

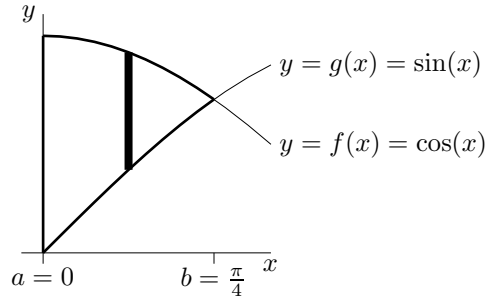
## Centroid Example

Find the centroid of the region bounded by  $y = \sin x$ ,  $y = \cos x$ ,  $x = 0$  and  $x = \frac{\pi}{4}$ .

**Solution.** We apply the formulae that the coordinates of the centroid (=centre of mass assuming constant density) of the region with top  $y = f(x)$ , bottom  $y = g(x)$ , left hand side  $x = a$  and right hand side  $x = b$  are

$$\bar{x} = \frac{\int_a^b x [f(x) - g(x)] dx}{\int_a^b [f(x) - g(x)] dx} \quad \bar{y} = \frac{\int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] dx}{\int_a^b [f(x) - g(x)] dx}$$

Before we apply these formulae, we recall where they came from. Assume that the region has density one. Consider a thin vertical slice, of width  $dx$ , running from  $(x, g(x))$  to  $(x, f(x))$ . It has area, and hence weight,



$[f(x) - g(x)] dx$ . On this slice  $x$  is essentially constant. So the formula for  $\bar{x}$  is just the formula for the (weighted) average of  $x$  over the whole region. On the slice  $y$  runs from  $g(x)$  to  $f(x)$ . The average value of  $y$  on the slice is  $\frac{1}{2}[f(x) + g(x)]$ . Because  $\frac{1}{2}[f(x) + g(x)][f(x) - g(x)] = \frac{1}{2}[f(x)^2 - g(x)^2]$ , the formula for  $\bar{y}$  is the formula for the average of  $y$  over the whole region.

In the given problem,  $a = 0$ ,  $b = \frac{\pi}{4}$ ,  $f(x) = \cos x$  and  $g(x) = \sin x$ . Subbing these in to the denominator of the formulae for  $\bar{x}$  and  $\bar{y}$  gives

$$\begin{aligned} \int_a^b [f(x) - g(x)] dx &= \int_0^{\pi/4} [\cos(x) - \sin(x)] dx = \left[ \sin(x) + \cos(x) \right]_0^{\pi/4} \\ &= \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] - [0 + 1] = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1 \end{aligned}$$

Subbing into the numerator of the formula for  $\bar{x}$  gives

$$\int_a^b x [f(x) - g(x)] dx = \int_0^{\pi/4} x [\cos(x) - \sin(x)] dx$$

To integrate this, use integration by parts with  $u = x$  and  $dv = [\cos x - \sin x] dx$ . So  $du = dx$ ,  $v = \sin x + \cos x$  and

$$\begin{aligned} \int_0^{\pi/4} x [\cos(x) - \sin(x)] dx &= x[\sin x + \cos x] \Big|_0^{\pi/4} - \int_0^{\pi/4} [\sin(x) + \cos(x)] dx \\ &= \frac{\pi}{4} \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] - \left[ -\cos(x) + \sin(x) \right]_0^{\pi/4} \\ &= \frac{2\pi}{4\sqrt{2}} - \left[ \left( -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (-1 + 0) \right] = \frac{\pi}{2\sqrt{2}} - 1 \end{aligned}$$

Finally, subbing into the numerator of the formula for  $\bar{y}$  gives

$$\int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] dx = \int_0^{\pi/4} \frac{1}{2} [\cos^2 x - \sin^2 x] dx = \int_0^{\pi/4} \frac{1}{2} \cos(2x) dx = \left[ \frac{1}{4} \sin(2x) \right]_0^{\pi/4} = \frac{1}{4}$$

Putting the formulae together

$$\bar{x} = \frac{\int_a^b x [f(x) - g(x)] dx}{\int_a^b [f(x) - g(x)] dx} = \frac{\frac{\pi}{2\sqrt{2}} - 1}{\sqrt{2} - 1} \quad \bar{y} = \frac{\int_a^b \frac{1}{2} [f(x)^2 - g(x)^2] dx}{\int_a^b [f(x) - g(x)] dx} = \frac{\frac{1}{4}}{\sqrt{2} - 1}$$