Complex Numbers and Exponentials

A complex number is nothing more than a point in the xy–plane. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) is defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1)\,(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

respectively. It is conventional to use the notation $x + iy$ (or in electrical engineering country $x + jy$) to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, the sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

The complex number i has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

For example, if $z = 1 + 2i$ and $w = 3 + 4i$, then

$$z + w = (1 + 2i) + (3 + 4i) = 4 + 6i$$
$$zw = (1 + 2i)(3 + 4i) \quad = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 & z_1z_2 &= z_2z_1 \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2z_3) &= (z_1z_2)z_3 \\ 0 + z_1 &= z_1 & 1z_1 &= z_1 \\ z_1(z_2 + z_3) &= z_1z_2 + z_1z_3 & (z_1 + z_2)z_3 &= z_1z_3 + z_2z_3 \end{aligned}$$

The negative of any complex number $z = x+iy$ is defined by $-z = -x+(-y)i$, and obeys $z+(-z) = 0$. The inverse of any complex number $z = x+iy$, other than 0, is defined by $\frac{1}{z} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ and obeys $\frac{1}{z}z = 1$. To verify that $\frac{1}{z}z = 1$, just multiply out

$$\left(\tfrac{x}{x^2+y^2} - \tfrac{y}{x^2+y^2}i\right)(x + iy) = \tfrac{x^2}{x^2+y^2} - \tfrac{xy}{x^2+y^2}i + \tfrac{xy}{x^2+y^2}i - \tfrac{y^2}{x^2+y^2}i^2 = \tfrac{x^2+y^2}{x^2+y^2} = 1$$

The definition $\frac{1}{z} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ is not as mysterious as it looks. It is easy to divide a complex number by a real number. For example

$$\tfrac{11+2i}{25} = \tfrac{11}{25} + \tfrac{2}{25}i$$

In general, there is a trick for rewriting any ratio of complex numbers as a ratio with a real denominator. For example, suppose that we want to find $\frac{1+2i}{3+4i}$. The trick is to multiply by $1 = \frac{3-4i}{3-4i}$. The number $3 - 4i$ was constructed by replacing i by $-i$ in the denominator $3 + 4i$. Since $(3 + 4i)(3 - 4i) = 9 - 12i + 12i + 16 = 25$

$$\tfrac{1+2i}{3+4i} = \tfrac{1+2i}{3+4i}\tfrac{3-4i}{3-4i} = \tfrac{(1+2i)(3-4i)}{25} = \tfrac{11+2i}{25} = \tfrac{11}{25} + \tfrac{2}{25}i$$

The complex conjugate of z is denoted $\bar{z}$ and is defined to be $\bar{z} = x - iy$. That is, to take the complex conjugate, one replaces every i by $-i$. Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2$$

is always a positive real number. In fact, it is the square of the distance from $x+iy$ (recall that this is the point (x, y) in the xy–plane) to 0 (which is the point $(0, 0)$). The distance from $z = x + iy$ to 0 is denoted $|z|$ and is called the absolute value, or modulus, of z. It is given by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Since $|z|^2 = z\bar{z}$, we have

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

for all complex numbers $z \neq 0$. Also, since $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$,

$$\begin{aligned} |z_1z_2| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\ &= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2} \\ &= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= |z_1||z_2| \end{aligned}$$

for all complex numbers z_1, z_2.

The notations $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ stand for the real and imaginary parts of the complex number z, respectively. If $z = x + iy$ (with x and y real) they are defined by

$$\mathrm{Re}\, z = x \qquad \mathrm{Im}\, z = y$$

Note that both $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ are real numbers. Just subbing in $\bar{z} = x - iy$ gives

$$\mathrm{Re}\, z = \tfrac{1}{2}(z + \bar{z}) \qquad \mathrm{Im}\, z = \tfrac{1}{2i}(z - \bar{z})$$

The Complex Exponential

Definition and Basic Properties. For any complex number $z = x + iy$ the exponential e^z, is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

In particular, $e^{iy} = \cos y + i \sin y$. Once again, this definition is not as mysterious as it looks. We could also define e^{iy} by the subbing x by iy in the Taylor series expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$e^{iy} = 1 + iy + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^5}{5!} + \tfrac{(iy)^6}{6!} + \cdots$$

The even terms in this expansion are

$$1 + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^6}{6!} + \cdots = 1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} - \tfrac{y^6}{6!} + \cdots = \cos y$$

and the odd terms in this expansion are

$$iy + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^5}{5!} + \cdots = i\Big(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} + \cdots\Big) = i \sin y$$

For any two complex numbers z_1 and z_2

$$\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}\left\{(\cos y_1\cos y_2 - \sin y_1\sin y_2) + i(\cos y_1\sin y_2 + \cos y_2\sin y_1)\right\} \\
&= e^{x_1+x_2}\left\{\cos(y_1+y_2) + i\sin(y_1+y_2)\right\} \\
&= e^{(x_1+x_2)+i(y_1+y_2)} \\
&= e^{z_1+z_2}
\end{aligned}$$

so that the familiar multiplication formula also applies to complex exponentials. For any complex number $a = \alpha + i\beta$ and real number t

$$e^{at} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned}
\tfrac{d}{dt}e^{at} &= \alpha e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + e^{\alpha t}[-\beta\sin(\beta t) + i\beta\cos(\beta t)] \\
&= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] \\
&= ae^{at}
\end{aligned}$$

is also the familiar one.

Relationship with sin **and** cos**.** When θ is a real number

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^{-i\theta} &= \cos\theta - i\sin\theta = \overline{e^{i\theta}}
\end{aligned}$$

are complex numbers of modulus one. Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations)

$$\begin{aligned}
\cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \ = \operatorname{Re} e^{i\theta} \\
\sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \operatorname{Im} e^{i\theta}
\end{aligned}$$

These formulae make it easy derive trig identities. For example

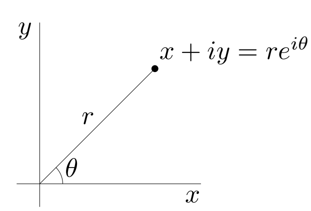
$$\begin{aligned}
\cos\theta\cos\phi &= \tfrac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) \\
&= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}) \\
&= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}) \\
&= \tfrac{1}{2}\big(\cos(\theta+\phi) + \cos(\theta-\phi)\big)
\end{aligned}$$

and, using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

$$\begin{aligned}
\sin^3\theta &= -\tfrac{1}{8i}\big(e^{i\theta} - e^{-i\theta}\big)^3 \\
&= -\tfrac{1}{8i}\big(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}\big) \\
&= \tfrac{3}{4}\tfrac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big) - \tfrac{1}{4}\tfrac{1}{2i}\big(e^{i3\theta} - e^{-i3\theta}\big) \\
&= \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin(3\theta)
\end{aligned}$$

Polar Coordinates. Let $z = x + iy$ be any complex number. Writing x and y in polar coordinates in the usual way gives

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$



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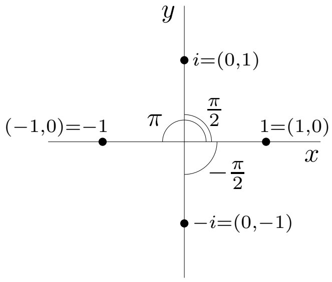


$$\begin{array}{rcccccl} 1 &=& e^{i0} &=& e^{2\pi i} &=& e^{2k\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ -1 &=& e^{i\pi} &=& e^{3\pi i} &=& e^{(1+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ i &=& e^{i\pi/2} &=& e^{\frac{5}{2}\pi i} &=& e^{(\frac{1}{2}+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ -i &=& e^{-i\pi/2} &=& e^{\frac{3}{2}\pi i} &=& e^{(-\frac{1}{2}+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \end{array}$$

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The n^{th} roots of unity are, by definition, all solutions z of

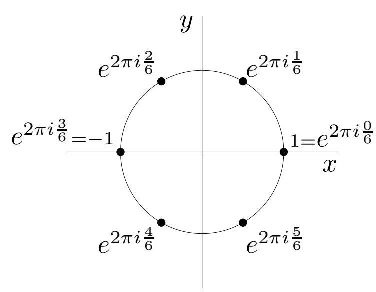
$$z^n = 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy–plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The n^{th} roots of unity are all complex numbers $e^{2\pi i\frac{k}{n}}$ with k integer. There are precisely n distinct n^{th} roots of unity because $e^{2\pi i\frac{k}{n}} = e^{2\pi i\frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi i\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The are n distinct nth roots of unity are

$$1\ ,\ e^{2\pi i\frac{1}{n}}\ ,\ e^{2\pi i\frac{2}{n}}\ ,\ e^{2\pi i\frac{3}{n}}\ ,\ \cdots\ ,\ e^{2\pi i\frac{n-1}{n}}$$



Exploiting Complex Exponentials in Calculus Computations

Example 1

$$\begin{aligned} \int e^x \cos x\ dx &= \tfrac{1}{2}\int e^x\left[e^{ix} + e^{-ix}\right]\ dx = \tfrac{1}{2}\int \left[e^{(1+i)x} + e^{(1-i)x}\right]\ dx \\ &= \tfrac{1}{2}\left[\tfrac{1}{1+i}e^{(1+i)x} + \tfrac{1}{1-i}e^{(1-i)x}\right] + C \end{aligned}$$

This form of the indefinite integral looks a little wierd because of the i's. But it is correct and it is pure real, despite the i's, because $\frac{1}{1-i}e^{(1-i)x}$ is the complex conjugate of $\frac{1}{1+i}e^{(1+i)x}$. We can convert the indefinite integral into a more familar form just by subbing back in $e^{\pm ix} = \cos x \pm i\sin x$, $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}$ and $\frac{1}{1-i} = \overline{\frac{1}{1+i}} = \frac{1+i}{2}$.

$$\begin{aligned} \int e^x \cos x\ dx &= \tfrac{1}{2}e^x\left[\tfrac{1}{1+i}e^{ix} + \tfrac{1}{1-i}e^{-ix}\right] + C \\ &= \tfrac{1}{2}e^x\left[\tfrac{1-i}{2}(\cos x + i\sin x) + \tfrac{1+i}{2}(\cos x - i\sin x)\right] + C \\ &= \tfrac{1}{2}e^x\cos x + \tfrac{1}{2}e^x\sin x + C \end{aligned}$$

Example 2 Using $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$,

$$\begin{aligned}\int \cos^4 x\ dx &= \tfrac{1}{2^4}\int \left[e^{ix}+e^{-ix}\right]^4\ dx = \tfrac{1}{2^4}\int \left[e^{4ix}+4e^{2ix}+6+4e^{-2i}+e^{-4ix}\right]\ dx \\ &= \tfrac{1}{2^4}\left[\tfrac{1}{4i}e^{4ix}+\tfrac{4}{2i}e^{2ix}+6x+\tfrac{4}{-2i}e^{-2ix}+\tfrac{1}{-4i}e^{-4ix}\right]+C \\ &= \tfrac{1}{2^4}\left[\tfrac{1}{2}\tfrac{1}{2i}(e^{4ix}-e^{-4ix})+\tfrac{4}{2i}(e^{2ix}-e^{-2ix})+6x\right]+C \\ &= \tfrac{1}{2^4}\left[\tfrac{1}{2}\sin 4x+4\sin 2x+6x\right]+C \\ &= \tfrac{1}{32}\sin 4x+\tfrac{1}{4}\sin 2x+\tfrac{3}{8}x+C\end{aligned}$$

Example 3 We shall now guess a solution to the differential equation

$$y'' + 2y' + 3y = \cos t \tag{1}$$

Equations like this arise, for example, in the study of the RLC circuit. We shall simplify the computation by exploiting that $\cos t = \operatorname{Re} e^{it}$. First, we shall guess a function $Y(t)$ obeying

$$Y'' + 2Y' + 3Y = e^{it} \tag{2}$$

Then, taking complex conjugates,

$$\bar{Y}'' + 2\bar{Y}' + 3\bar{Y} = e^{-it} \tag{$\bar{2}$}$$

and, adding $\frac{1}{2}(2)$ and $\frac{1}{2}(\bar{2})$ together will give

$$(\operatorname{Re} Y)'' + 2(\operatorname{Re} Y)' + 3(\operatorname{Re} Y) = \operatorname{Re} e^{it} = \cos t$$

which shows that $\operatorname{Re} Y(t)$ is a solution to (1). Let's try $Y(t) = Ae^{it}$. This is a solution of (2) if and only if

$$\begin{aligned} & \tfrac{d^2}{dt^2}\left(Ae^{it}\right) + 2\tfrac{d}{dt}\left(Ae^{it}\right) + 3Ae^{it} = e^{it} \\ \iff & \quad (2+2i)Ae^{it} = e^{it} \\ \iff & \quad A = \tfrac{1}{2+2i}\end{aligned}$$

So we have found a solution to (2) and $\operatorname{Re} \frac{e^{it}}{2+2i}$ is a solution to (1). To simplify this, write $2+2i$ in polar coordinates. So

$$2+2i = 2\sqrt{2}e^{i\frac{\pi}{4}} \Rightarrow \tfrac{e^{it}}{2+2i} = \tfrac{e^{it}}{2\sqrt{2}e^{i\frac{\pi}{4}}} = \tfrac{1}{2\sqrt{2}}e^{i(t-\frac{\pi}{4})} \Rightarrow \operatorname{Re}\tfrac{e^{it}}{2+2i} = \tfrac{1}{2\sqrt{2}}\cos(t-\tfrac{\pi}{4})$$

Example 4 In this example, we shall find $\int \sqrt{x^2-1}\ dx$.

First, here is some motivation for the substitution that I shall use. To integrate $\int \sqrt{1-x^2}\ dx$, we substitute $x = \cos t$, since it is easy to take the square root in $\sqrt{1-x^2} = \sqrt{1-\cos^2 t} = \sqrt{\sin^2 t}$. Now that we know about complex numbers, we are no longer afraid of taking the square root of negative numbers. Consequently, we can still substitute $x = \cos t$ into $\sqrt{x^2-1} = \sqrt{\cos^2 t - 1} = \sqrt{-\sin^2 t} = \sqrt{-1}\sqrt{\sin^2 t} = \pm i \sin t$.

In any real application, the domain of integration for $\int \sqrt{x^2-1}\ dx$ will only include x's obeying $x^2 \geq 1$, so that $\sqrt{x^2-1}$ is real. This looks like it causes problems for the substitution $x = \cos t$, because we are used to thinking that $\cos t$ only takes values between -1 and 1. But the restriction $-1 \leq \cos t \leq 1$ is only valid when t is real. Allowing t to be complex allows $\cos t$ to take all possible complex values. In fact, I claim that as t runs over all pure imaginary values (that is $t = iy$ with y real), $\cos t$ takes all real values bigger than $+1$. To see this, set $z = it$. Then as t runs over all pure imaginary values, z runs over all pure real values. When

$z = 0$, $\cos t = \frac{1}{2}\big(e^{it} + e^{-it}\big) = \frac{1}{2}\big(e^z + e^{-z}\big)$ takes the value 1. As z increases, $\frac{1}{2}\big(e^z + e^{-z}\big)$ increases (because $\frac{d}{dz}\frac{1}{2}\big(e^z + e^{-z}\big) = \frac{1}{2}\big(e^z - e^{-z}\big) > 0$ for $z > 0$) and as z approachs infinity, so does $\frac{1}{2}\big(e^z + e^{-z}\big)$. Thus as z runs through the real numbers from 0 to infinity, $\frac{1}{2}\big(e^z + e^{-z}\big)$ runs through the real numbers from 1 to infinity. The function $\frac{1}{2}\big(e^z + e^{-z}\big)$ is called the hyperbolic cosine of z and is denoted $\cosh z$. Similarly, the hyperbolic sine of z is $\sinh z = \frac{1}{2}\big(e^z - e^{-z}\big)$. The relationship between hyperbolic and regular sine and cosine is

$$\cos y = \cosh iy \qquad i\sin y = \sinh iy$$

For every trig identity, there is a corresponding identity for sinh and cosh. Just the signs change. For example $\sin^2 x + \cos^2 x = 1$, but $\cosh^2 x - \sinh^2 x = 1$. The identities are checked by just subbing in $\sinh z = \frac{1}{2}\big(e^z - e^{-z}\big)$ and $\cosh z = \frac{1}{2}\big(e^z + e^{-z}\big)$. Similarly, the derivative rules for sinh and cosh are the same as those for sin and cos, up to signs. For example, while $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\cosh x = \sinh x$.

Now the evaluation of the integral. Suppose that we want $x \geq 1$. Sub in $x = \cosh z = \frac{1}{2}\big(e^z + e^{-z}\big)$ with $z \geq 0$. (If we wanted $x \leq -1$, we would sub in $x = -\cosh z$.) I'll write everything out explicitly in terms of exponentials. The formulae would be shorter, if I wrote everything in terms of $\cosh x$ and $\sinh x$.

$$\begin{aligned}
x &= \tfrac{1}{2}\big(e^z + e^{-z}\big) \\
dx &= \tfrac{1}{2}\big(e^z - e^{-z}\big)dz \\
x^2 - 1 &= \tfrac{1}{4}\big(e^z + e^{-z}\big)^2 - 1 = \tfrac{1}{4}\big(e^{2z} + 2 + e^{-2z}\big) - 1 = \tfrac{1}{4}\big(e^{2z} - 2 + e^{-2z}\big) = \tfrac{1}{4}\big(e^z - e^{-z}\big)^2 \\
\sqrt{x^2 - 1} &= \tfrac{1}{2}\big(e^z - e^{-z}\big) \\
\sqrt{x^2 - 1}\ dx &= \tfrac{1}{4}\big(e^z - e^{-z}\big)^2\ dz = \tfrac{1}{4}\big(e^{2z} - 2 + e^{-2z}\big)\ dz \\
\int \sqrt{x^2 - 1}\ dx &= \tfrac{1}{4}\int \big(e^{2z} - 2 + e^{-2z}\big)\ dz = \tfrac{1}{4}\big(\tfrac{1}{2}e^{2z} - 2z - \tfrac{1}{2}e^{-2z}\big) + C
\end{aligned}$$

Now we have to sub back in what z is in terms of x. That is, we have to solve $x = \frac{1}{2}\big(e^z + e^{-z}\big)$ for z as a function of x.

$$x = \tfrac{1}{2}\big(e^z + e^{-z}\big) \iff 2x = e^z + e^{-z} \iff 2xe^z = e^{2z} + 1 \iff e^{2z} - 2xe^z + 1 = 0$$

Think of this as the quadratic equation $Q^2 - 2xQ + 1 = 0$ for $Q = e^z$. The quadratic equation $Q^2 - 2xQ + 1 = 0$ has two solutions: $Q = \frac{1}{2}\big(2x \pm \sqrt{4x^2 - 4}\big) = x \pm \sqrt{x^2 - 1}$. Note that if we divide the equation $e^{2z} - 2xe^z + 1 = 0$ by e^{2z} we get $e^{-2z} - 2xe^{-z} + 1 = 0$, which is exactly the same quadratic equation for $Q' = e^{-z}$ as we had for Q. One of the two solutions $x \pm \sqrt{x^2 - 1}$ is e^z and the other is e^{-z}. As we want $z \geq 0$, so that $e^z \geq e^{-z}$, we have to choose $e^z = x + \sqrt{x^2 - 1}$ and $e^{-z} = x - \sqrt{x^2 - 1}$. As a check, note that

$$\big(x + \sqrt{x^2 - 1}\big)\big(x - \sqrt{x^2 - 1}\big) = x^2 - (x^2 - 1) = 1 \quad \Rightarrow \quad \tfrac{1}{x + \sqrt{x^2 - 1}} = x - \sqrt{x^2 - 1}$$

Subbing in $e^z = x + \sqrt{x^2 - 1}$ and $e^{-z} = x - \sqrt{x^2 - 1}$ and $z = \ln\big(x + \sqrt{x^2 - 1}\big)$,

$$\begin{aligned}
\int \sqrt{x^2 - 1}\ dx &= \tfrac{1}{4}\big[\tfrac{1}{2}e^{2z} - 2z - \tfrac{1}{2}e^{-2z}\big] + C \\
&= \tfrac{1}{4}\big[\tfrac{1}{2}\big(x + \sqrt{x^2 - 1}\big)^2 - 2\ln\big(x + \sqrt{x^2 - 1}\big) - \tfrac{1}{2}\big(x - \sqrt{x^2 - 1}\big)^2\big] + C \\
&= \boxed{\tfrac{1}{2}\big[x\sqrt{x^2 - 1} - \ln\big(x + \sqrt{x^2 - 1}\big)\big] + C}
\end{aligned}$$

As a check, note that

$$\begin{aligned}
\tfrac{d}{dx}\tfrac{1}{2}\big[x\sqrt{x^2 - 1} - \ln\big(x + \sqrt{x^2 - 1}\big)\big] &= \tfrac{1}{2}\Big[\sqrt{x^2 - 1} + x\tfrac{x}{\sqrt{x^2 - 1}} - \tfrac{1 + x/\sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\Big] \\
&= \tfrac{1}{2}\Big[\tfrac{x^2 - 1}{\sqrt{x^2 - 1}} + \tfrac{x^2}{\sqrt{x^2 - 1}} - \tfrac{1}{\sqrt{x^2 - 1}}\tfrac{\sqrt{x^2 - 1} + x}{x + \sqrt{x^2 - 1}}\Big] \\
&= \tfrac{1}{2}\Big[\tfrac{x^2 - 1}{\sqrt{x^2 - 1}} + \tfrac{x^2}{\sqrt{x^2 - 1}} - \tfrac{1}{\sqrt{x^2 - 1}}\Big] \\
&= \tfrac{1}{2}\Big[\tfrac{2x^2 - 2}{\sqrt{x^2 - 1}}\Big] = \tfrac{x^2 - 1}{\sqrt{x^2 - 1}} = \sqrt{x^2 - 1}
\end{aligned}$$

as desired.

Example 5 In this example, we shall find $\int \frac{x+2}{x^2+2x+5}\ dx$. Using complex numbers, any polynomial can be written as a product of linear factors. This allows us to eliminate quadratic denominators from the partial fractions procedure. This example illustrates how.

We first have to factor the denominator x^2+2x+5. We can use the high school formula for the roots of a quadratic equation: $\frac{-2\pm\sqrt{2^2-4\times 5}}{2} = \frac{-2\pm\sqrt{4-20}}{2} = -1\pm\sqrt{-4} = -1\pm 2i$. Or we can complete the square

$$x^2+2x+5 = (x+1)^2+4 = (x+1)^2-(2i)^2 = [(x+1)-2i][(x+1)+2i] = [x+1-2i][x+1+2i]$$

Next we write the integrand in the form

$$\frac{x+2}{x^2+2x+5} = \frac{x+2}{(x+1-2i)(x+1+2i)} = \frac{a}{x+1-2i} + \frac{b}{x+1+2i}$$

with the constants a and b chosen so that

$$\frac{a}{x+1-2i} + \frac{b}{x+1+2i} = \frac{a(x+1+2i)+b(x+1-2i)}{(x+1-2i)(x+1+2i)} = \frac{x+2}{(x+1-2i)(x+1+2i)} \text{ i.e. so that } a(x+1+2i)+b(x+1-2i) = x+2$$

This has to be true for all x. We can solve easily for a if we choose $x+1=2i$ and we can solve easily for b if we choose $x+1=-2i$:

$$\begin{array}{llllll} x+1=2i & \Rightarrow & a(2i+2i)+b(2i-2i)=2i+1 & \Rightarrow & 4i\,a = 1+2i & \Rightarrow\quad a = \frac{1+2i}{4i} = \frac{1}{2}-\frac{1}{4}i \\ x+1=-2i & \Rightarrow & a(-2i+2i)+b(-2i-2i)=-2i+1 & \Rightarrow & -4i\,b = 1-2i & \Rightarrow\quad b = -\frac{1-2i}{4i} = \frac{1}{2}+\frac{1}{4}i \end{array}$$

since $\frac{1}{i} = -i$. As a check, we observe that, with $a = \frac{1}{2}-\frac{1}{4}i$ and $b = \frac{1}{2}+\frac{1}{4}i$,

$$\begin{aligned} a(x+1+2i)+b(x+1-2i) &= \big(\tfrac{1}{2}-\tfrac{1}{4}i\big)(x+1+2i) + \big(\tfrac{1}{2}+\tfrac{1}{4}i\big)(x+1-2i) \\ &= (x+1)\big(\tfrac{1}{2}-\tfrac{1}{4}i+\tfrac{1}{2}+\tfrac{1}{4}i\big) + 2i\big(\tfrac{1}{2}-\tfrac{1}{4}i-\tfrac{1}{2}-\tfrac{1}{4}i\big) \\ &= x+1+2i\big(-\tfrac{1}{2}i\big) = x+2 \end{aligned}$$

as desired. The integral is now easy,

$$\begin{aligned} \int \frac{x+2}{x^2+2x+5}\ dx &= \int \Big[\frac{a}{x+1-2i} + \frac{b}{x+1+2i}\Big]\ dx = a\ln(x+1-2i) + b\ln(x+1+2i) + C \\ &= \big(\tfrac{1}{2}-\tfrac{1}{4}i\big)\ln(x+1-2i) + \big(\tfrac{1}{2}+\tfrac{1}{4}i\big)\ln(x+1+2i) + C \end{aligned}$$

though the answer looks a little wierd because of the complex numbers.

One can eliminate the complex numbers by using the fact that

$$\ln(X\pm iY) = \ln\sqrt{X^2+Y^2} \pm i\tan^{-1}\frac{Y}{X} \tag{L}$$

To derive (L), let $\ln(X\pm iY) = U\pm iV$, with U and V real. Then U and V are to be determined by $e^{U\pm iV} = X\pm iY$ or $e^U\big(\cos V \pm i\sin V\big) = X\pm iY$ or $e^U\cos V = X$, $e^U\sin V = Y$. Dividing the last two equations gives $\tan V = \frac{Y}{X}$ and adding the squares of the last two equations together gives $e^{2U} = X^2+Y^2$.

Applying (L) with $X = x+1$ and $Y=2$ gives

$$\begin{aligned} \big(\tfrac{1}{2}-\tfrac{1}{4}i\big)\ln(x+1-2i) + \big(\tfrac{1}{2}+\tfrac{1}{4}i\big)\ln(x+1+2i) &= \big(\tfrac{1}{2}-\tfrac{1}{4}i\big)\big(\sqrt{x^2+2x+5} - i\tan^{-1}\tfrac{2}{x+1}\big) \\ &\quad + \big(\tfrac{1}{2}+\tfrac{1}{4}i\big)\big(\sqrt{x^2+2x+5} + i\tan^{-1}\tfrac{2}{x+1}\big) \\ &= \sqrt{x^2+2x+5} - \tfrac{1}{2}\tan^{-1}\tfrac{2}{x+1} \end{aligned}$$