# A Careful Area Computation

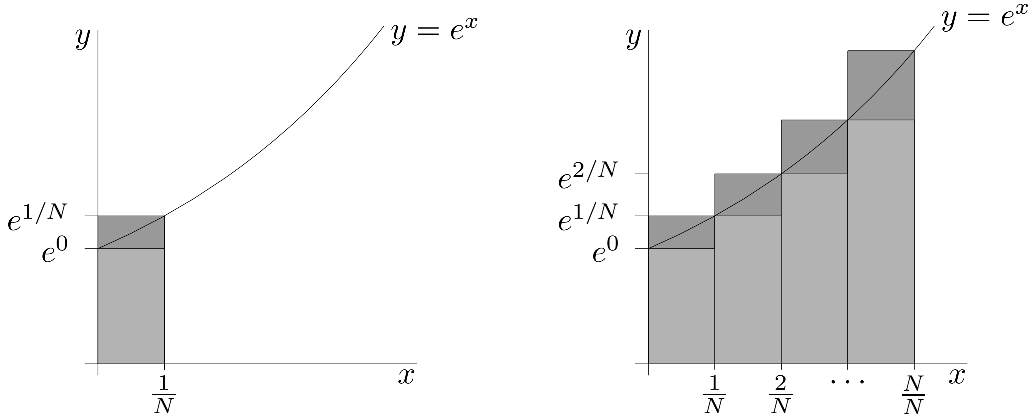
We are going to carefully compute the **exact** area of the region $0 \le y \le e^x \le 1$, $0 \le x \le 1$. There will be no uncontrolled approximations.

Because derivative $\frac{d}{dx}e^x = e^x$ is always positive, the function $e^x$ increases as $x$ increases. Consequently, the smallest and largest values of $e^x$ on the interval $a \le x \le b$ are $e^a$ and $e^b$, respectively. In particular, for $0 \le x \le \frac{1}{N}$, $e^x$ takes values only between $e^0$ and $e^{1/N}$. As a result, the set

$$\left\{ \ (x,y) \ \middle|\ 0 \le x \le \tfrac{1}{N},\ 0 \le y \le e^x \ \right\}$$

contains the rectangle of $0 \le x \le \frac{1}{N}$, $0 \le y \le e^0$ (the lighter rectangle in the figure on the left below) and is contained in the rectangle $0 \le x \le \frac{1}{N}$, $0 \le y \le e^{1/N}$ (the largest rectangle in the figure on the left below). Hence

$$\tfrac{1}{N}e^0 \le \text{Area}\left\{ \ (x,y) \ \middle|\ 0 \le x \le \tfrac{1}{N},\ 0 \le y \le e^x \ \right\} \le \tfrac{1}{N}e^{1/N} \tag{1}$$




Similarly, as in the figure on the right above,

$$\begin{aligned}
\tfrac{1}{N}e^{1/N} &\le \text{Area}\left\{ \ (x,y) \ \middle|\ \tfrac{1}{N} \le x \le \tfrac{2}{N},\ 0 \le y \le e^x \ \right\} &&\le \tfrac{1}{N}e^{2/N} \\
\tfrac{1}{N}e^{2/N} &\le \text{Area}\left\{ \ (x,y) \ \middle|\ \tfrac{2}{N} \le x \le \tfrac{3}{N},\ 0 \le y \le e^x \ \right\} &&\le \tfrac{1}{N}e^{3/N} \\
&\ \ \vdots \qquad\qquad\qquad \vdots &&\ \ \vdots \\
\tfrac{1}{N}e^{(N-1)/N} &\le \text{Area}\left\{ \ (x,y) \ \middle|\ \tfrac{N-1}{N} \le x \le \tfrac{N}{N},\ 0 \le y \le e^x \ \right\} &&\le \tfrac{1}{N}e^{N/N}
\end{aligned} \tag{2}$$

Adding (1) and all of the lines of (2) together gives

$$\begin{aligned}
&\tfrac{1}{N}\big(1 + e^{\frac{1}{N}} + \cdots + e^{\frac{N-1}{N}}\big) \\
&\qquad \le \text{Area}\left\{ \ (x,y) \ \middle|\ 0 \le x \le 1,\ 0 \le y \le e^x \ \right\} \\
&\qquad\qquad \le \tfrac{1}{N}\big(e^{\frac{1}{N}} + e^{\frac{2}{N}} + \cdots + e^{\frac{N}{N}}\big) \\
&\qquad\qquad = \tfrac{1}{N}e^{\frac{1}{N}}\big(1 + e^{\frac{1}{N}} + \cdots + e^{\frac{N-1}{N}}\big)
\end{aligned}$$

Using $1+r+\cdots+r^m = \frac{1-r^{m+1}}{1-r}$ with $r = e^{1/N}$ and $m = N-1$, so that $r^{m+1} = \big(e^{1/N}\big)^N = e$,

$$\frac{1}{N}\frac{1-e}{1-e^{1/N}} \le \text{Area}\big\{\ (x,y)\ \big|\ 0\le x\le 1,\ 0\le y\le e^x\ \big\} \le \frac{1}{N}e^{1/N}\frac{1-e}{1-e^{1/N}}$$

Thus the exact area must be at least as large as $\frac{1}{N}\frac{1-e}{1-e^{1/N}}$ for every single integer $N\ge 1$. So the exact area must also be at least as large as

$$\lim_{N\to\infty}\tfrac{1}{N}\tfrac{1-e}{1-e^{1/N}} = (1-e)\lim_{x=\frac{1}{N}\to 0}\tfrac{x}{1-e^x} = (1-e)\lim_{x\to 0}\tfrac{1}{-e^x} = e-1$$

by L'Hôpital's rule. Similarly, the exact area must be smaller than (or equal to) $\frac{1}{N}e^{\frac{1}{N}}\frac{1-e}{1-e^{1/N}}$ for every single natural number $N$. So the exact area must also be smaller than or equal to

$$\lim_{N\to\infty}\tfrac{1}{N}e^{\frac{1}{N}}\tfrac{1-e}{1-e^{1/N}} = (1-e)\lim_{x\to 0}e^x\tfrac{x}{1-e^x} = (1-e)\lim_{x\to 0}e^x\lim_{x\to 0}\tfrac{x}{1-e^x} = e-1$$

We have now shown that

$$e-1 \le \text{Area}\big\{\ (x,y)\ \big|\ 0\le y\le e^x,\ 0\le x\le 1\ \big\} \le e-1$$

so that the area must be exactly $e-1$.