height $f\left(\frac{x_{j-1}+x_j}{2}\right)$, which is the exact height at the midpoint of the range of x. The area of the approximating rectangle is $f\left(\frac{x_{j-1}+x_j}{2}\right)\Delta x$. To save writing, set





$\bar{x}_j = \frac{x_{j-1}+x_j}{2}$ So the Midpoint Rule approximates each subintegral by

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx f(\bar{x}_j)\Delta x$$

and the full integral by

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^{x_1} f(x)\,dx \quad + \int_{x_1}^{x_2} f(x)\,dx \quad + \cdots \; + \int_{x_{n-1}}^{x_n} f(x)\,dx \\ &\approx \quad f(\bar{x}_1)\Delta x + \quad f(\bar{x}_2)\Delta x + \cdots + \quad f(\bar{x}_n)\Delta x\end{aligned}$$

In summary, the Midpoint Rule approximates

$$\boxed{\int_a^b f(x)\,dx \approx \Big[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\Big]\Delta x}$$

where

$$\begin{aligned}&\Delta x = \tfrac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \cdots, \quad x_{n-1} = b - \Delta x, \quad x_n = b \\ &\qquad\qquad\qquad \bar{x}_1 = \tfrac{x_0+x_1}{2}, \quad \bar{x}_2 = \tfrac{x_1+x_2}{2}, \quad \cdots, \quad \bar{x}_n = \tfrac{x_{n-1}+x_n}{2}\end{aligned}$$

For example, here is the approximation for $\int_0^\pi \sin x\,dx$ with $n = 8$. First note that $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{\pi}{8} \qquad x_2 = \tfrac{2\pi}{8} \qquad \cdots \qquad x_7 = \tfrac{7\pi}{8} \qquad x_8 = \tfrac{8\pi}{8} = \pi$$

Consequently,

$$\bar{x}_1 = \tfrac{\pi}{16} \qquad \bar{x}_2 = \tfrac{3\pi}{16} \qquad \cdots \qquad \bar{x}_7 = \tfrac{13\pi}{16} \qquad \bar{x}_8 = \tfrac{15\pi}{16}$$

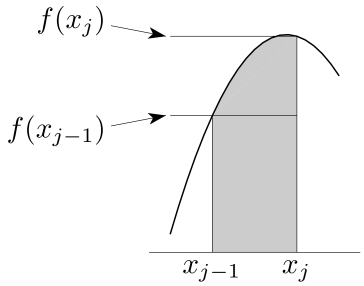
and

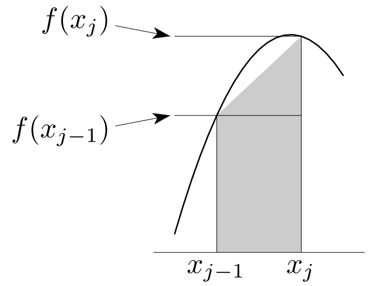
$$\begin{aligned}\int_0^\pi \sin x\,dx &\approx \Big[\sin(\bar{x}_1) + \sin(\bar{x}_2) + \cdots + \sin(\bar{x}_8)\Big]\Delta x \\ &= \Big[\sin(\tfrac{\pi}{16}) + \sin(\tfrac{3\pi}{16}) + \sin(\tfrac{5\pi}{16}) + \sin(\tfrac{7\pi}{16}) + \sin(\tfrac{9\pi}{16}) + \sin(\tfrac{11\pi}{16}) + \sin(\tfrac{13\pi}{16}) + \sin(\tfrac{15\pi}{16})\Big]\tfrac{\pi}{8} \\ &= \Big[0.1951 + 0.5556 + 0.8315 + 0.9808 + 0.9808 + 0.8315 + 0.5556 + 0.1951\Big] \times 0.3927 \\ &= 5.1260 \times 0.3927 \\ &= 2.013\end{aligned}$$

The exact answer is $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$. So with eight steps of the Midpoint Rule we achieved $100\frac{2.013-2}{2} = 0.65\%$ accuracy.

The Trapezoidal Rule

The Trapezoidal Rule approximates $\int_{x_{j-1}}^{x_j} f(x)\,dx$ by the area of a trapezoid. A trapezoid is a four sided polygon, like a rectangle. But, unlike a rectangle, the top and bottom of a trapezoid need not be parallel. The trapezoid used to approximate $\int_{x_{j-1}}^{x_j} f(x)\,dx$ has width $x_j - x_{j-1} = \Delta x$. Its left hand side has height $f(x_{j-1})$ and its right hand side has height $f(x_j)$. The area of a trapezoid is





its width times its average height. So the Trapezoidal Rule approximates

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx \tfrac{f(x_{j-1})+f(x_j)}{2}\Delta x$$

and the full integral by

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^{x_1} f(x)\,dx \quad + \int_{x_1}^{x_2} f(x)\,dx \quad + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx \\ &\approx \tfrac{f(x_0)+f(x_1)}{2}\Delta x + \tfrac{f(x_1)+f(x_2)}{2}\Delta x + \cdots + \tfrac{f(x_{n-1})+f(x_n)}{2}\Delta x \\ &= \Big[\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\Big]\Delta x\end{aligned}$$

In summary, the Trapezoidal Rule approximates

$$\boxed{\int_a^b f(x)\,dx \approx \Big[\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\Big]\Delta x}$$

where

$$\Delta x = \tfrac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \cdots, \quad x_{n-1} = b - \Delta x, \quad x_n = b$$

As an example we again approximate $\int_0^\pi \sin x\,dx$ with $n = 8$. We still have $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{\pi}{8} \qquad x_2 = \tfrac{2\pi}{8} \qquad \cdots \qquad x_7 = \tfrac{7\pi}{8} \qquad x_8 = \tfrac{8\pi}{8} = \pi$$

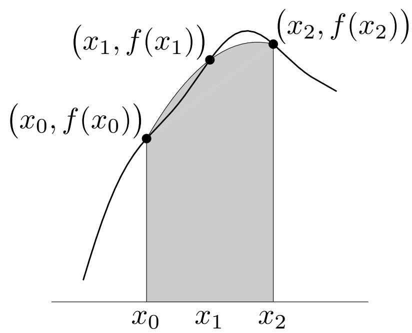
Consequently,

$$\begin{aligned}\int_0^\pi \sin x\,dx &\approx \Big[\tfrac{1}{2}\sin(x_0) + \sin(x_1) + \cdots + \sin(x_7) + \tfrac{1}{2}\sin(x_8)\Big]\Delta x \\ &= \Big[\tfrac{1}{2}\sin(0) + \sin(\tfrac{\pi}{8}) + \sin(\tfrac{2\pi}{8}) + \sin(\tfrac{3\pi}{8}) + \sin(\tfrac{4\pi}{8}) + \sin(\tfrac{5\pi}{8}) + \sin(\tfrac{6\pi}{8}) + \sin(\tfrac{7\pi}{8}) + \tfrac{1}{2}\sin(\tfrac{8\pi}{8})\Big]\tfrac{\pi}{8} \\ &= \Big[\tfrac{1}{2}\times 0 + 0.3827 + 0.7071 + 0.9239 + 1.0000 + 0.9239 + 0.7071 + 0.3827 + \tfrac{1}{2}\times 0\Big] \times 0.3927 \\ &= 5.0274 \times 0.3927 \\ &= 1.974\end{aligned}$$

The exact answer is $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$. So with eight steps of the Trapezoidal Rule we achieved $100\frac{|1.974-2|}{2} = 1.3\%$ accuracy.

Simpson's Rule

Simpson's Rule approximates $\int_{x_0}^{x_2} f(x)\,dx$ by the area under the part of a parabola with x running from x_0 to x_2. The parabola used passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$. It then approximates $\int_{x_2}^{x_4} f(x)\,dx$ by the area under the part of a parabola with



$x_2 \le x \le x_4$. This parabola passes through the three points $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$. And so on. Because Simspon's rule does the approximation two slices at a time, n **must be even**.

To derive Simpson's rule formula, we first find the equation of the parabola that passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$. Then we find the area under the part of that parabola with $x_0 \le x \le x_2$. We can make the formulae look less complicated by writing the equation of the parabola in the form

$$y = A(x - x_1)^2 + B(x - x_1) + C$$

The three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ lie on this parabola if and only if

$$\begin{aligned} A\big(x_0 - x_1)^2 + B(x_0 - x_1) + C &= f(x_0) \\ A\big(x_1 - x_1)^2 + B(x_1 - x_1) + C &= f(x_1) \\ A\big(x_2 - x_1)^2 + B(x_2 - x_1) + C &= f(x_2) \end{aligned}$$

Because $x_1 - x_1 = 0$, the middle equation simplifies to $C = f(x_1)$. Because $x_0 - x_1 = -\Delta x$, $x_2 - x_1 = \Delta x$ and $C = f(x_1)$, the first and third equations simplify to

$$\begin{aligned} \Delta x^2\, A - \Delta x\, B &= f(x_0) - f(x_1) \\ \Delta x^2\, A + \Delta x\, B &= f(x_2) - f(x_1) \end{aligned}$$

Adding the two equations together gives $2\Delta x^2\, A = f(x_0) - 2f(x_1) + f(x_2)$. Subtracting the first equation from the second gives $2\Delta x\, B = f(x_2) - f(x_0)$. We now know the desired parabola.

$$A = \tfrac{1}{2\Delta x^2}\big(f(x_0) - 2f(x_1) + f(x_2)\big) \qquad B = \tfrac{1}{2\Delta x}\big(f(x_2) - f(x_0)\big) \qquad C = f(x_1)$$

The area under the part of this parabola with $x_0 \le x \le x_2$ is

$$\begin{aligned}
\int_{x_0}^{x_2} \Big[A(x-x_1)^2 + B(x-x_1) + C\Big]\,dx &= \int_{-\Delta x}^{\Delta x} \Big[At^2 + Bt + C\Big]\,dt \qquad \text{where } t = x - x_1 \\
&= 2\int_0^{\Delta x} \Big[At^2 + C\Big]\,dt \qquad \text{since } Bt \text{ is odd and } At^2 + C \text{ is even} \\
&= 2\Big[\tfrac{1}{3}At^3 + Ct\Big]_0^{\Delta x} \\
&= \tfrac{2}{3}A\Delta x^3 + 2C\Delta x \\
&= \tfrac{1}{3}\Delta x\big[f(x_0) - 2f(x_1) + f(x_2)\big] + 2f(x_1)\Delta x \\
&= \tfrac{1}{3}\Delta x\big[f(x_0) + 4f(x_1) + f(x_2)\big]
\end{aligned}$$

So Simpson's rule approximates

$$\int_{x_0}^{x_2} f(x)\,dx \approx \tfrac{1}{3}\Delta x\big[f(x_0) + 4f(x_1) + f(x_2)\big]$$

and

$$\int_{x_2}^{x_4} f(x)\,dx \approx \tfrac{1}{3}\Delta x\big[f(x_2) + 4f(x_3) + f(x_4)\big]$$

and so on. All together

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx \\
&\approx \tfrac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + f(x_2)\big] + \tfrac{\Delta x}{3}\big[f(x_2) + 4f(x_3) + f(x_4)\big] \\
&\quad + \tfrac{\Delta x}{3}\big[f(x_4) + 4f(x_5) + f(x_6)\big] + \cdots + \tfrac{\Delta x}{3}\big[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big] \\
&= \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]\tfrac{\Delta x}{3}
\end{aligned}$$

In summary, Simpson's rule approximates

$$\boxed{\int_a^b f(x)\,dx \approx \Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]\tfrac{\Delta x}{3}}$$

where n is even and

$$\Delta x = \tfrac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \cdots, \quad x_{n-1} = b - \Delta x, \quad x_n = b$$

As an example we approximate $\int_0^\pi \sin x\,dx$ with $n = 8$, yet again. Under Simpson's rule

$$\begin{aligned}
\int_0^\pi \sin x\,dx &\approx \Big[\sin(x_0) + 4\sin(x_1) + 2\sin(x_2) + \cdots + 4\sin(x_7) + \sin(x_8)\Big]\tfrac{\Delta x}{3} \\
&= \Big[\sin(0) + 4\sin(\tfrac{\pi}{8}) + 2\sin(\tfrac{2\pi}{8}) + 4\sin(\tfrac{3\pi}{8}) + 2\sin(\tfrac{4\pi}{8}) \\
&\qquad + 4\sin(\tfrac{5\pi}{8}) + 2\sin(\tfrac{6\pi}{8}) + 4\sin(\tfrac{7\pi}{8}) + \sin(\tfrac{8\pi}{8})\Big]\tfrac{\pi}{8\times 3} \\
&= \Big[0 + 4\times 0.382683 + 2\times 0.707107 + 4\times 0.923880 + 2\times 1.0 \\
&\qquad + 4\times 0.923880 + 2\times 0.707107 + 4\times 0.382683 + 0\Big]\tfrac{\pi}{8\times 3} \\
&= 15.280932 \times 0.130900 \\
&= 2.00027
\end{aligned}$$

With only eight steps of Simpson's rule we achieved $100\frac{2.00027-2}{2} = 0.014\%$ accuracy.

These notes have derived the midpoint, trapezoidal and Simpson's rules for approximating the values of definite integrals. So far we have not attempted to see how efficient and how accurate the algorithms are. A first look at those questions is provided in the notes "Simple Numerical Integrators – Error Behaviour".