Trignometric Integrals

Integrals of polynomials of the trigonometric functions $\sin x$, $\cos x$, $\tan x$ and so on, are generally evaluated by using a combination of simple substitutions and trigonometric identities. There are of course a very large number of trigonometric identities, but usually we use only a handful of them. The most important are:

$$\sin^2 x + \cos^2 x = 1 \tag{1a}$$

$$\sin(2x) = 2\sin x \cos x \tag{1b}$$

$$\cos(2x) = \cos^2 x - \sin^2 x \tag{1c}$$

$$= 2\cos^2 x - 1 \tag{1d}$$

$$= 1 - 2\sin^2 x \tag{1e}$$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \tag{1f}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \tag{1g}$$

Identities (1d) and (1e) follow easily from (1c) by using $\sin^2 x + \cos^2 x = 1$. Identities (1f) and (1g) follow directly from (1e) and (1d), respectively.

Integrating $\int \sin^m x \cos^n x \; dx$

If n is an odd integer this can be integrated by substituting $u = \sin x$, $du = \cos x \; dx$ and then using $\cos^2 x = 1 - \sin^2 x = 1 - u^2$ to convert all remaining $\cos x$'s to u's. Here is an example.

Example 1 ($\int \sin^2 x \cos^3 x \; dx$)

Start by factoring off one power of $\cos x$ to combine with dx to get $\cos x \; dx = du$.

$$\begin{aligned}
\int \sin^2 x \cos^3 x \; dx &= \int \sin^2 x \cos^2 x \;\; \cos x \; dx \\
&= \int u^2 \; (1-u^2) \; du \qquad \text{with } u = \sin x,\ du = \cos x \; dx,\ \cos^2 x = 1 - u^2 \\
&= \frac{u^3}{3} - \frac{u^5}{5} + C \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C
\end{aligned}$$

Example 1

Of course if m is an odd integer we can use the same strategy with the roles of $\sin x$ and $\cos x$ interchanged. That is, we substitute $u = \cos x$, $du = -\sin x \; dx$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

If m and n are both even, the strategy is to use the trig identities (1f) and (1g) to get back to the m or n odd case. Here are a couple of examples that arise quite commonly in applications.

Example 2 ($\int \cos^2 x\ dx$)

By (1g)

$$\int \cos^2 x\ dx = \frac{1}{2}\int \big[1+\cos(2x)\big]\ dx = \frac{1}{2}\Big[x+\frac{1}{2}\sin(2x)\Big]+C$$

Example 2

Example 3 ($\int \cos^4 x\ dx$)

First we'll prepare the integrand $\cos^4 x$ for easy integration by applying (1g) a couple times. We have already used (1g) once to get

$$\cos^2 x = \frac{1}{2}\big[1+\cos(2x)\big]$$

Squaring it gives

$$\cos^4 x = \frac{1}{4}\big[1+\cos(2x)\big]^2 = \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)$$

Now by (1g) a second time

$$\begin{aligned}\cos^4 x &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\ \frac{1+\cos(4x)}{2}\\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

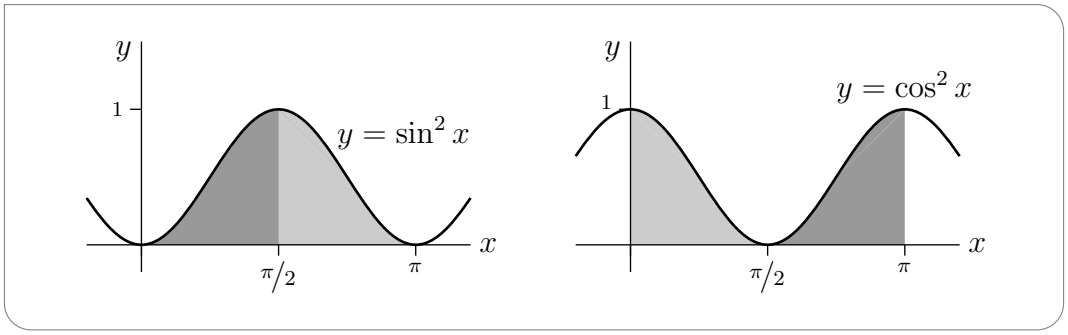
Now it's easy to integrate

$$\begin{aligned}\int \cos^4 x\ dx &= \frac{3}{8}\int dx+\frac{1}{2}\int \cos(2x)\ dx+\frac{1}{8}\int \cos(4x)\ dx\\ &= \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C\end{aligned}$$

Example 3

Example 4 ($\int_0^\pi \cos^2 x\ dx$ and $\int_0^\pi \sin^2 x\ dx$)

Of course we can compute the definite integral $\int_0^\pi \cos^2 x\ dx$ by using the antiderivative for $\cos^2 x$ that we found in Example 2. But here is a trickier way to evaluate that integral, and also the integral $\int_0^\pi \sin^2 x\ dx$ at the same time, very quickly without needing the antiderivative of Example 2. We just need to observe that $\int_0^\pi \cos^2 x\ dx$ and $\int_0^\pi \sin^2 x\ dx$ are equal because they represent the same area — look at the graphs below — the darkly shaded regions in the two graphs have the same area and the lightly shaded regions in the two graphs have the same area.



Consequently,

$$\begin{aligned}\int_0^\pi \cos^2 x\ dx = \int_0^\pi \sin^2 x\ dx &= \frac{1}{2}\left[\int_0^\pi \sin^2 x\ dx + \int_0^\pi \cos^2 x\ dx\right]\\ &= \frac{1}{2}\int_0^\pi \left[\sin^2 x + \cos^2 x\right]\ dx\\ &= \frac{1}{2}\int_0^\pi dx\\ &= \frac{\pi}{2}\end{aligned}$$

Example 4

Integrating $\int \tan^m x \sec^n x\ dx$

The strategy for dealing with these integrals is similar to the strategy that we used to evaluate integrals of the form $\int \sin^m x \cos^n x\ dx$. It uses

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\ \tan x \qquad 1+\tan^2 x = \sec^2 x$$

(There is no need to memorize $1+\tan^2 x = \sec^2 x$. To derive it very quickly just divide $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$.) To integrate $\int \tan^m x \sec^n x\ dx$,

- if m is odd, write $\tan^m x\ \sec^n x\ dx = \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n dx = \frac{\sin^{m-1} x}{\cos^{n+m} x}\ \sin x\, dx$ and substitute $u = \cos x$, $du = -\sin x\ dx$, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$. See Examples 5 and 6.

- Alternatively, if m is odd and $n \geq 1$, move one factor of $\sec x\ \tan x$ to the side so that you can see $\sec x\ \tan x\ dx$ in the integral, and substitute $u = \sec x$, $du = \sec x\ \tan x\, dx$ and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$. See Example 7.

- If n is even with $n \geq 2$, move one factor of $\sec^2 x$ to the side so that you can see $\sec^2 x\ dx$ in the integral, and substitute $u = \tan x$, $du = \sec^2 x\, dx$ and $\sec^2 x = 1+\tan^2 x = 1+u^2$. See Example 8.

- If $n = 0$ and m is even we can still use the $u = \tan x$ substitution, after using $\tan^2 x = \sec^2 x - 1$ (possibly more than once) to create a $\sec^2 x$. See Example 9.

- There is still one more case, namely n odd and m even. There are strategies like those above for treating this case. But they are more complicated and also involve more tricks (that basically have to be memorized). Examples using them are provided in the optional section entitled "Integrating $\sec x$, $\csc x$, $\sec^3 x$ and $\csc^3 x$", below. A more straight forward strategy uses another technique called "partial fractions". We shall return to this strategy after we have learned about partial fractions. See Examples 5 and 6 in the notes "Partial Fractions".

Example 5 ($\int \tan x \; dx$)

Write the integrand $\tan x = \frac{1}{\cos x} \sin x$. We can substitute $u = \cos x$, $du = -\sin x \, dx$ just as we did in treating integrands of the form $\sin^m x \, \cos^n x$ with m odd.

$$\begin{aligned}\int \tan x \, dx &= \int \frac{1}{\cos x} \sin x \, dx = \int \frac{1}{u} \, \frac{du}{-1} = -\ln|u| + C = -\ln|\cos x| + C \\ &= \ln|\cos x|^{-1} + C = \ln|\sec x| + C\end{aligned}$$

Example 5

Example 6 ($\int \tan^3 x \; dx$)

Write the integrand $\tan^3 x = \frac{\sin^2 x}{\cos^3 x} \sin x$. Again substitute $u = \cos x$, $du = -\sin x \, dx$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}\int \tan^3 x \, dx &= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, dx = \int \frac{1 - u^2}{u^3} \, \frac{du}{-1} = \frac{u^{-2}}{2} + \ln|u| + C \\ &= \frac{1}{2} \sec^2 x + \ln|\cos x| + C\end{aligned}$$

Example 6

Example 7 ($\int \tan^3 x \sec^4 x \; dx$)

Start by factoring off one copy of $\sec x \tan x$ and combine it with dx to form $\sec x \tan x \; dx$, which will be du. Then, substituting $u = \sec x$, $du = \sec x \tan x \; dx$ and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$,

$$\begin{aligned}\int \tan^3 x \sec^4 x \; dx &= \int \underbrace{\tan^2 x}_{u^2-1} \; \underbrace{\sec^3 x}_{u^3} \; \underbrace{\sec x \tan x \; dx}_{du} \\ &= \int [u^2 - 1] u^3 \; du\end{aligned}$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$
$$= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$$

Example 7

Example 8 ($\int \sec^4 x\ dx$)

Start by factoring off one copy of $\sec^2 x$ and combine it with dx to form $\sec^2 x\ dx$, which will be du. Then, substituting $u = \tan x$, $du = \sec^2 x\ dx$ and $\sec^2 x = 1 + \tan^2 x = 1 + u^2$,

$$\int \sec^4 x\ dx = \int \underbrace{\sec^2 x}_{1+u^2}\ \underbrace{\sec^2 x\ dx}_{du}$$
$$= \int \left[1 + u^2\right]\ du$$
$$= u + \frac{u^3}{3} + C$$
$$= \tan x + \frac{1}{3}\tan^3 x + C$$

Example 8

Example 9 ($\int \tan^4 x\ dx$)

By way of preparation, we try to create a $\sec^2 x$ from $\tan^4 x$, by using $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}\tan^4 x &= \tan^2 x \tan^2 x = \tan^2 x\left[\sec^2 x - 1\right]\\ &= \tan^2 x \sec^2 x - \tan^2 x\\ &= \tan^2 x \sec^2 x - \sec^2 x + 1\end{aligned}$$

Now we can subtitute $u = \tan x$, $du = \sec^2 x\ dx$.

$$\begin{aligned}\int \tan^4 x\ dx &= \int \underbrace{\tan^2 x}_{u^2}\ \underbrace{\sec^2 x\ dx}_{du} - \int \underbrace{\sec^2 x\ dx}_{du} + \int dx\\ &= \int u^2\ du - \int du + \int dx\\ &= \frac{u^3}{3} - u + x + C\\ &= \frac{\tan^3 x}{3} - \tan x + x + C\end{aligned}$$

Example 9

Of course we have not considered $\cot x$ and $\csc x$. But they can be treated in much the same way as $\tan x$ and $\sec x$ were.

Integrating $\sec x$, $\csc x$, $\sec^3 x$ and $\csc^3 x$ (Optional)

Example 10 ($\int \sec x \; dx$ — by trickery)

The standard trick used to integrate $\sec x$ is to multiply the integrand by $1 = \frac{\sec x + \tan x}{\sec x + \tan x}$ and then substitute $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\, dx$.

$$\begin{aligned}
\int \sec x \; dx &= \int \sec x \; \frac{\sec x + \tan x}{\sec x + \tan x} \; dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx \\
&= \int \frac{du}{u} \\
&= \ln |u| + C \\
&= \ln |\sec x + \tan x| + C
\end{aligned}$$

Example 10

There is a second method for integrating $\int \sec x \; dx$, that is more tedious, but more straight forward. In particular it does not involve a memorized trick. The integral $\int \sec x \, dx$ is converted into the integral $\int \frac{du}{1-u^2}$ by using the substitution $u = \sin x$, $du = \cos x \, dx$. The integral $\int \frac{du}{1-u^2}$ is then integrated by the method of partial fractions, which we shall learn about in the notes "Partial Fractions". The details are in Example 5 in those notes. This second method gives the answer

$$\int \sec x \; dx = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C$$

which appears to be different than the answer in Example 10. But they are really the same (of course) since

$$\begin{aligned}
&\frac{1 + \sin x}{1 - \sin x} = \frac{(1 + \sin x)^2}{1 - \sin^2 x} = \frac{(1 + \sin x)^2}{\cos^2 x} \\
\implies &\frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} = \frac{1}{2} \ln \frac{(1 + \sin x)^2}{\cos^2 x} \Big| = \ln \left| \frac{\sin x + 1}{\cos x} \right| = \ln |\tan x + \sec x|
\end{aligned}$$

Example 11 ($\int \csc x \; dx$ — by the $u = \tan \frac{x}{2}$ substitution)

The integral $\int \csc x \; dx$ may also be evaluated by both the methods above. That is either

- by multiplying the integrand by $1 = \frac{\cot x - \csc x}{\cot x - \csc x}$ and then substituting $u = \cot x - \csc x$, $du = (-\csc^2 x + \csc x \cot x)\, dx$ or

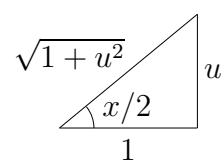
- by substituting $u = \cos x$, $du = -\sin x\, dx$ to give $\int \csc x\ dx = -\int \frac{du}{1-u^2}$ and then using the method of partial fractions.

These two methods give the answers

$$\int \csc x\ dx = \ln|\cot x - \csc x| + C = -\frac{1}{2}\ln\frac{1+\cos x}{1-\cos x} + C \tag{2}$$

In this example, we shall evaluate $\int \csc x\ dx$ by yet a third method, which can be used to integrate rational functions of $\sin x$ and $\cos x$. A rational function of $\sin x$ and $\cos x$ is a ratio with both the numerator and denominator being finite sums of terms of the form $a \sin^m x \cos^n x$, where a is a constant and m and n are positive integers. This method uses the substitution

$$\begin{aligned}
x &= 2\arctan u \qquad \text{i.e. } u = \tan\frac{x}{2} \\
dx &= \frac{2}{1+u^2}du \\
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2} \\
\cos x &= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+u^2} - \frac{u^2}{1+u^2} = \frac{1-u^2}{1+u^2}
\end{aligned}$$



This substitution converts $\int \csc x\, dx$ into

$$\begin{aligned}
\int \csc x\ dx &= \int \frac{1}{\sin x}\ dx = \int \frac{1+u^2}{2u}\ \frac{2}{1+u^2}du = \int \frac{1}{u}\ du = \ln|u| + C \\
&= \ln\left|\tan\frac{x}{2}\right| + C
\end{aligned}$$

To see that this answer is really the same as that in (2), note that

$$\cot x - \csc x = \frac{\cos x - 1}{\sin x} = \frac{-2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = -\tan\frac{x}{2}$$

Example 11

Example 12 ($\int \sec^3 x\ dx$ — by trickery)

The standard trick used to evaluate $\int \sec^3 x\ dx$ is integration by parts with $u = \sec x$, $dv = \sec^2 x\ dx$, $du = \sec x \tan x\ dx$, $v = \tan x$.

$$\begin{aligned}
\int \sec^3 x\ dx &= \int \underbrace{\sec x}_{u}\ \underbrace{\sec^2 x\ dx}_{dv} \\
&= \underbrace{\sec x}_{u}\ \underbrace{\tan x}_{v} - \int \underbrace{\tan x}_{v}\ \underbrace{\sec x \tan x\ dx}_{du}
\end{aligned}$$

Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\begin{aligned}\int \sec^3 x\ dx &= \sec x\ \tan x - \int [\sec^3 x - \sec x]\ dx \\ &= \sec x\ \tan x + \ln|\sec x + \tan x| + C - \int \sec^3 x\ dx\end{aligned}$$

where we used $\int \sec x\ dx = \ln|\sec x + \tan x| + C$. Now moving the $\int \sec^3 x\ dx$ from the right hand side to the left hand side

$$\begin{aligned}2\int \sec^3 x\ dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\ \Rightarrow \int \sec^3 x\ dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

for a new arbitrary constant C (which is just one half the old one).

Example 12

The integral $\int \sec^3\ dx$ can also be evaluated by two other methods.

- Substitute $u = \sin x$, $du = \cos x\ dx$ to convert $\int \sec^3 x\ dx$ into $\int \frac{du}{[1-u^2]^2}$ and evaluate the latter using the method of partial fractions. This is done in Example 6 in the notes "Partial Fractions".
- Use the $u = \tan\frac{x}{2}$ substitution. We use this method to evaluate $\int \csc^3 x\ dx$ in Example 13, below.

Example 13 ($\int \csc^3 x\ dx$ – by the $x = \tan\frac{x}{2}$ substitution)

As another example of the

$$u = \tan\frac{x}{2} \qquad dx = \frac{2}{1+u^2}du \qquad \sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2}$$

substitution, that we used in Example 11, we evaluate

$$\begin{aligned}\int \csc^3 x\ dx &= \int \frac{1}{\sin^3 x}\ dx = \int \left(\frac{1+u^2}{2u}\right)^3 \frac{2}{1+u^2}\ du = \frac{1}{4}\int \frac{1+2u^2+u^4}{u^3}\ du \\ &= \frac{1}{4}\left\{\frac{u^{-2}}{-2} + 2\ln|u| + \frac{u^2}{2}\right\} + C \\ &= \frac{1}{8}\left\{-\cot^2\frac{x}{2} + 4\ln\left|\tan\frac{x}{2}\right| + \tan^2\frac{x}{2}\right\} + C\end{aligned}$$

This is a perfectly acceptable answer. But if you don't like the $\frac{x}{2}$'s, they may be eliminated by using

$$\begin{aligned}\tan^2\frac{x}{2} - \cot^2\frac{x}{2} &= \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} - \frac{\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}} = \frac{\sin^4\frac{x}{2} - \cos^4\frac{x}{2}}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}} \\ &= \frac{\sin^2\frac{x}{2} - \cos^2\frac{x}{2}}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}} && \text{since } \sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1 \\ &= \frac{-\cos x}{\frac{1}{4}\sin^2 x} && \text{by (1b) and (1c)}\end{aligned}$$

and

$$\begin{aligned}\tan\frac{x}{2} &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} \\ &= \frac{\frac{1}{2}[1-\cos x]}{\frac{1}{2}\sin x} \qquad \text{by (1b) and (1c)}\end{aligned}$$

So we may also write

$$\int \csc^3 x\ dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

Example 13