Centre of Mass and Torque

Centre of Mass

If you support a body at its center of mass (in a uniform gravitational field) it balances perfectly. That’s the definition of the center of mass of the body. If the body consists of a finite number of masses \( m_1, \ldots, m_n \) attached to an infinitely strong, weightless (idealized) rod with mass number \( i \) attached at position \( x_i \), then the center of mass is at the (weighted) average value of \( x \):

\[
\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}
\]

The denominator \( m = \sum_{i=1}^{n} m_i \) is the total mass of the body. This formula for the center of mass is derived in the following (optional) section. See (6).

For many (but certainly not all) purposes an (extended rigid) body acts like a point particle located at its center of mass. For example it is very common to treat the Earth as a point particle. Here is a more detailed example in which we think of a body as being made up of a number of component parts and compute the center of mass of the body as a whole by using the center of masses of the component parts. Suppose that we have a dumbbell which consists of

- a left end made up of particles of masses \( m_{l,1}, \ldots, m_{l,3} \) located at \( x_{l,1}, \ldots, x_{l,3} \) and
- a right end made up of particles of masses \( m_{r,1}, \ldots, m_{r,4} \) located at \( x_{r,1}, \ldots, x_{r,4} \) and
- an infinitely strong, weightless (idealized) rod joining all of the particles.

Then the mass and center of mass of the left end are

\[
M_l = m_{l,1} + \cdots + m_{l,3} \quad \bar{X}_l = \frac{m_{l,1} x_{l,1} + \cdots + m_{l,3} x_{l,3}}{M_l}
\]

and the mass and center of mass of the right end are

\[
M_r = m_{r,1} + \cdots + m_{r,4} \quad \bar{X}_r = \frac{m_{r,1} x_{r,1} + \cdots + m_{r,4} x_{r,4}}{M_r}
\]

The mass and center of mass of the entire dumbbell are

\[
M = m_{l,1} + \cdots + m_{l,3} + m_{r,1} + \cdots + m_{r,4} = M_l + M_r
\]

\[
\bar{x} = \frac{m_{l,1} x_{l,1} + \cdots + m_{l,3} x_{l,3} + m_{r,1} x_{r,1} + \cdots + m_{r,4} x_{r,4}}{M}
\]

\[
= \frac{M_l \bar{X}_l + M_r \bar{X}_r}{M_r + M_l}
\]
So we can compute the center of mass of the entire dumbbell by treating it as being made up of two point particles, one of mass $M_l$ located at the center of mass of the left end, and one of mass $M_r$ located at the center of mass of the right end.

Now we’ll extend the above ideas to cover more general classes of bodies. If the body consists of mass distributed continuously along a straight line, say with mass density $\rho(x) \text{kg/m}$ and with $x$ running from $a$ to $b$, rather than consisting of a finite number of point masses, the formula for the center of mass becomes

$$\bar{x} = \frac{\int_{a}^{b} x \, \rho(x) \, dx}{\int_{a}^{b} \rho(x) \, dx}$$

Think of $\rho(x) \, dx$ as the mass of the “almost point particle” between $x$ and $x + dx$.

If the body is a two dimensional object, like a metal plate, lying in the $xy$–plane, its center of mass is a point $(\bar{x}, \bar{y})$ with $\bar{x}$ being the (weighted) average value of the $x$–coordinate over the body and $\bar{y}$ being the (weighted) average value of the $y$–coordinate over the body. To be concrete, suppose the body fills the region

$$\{ (x, y) \mid a \leq x \leq b, \ B(x) \leq y \leq T(x) \}$$

in the $xy$–plane. For simplicity, we will assume that the density of the body is a constant, say $\rho$. When the density is constant, the center of mass is also called the centroid and is thought of as the geometric center of the body.

To find the centroid of the body, we use the use our standard “slicing” strategy. We slice the body into thin vertical strips, as illustrated in the figure below. Here is a detailed description of a generic strip.

- The strip has width $dx$.
- Each point of the strip has essentially the same $x$–coordinate. Call it $x$.
- The top of the strip is at $y = T(x)$ and the bottom of the strip is at $y = B(x)$.
- So the strip has
  - height $T(x) - B(x)$
\[
- \text{area } [T(x) - B(x)] \, dx \\
- \text{mass } \rho[T(x) - B(x)] \, dx \\
- \text{centroid, i.e. middle point, } (x, \frac{B(t) + T(x)}{2}).
\]

In computing the centroid of the entire body, we may treat each strip as a single particle of mass \( \rho[T(x) - B(x)] \, dx \) located at \( (x, \frac{B(t) + T(x)}{2}) \). So the mass of the entire body is

\[ M = \rho \int_a^b [T(x) - B(x)] \, dx = \rho A \tag{1a} \]

where \( A = \int_a^b [T(x) - B(x)] \, dx \) is the area of the region. The coordinates of the centroid are

\[ \bar{x} = \frac{\int_a^b x \rho[T(x) - B(x)] \, dx}{\int_a^b \rho[T(x) - B(x)] \, dx} = \frac{\int_a^b x[T(x) - B(x)] \, dx}{A} \tag{1b} \]

\[ \bar{y} = \frac{\int_a^b \frac{B(x) + T(x)}{2} \rho[T(x) - B(x)] \, dx}{\int_a^b \rho[T(x) - B(x)] \, dx} = \frac{\int_a^b [T(x)^2 - B(x)^2] \, dx}{2A} \tag{1c} \]

We can of course also slice up the body using horizontal slices. If the body has constant density \( \rho \) and fills the region

\[ \{ (x, y) \mid L(y) \leq x \leq R(y), \ c \leq y \leq d \} \]

then the same computation as above gives the mass of the body to be

\[ M = \rho \int_c^d [R(y) - L(y)] \, dy = \rho A \tag{2a} \]
where \( A = \int_c^d [R(y) - L(y)] \, dy \) is the area of the region, and gives the coordinates of the centroid to be

\[
\bar{x} = \frac{\int_c^d \frac{R(y)+L(y)}{2} \rho[R(y)-L(y)] \, dy}{M} = \frac{\int_c^d [R(y)^2 - L(y)^2] \, dx}{2A} \quad (2b)
\]

\[
\bar{y} = \frac{\int_c^d y \rho[R(y)-L(y)] \, dy}{M} = \frac{\int_c^d y[R(y)-L(y)] \, dy}{A} \quad (2c)
\]

### Example 1

Find the \( x \)-coordinate of the centroid (centre of gravity) of the plane region \( R \) that lies in the first quadrant \( x \geq 0, \ y \geq 0 \) and inside the ellipse \( 4x^2 + 9y^2 = 36 \). (The area bounded by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \( \pi ab \) square units.)

\[
4x^2 + 9y^2 = 36
\]

**Solution.** In standard form \( 4x^2 + 9y^2 = 36 \) is \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \). So, on \( R \), \( x \) runs from 0 to 3 and \( R \) has area \( A = \frac{1}{4} \pi \times 3 \times 2 = \frac{3}{2} \pi \). For each fixed \( x \), between 0 and 3, \( y \) runs from 0 to \( 2\sqrt{1-\frac{x^2}{9}} \). So, applying (1.b) with \( a = 0 \), \( b = 3 \), \( T(x) = 2\sqrt{1-\frac{x^2}{9}} \) and \( B(x) = 0 \),

\[
\bar{x} = \frac{1}{A} \int_0^3 x T(x) \, dx = \frac{1}{A} \int_0^3 x^2 \sqrt{1-\frac{x^2}{9}} \, dx = \frac{4}{3 \pi} \int_0^3 x \sqrt{1-\frac{x^2}{9}} \, dx
\]

Sub in \( u = 1 - \frac{x^2}{9} \), \( du = -\frac{2}{9} x \, dx \).

\[
\bar{x} = -\frac{9}{2 \pi} \int_0^1 \sqrt{u} \, du = -\frac{9}{2 \pi} \left[ \frac{u^{3/2}}{3/2} \right]_0^1 = -\frac{9}{2 \pi} \left[ \frac{2}{3} \right] = \frac{4}{\pi}
\]

### Example 2

Find the centroid of the quarter circular disk \( x \geq 0, \ y \geq 0, x^2 + y^2 \leq r^2 \).

\[
x^2 + y^2 = r^2
\]
Solution. By symmetry, $\bar{x} = \bar{y}$. The area of the quarter disk is $A = \frac{1}{4}\pi r^2$. By (1.b) with $a = 0$, $b = r$, $T(x) = \sqrt{r^2 - x^2}$ and $B(x) = 0$,

$$\bar{x} = \frac{1}{A} \int_0^r x \sqrt{r^2 - x^2} \, dx$$

To evaluate the integral, sub in $u = r^2 - x^2$, $du = -2x \, dx$.

$$\int_0^r x \sqrt{r^2 - x^2} \, dx = \int_0^{r^2} \sqrt{u} \frac{du}{2} = -\frac{1}{2} \left[ u^{3/2} \right]_0^0 = \frac{r^3}{3}$$

So

$$\bar{x} = \frac{4}{\pi r^2} \left[ \frac{r^3}{3} \right] = \frac{4r}{3\pi}$$

As we observed above, we should have $\bar{x} = \bar{y}$. But, just for practice, let’s compute $\bar{y}$ by the integral formula (1.c), again with $a = 0$, $b = r$, $T(x) = \sqrt{r^2 - x^2}$ and $B(x) = 0$,

$$\bar{y} = \frac{1}{2A} \int_0^r \left( \sqrt{r^2 - x^2} \right)^2 \, dx = \frac{2}{\pi r^2} \int_0^r \left( r^2 - x^2 \right) \, dx$$

$$= \frac{2}{\pi r^2} \left[ r^2x - \frac{x^3}{3} \right]_0^r = \frac{2}{\pi r^2} \frac{2r^3}{3} = \frac{4r}{3\pi}$$

as expected.

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Example 2

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Example 3

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Find the centroid of the region $R$ in the diagram.

Solution. By symmetry, $\bar{x} = \bar{y}$. The region $R$ is a $2 \times 2$ square with one quarter of a circle of radius 1 removed and so has area $2 \times 2 - \frac{1}{4}\pi = \frac{16}{4} - \frac{\pi}{4}$. The top of $R$ is $y = T(x) = 2$. The
bottom is \( y = B(x) \) with \( B(x) = \sqrt{1 - x^2} \) when \( 0 \leq x \leq 1 \) and \( B(x) = 0 \) when \( 1 \leq x \leq 2 \). So

\[
\bar{y} = \bar{x} = \frac{1}{A} \left[ \int_0^1 x[2 - \sqrt{1 - x^2}] \, dx + \int_1^2 x[2 - 0] \, dx \right]
\]

\[
= \frac{4}{16 - \pi} \left[ x^2 \big|_0^1 + x^2 \big|_1^2 - \int_0^1 x\sqrt{1 - x^2} \, dx \right]
\]

\[
= \frac{4}{16 - \pi} \left[ 4 - \frac{1}{3} \right] \quad \text{by (3) with } r = 1
\]

\[
= \frac{44}{48 - 3\pi}
\]

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**Example 3**

Prove that the centroid of any triangle is located at the point of intersection of the medians. A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.

**Solution.** Choose a coordinate system so that the vertices of the triangle are located at \((a, 0)\), \((0, b)\) and \((c, 0)\). (In the figure below, \(a\) is negative.) The line joining \((a, 0)\) and \((0, b)\) has equation \(bx + ay = ab\). (Check that \((a, 0)\) and \((0, b)\) both really are on this line.) The line joining \((c, 0)\) and \((0, b)\) has equation \(bx + cy = bc\). (Check that \((c, 0)\) and \((0, b)\) both really are on this line.) Hence for each fixed \( y \) between 0 and \( b \), \( x \) runs from \( a - \frac{2}{b}y \) to \( c - \frac{2}{b}y \).

We’ll use horizontal strips to compute \( \bar{x} \) and \( \bar{y} \). We could just apply (2) with \( c = 0 \), \( d = b \), \( R(y) = \frac{2}{b}(b - y) \) (which is gotten by solving \( bx + cy = bc \) for \( x \)) and \( L(y) = \frac{2}{b}(b - y) \) (which is gotten by solving \( bx + ay = ab \) for \( x \)).

But rather than memorizing or looking up those formulae, we’ll derive them for this example. So consider a thin strip at height \( y \) as illustrated in the figure above.
As the area of the triangle is $A = \frac{1}{2}(c - a)b$,

$$\bar{y} = \frac{1}{A} \int_{a}^{b} y \ell(y) \, dy = \frac{2}{(c - a)b} \int_{a}^{b} y \frac{c-a}{b}(b-y) \, dy = \frac{2}{b^2} \int_{0}^{b} (by - y^2) \, dy = \frac{2}{b^2} \left( \frac{b^2}{2} - \frac{b^3}{3} \right)$$

$$= \frac{2b^3}{6b} = \frac{b}{3}$$

$$\bar{x} = \frac{1}{A} \int_{a}^{b} \frac{a+c}{2b}(b-y) \ell(y) \, dy = \frac{2}{(c-a)b} \int_{a}^{b} \frac{a+c}{2b}(b-y) \frac{c-a}{b}(b-y) \, dy = \frac{a+c}{b^3} \int_{0}^{b} (y-b)^2 \, dy$$

$$= \frac{a+c}{b^3} \left[ \frac{1}{3}(y-b)^3 \right]_{0}^{b} = \frac{a+c}{b^3} \left( \frac{b^3}{3} \right) = \frac{a+c}{3}$$

We have found that the centroid of the triangle is at $(\bar{x}, \bar{y}) = \left( \frac{a+c}{3}, \frac{b}{3} \right)$. We shall now show that this point lies on all three medians.

• One vertex is at $(a, 0)$. The opposite side runs from $(0, b)$ and $(c, 0)$ and so has midpoint $\frac{1}{2}(c, b)$. The line from $(a, 0)$ to $\frac{1}{2}(c, b)$ has slope $\frac{b/2}{c/2-a} = \frac{b}{b-a}$ and so has equation

$$y = \frac{b}{b-a}(x-a). \quad \text{As} \quad \frac{b}{b-a}(\bar{x} - a) = \frac{b}{b-a}(\frac{a+c}{3} - a) = \frac{b}{3}(c + a - 3a) = \frac{b}{3} = \bar{y}, \quad \text{the centroid does indeed lie on this median. In this computation we have implicitly assumed that} \quad c \neq 2a \quad \text{so that the denominator} \quad c - 2a \neq 0. \quad \text{In the event that} \quad c = 2a, \quad \text{the median runs from} \quad (a, 0) \quad \text{to} \quad \left( a, \frac{b}{2} \right) \quad \text{and so has equation} \quad x = a. \quad \text{When} \quad c = 2a \quad \text{we also have} \quad \bar{x} = \frac{a+c}{3} = a, \quad \text{so that the centroid still lies on the median.}$$

• Another vertex is at $(c, 0)$. The opposite side runs from $(a, 0)$ and $(0, b)$ and so has midpoint $\frac{1}{2}(a, b)$. The line from $(c, 0)$ to $\frac{1}{2}(a, b)$ has slope $\frac{b/2}{a/2-c} = \frac{b}{a-2c}$ and so has equation

$$y = \frac{b}{a-2c}(x-c). \quad \text{As} \quad \frac{b}{a-2c}(\bar{x} - c) = \frac{b}{a-2c}(\frac{a+c}{3} - c) = \frac{b}{3}(a+c-3c) = \frac{b}{3} = \bar{y}, \quad \text{the centroid does indeed lie on this median. In this computation we have implicitly assumed that} \quad a \neq 2c \quad \text{so that the denominator} \quad a - 2c \neq 0. \quad \text{In the event that} \quad a = 2c, \quad \text{the median runs from} \quad (c, 0) \quad \text{to} \quad \left( c, \frac{b}{2} \right) \quad \text{and so has equation} \quad x = c. \quad \text{When} \quad a = 2c \quad \text{we also have} \quad \bar{x} = \frac{a+c}{3} = c, \quad \text{so that the centroid still lies on the median.}$$

• The third vertex is at $(0, b)$. The opposite side runs from $(a, 0)$ and $(c, 0)$ and so has midpoint $(\frac{a+c}{2}, 0)$. The line from $(0, b)$ to $(\frac{a+c}{2}, 0)$ has slope $\frac{-b}{(a+c)/2} = -\frac{2b}{a+c}$ and so has equation

$$y = b - \frac{2b}{a+c}x. \quad \text{As} \quad b - \frac{2b}{a+c}\bar{x} = b - \frac{2b}{a+c} \frac{a+c}{3} = \frac{b}{3} = \bar{y}, \quad \text{the centroid does indeed lie on this median. This time, we have implicitly assumed that} \quad a + c \neq 0. \quad \text{In the event that} \quad a + c = 0, \quad \text{the median runs from} \quad (0, b) \quad \text{to} \quad (0, 0) \quad \text{and so has equation} \quad x = 0. \quad \text{When} \quad a + c = 0 \quad \text{we also have} \quad \bar{x} = \frac{a+c}{3} = 0, \quad \text{so that the centroid still lies on the median.}
Newton’s law of motion says that the position $x(t)$ of a single particle moving under the influence of a force $F$ obeys $mx''(t) = F$. Similarly, the positions $x_i(t)$, $1 \leq i \leq n$, of a set of particles moving under the influence of forces $F_i$ obey $mx_i''(t) = F_i$, $1 \leq i \leq n$. Often systems of interest consist of some small number of rigid bodies. Suppose that we are interested in the motion of a single rigid body, say a piece of wood. The piece of wood is made up of a huge number of atoms. So the system of equations determining the motion of all of the individual atoms in the piece of wood is huge. On the other hand, because the piece of wood is rigid, its configuration is completely determined by the position of, for example, its centre of mass and its orientation. (Rather than get into what is precisely meant by “orientation”, let’s just say that it is certainly determined by, for example, the positions of a few of the corners of the piece of wood). It is possible to extract from the huge system of equations that determine the motion of all of the individual atoms, a small system of equations that determine the motion of the centre of mass and the orientation. We can avoid some vector analysis, that is beyond the scope of this course, by assuming that our rigid body is moving in two rather than three dimensions.

So, imagine a piece of wood moving in the $xy$–plane. Furthermore, imagine that the piece of wood consists of a huge number of particles joined by a huge number of weightless but very strong steel rods. The steel rod joining particle number one to particle number two just represents a force acting between particles number one and two. Suppose that

- there are $n$ particles, with particle number $i$ having mass $m_i$,
- at time $t$, particle number $i$ has $x$–coordinate $x_i(t)$ and $y$–coordinate $y_i(t)$,
- at time $t$, the external force (gravity and the like) acting on particle number $i$ has $x$–coordinate $H_i(t)$ and $y$–coordinate $V_i(t)$. Here $H$ stands for horizontal and $V$ stands for vertical.
- at time $t$, the force acting on particle number $i$, due to the steel rod joining particle number $i$ to particle number $j$ has $x$–coordinate $H_{i,j}(t)$ and $y$–coordinate $V_{i,j}(t)$. If there is no steel rod joining particles number $i$ and $j$, just set $H_{i,j}(t) = V_{i,j}(t) = 0$. In particular, $H_{i,i}(t) = V_{i,i}(t) = 0$.

The only assumptions that we shall make about the steel rod forces are

(A1) for each $i \neq j$, $H_{i,j}(t) = -H_{j,i}(t)$ and $V_{i,j}(t) = -V_{j,i}(t)$. In words, the steel rod joining particles $i$ and $j$ applies equal and opposite forces to particles $i$ and $j$. 

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for each \( i \neq j \), there is a function \( M_{i,j}(t) \) such that \( H_{i,j}(t) = M_{i,j}(t)\left[ x_i(t) - x_j(t) \right] \) and \( V_{i,j}(t) = M_{i,j}(t)\left[ y_i(t) - y_j(t) \right] \). In words, the force due to the rod joining particles \( i \) and \( j \) acts parallel to the line joining particles \( i \) and \( j \). For (A1) to be true, we need \( M_{i,j}(t) = M_{j,i}(t) \).

Newton’s law of motion, applied to particle number \( i \), now tells us that

\[
m_i x_i''(t) = H_i(t) + \sum_{j=1}^{n} H_{i,j}(t) \quad (X_i)
\]

\[
m_i y_i''(t) = V_i(t) + \sum_{j=1}^{n} V_{i,j}(t) \quad (Y_i)
\]

Adding up all the equations \((X_i)\), for \( i = 1, 2, 3, \ldots, n \) and adding up all the equations \((Y_i)\), for \( i = 1, 2, 3, \ldots, n \) gives

\[
\sum_{i=1}^{n} m_i x_i''(t) = \sum_{i=1}^{n} H_i(t) + \sum_{1 \leq i,j \leq n} H_{i,j}(t) \quad (\Sigma_i X_i)
\]

\[
\sum_{i=1}^{n} m_i y_i''(t) = \sum_{i=1}^{n} V_i(t) + \sum_{1 \leq i,j \leq n} V_{i,j}(t) \quad (\Sigma_i Y_i)
\]

The sum \( \sum_{1 \leq i,j \leq n} H_{i,j}(t) \) contains \( H_{1,2}(t) \) exactly once and it also contains \( H_{2,1}(t) \) exactly once and these two terms cancel exactly, by assumption (A1). In this way, all terms in \( \sum_{1 \leq i,j \leq n} H_{i,j}(t) \) with \( i \neq j \) exactly cancel. All terms with \( i = j \) are assumed to be zero. So \( \sum_{1 \leq i,j \leq n} H_{i,j}(t) = 0 \). Similarly, \( \sum_{1 \leq i,j \leq n} V_{i,j}(t) = 0 \), so the equations \((\Sigma_i X_i)\) and \((\Sigma_i Y_i)\) simplify to

\[
\sum_{i=1}^{n} m_i x_i''(t) = \sum_{i=1}^{n} H_i(t) \quad (\Sigma_i X_i)
\]

\[
\sum_{i=1}^{n} m_i y_i''(t) = \sum_{i=1}^{n} V_i(t) \quad (\Sigma_i Y_i)
\]

Denote by

\[
M = \sum_{i=1}^{n} m_i
\]

the total mass of the system, by

\[
X(t) = \frac{1}{M} \sum_{i=1}^{n} m_i x_i(t) \quad \text{and} \quad Y(t) = \frac{1}{M} \sum_{i=1}^{n} m_i y_i(t)
\]

the \( x \)– and \( y \)–coordinates of the centre of mass of the system at time \( t \) and by

\[
H(t) = \sum_{i=1}^{n} H_i(t) \quad \text{and} \quad V(t) = \sum_{i=1}^{n} V_i(t)
\]
the \( x \)- and \( y \)-coordinates of the total external force acting on the system at time \( t \). In this notation, the equations \( (\Sigma_i X_i) \) and \( (\Sigma_i Y_i) \) are

\[
MX''(t) = H(t) \quad MY''(t) = V(t)
\]

(4)

So the centre of mass of the system moves just like a single particle of mass \( M \) subject to the total external force.

Now multiply equation \( (Y_i) \) by \( x_i(t) \), subtract from it equation \( (X_i) \) multiplied by \( y_i(t) \), and sum over \( i \). This gives the equation \( \sum_i \left[ x_i(t) (Y_i) - y_i(t) (X_i) \right] \):

\[
\sum_{i=1}^n m_i [x_i(t)y_i''(t) - y_i(t)x_i''(t)] = \sum_{i=1}^n \left[ x_i(t) V_i(t) - y_i(t) H_i(t) \right] + \sum_{1 \leq i,j \leq n} \left[ x_i(t) V_{i,j}(t) - y_i(t) H_{i,j}(t) \right]
\]

By the assumption (A2)

\[
x_1(t)V_{1,2}(t) - y_1(t)H_{1,2}(t) = x_1(t)M_{1,2}(t)\left[ y_1(t) - y_2(t) \right] - y_1(t)M_{1,2}(t)\left[ x_1(t) - x_2(t) \right] = M_{1,2}(t)\left[ y_1(t)x_2(t) - x_1(t)y_2(t) \right]
\]

\[
x_2(t)V_{2,1}(t) - y_2(t)H_{2,1}(t) = x_2(t)M_{2,1}(t)\left[ y_2(t) - y_1(t) \right] - y_2(t)M_{2,1}(t)\left[ x_2(t) - x_1(t) \right] = M_{2,1}(t)\left[ -y_1(t)x_2(t) + x_1(t)y_2(t) \right] = M_{1,2}(t)\left[ -y_1(t)x_2(t) + x_1(t)y_2(t) \right]
\]

So the \( i = 1, j = 2 \) term in \( \sum_{1 \leq i,j \leq n} \left[ x_i(t)V_{i,j}(t) - y_i(t)H_{i,j}(t) \right] \) exactly cancels the \( i = 2, j = 1 \) term. In this way all of the terms in \( \sum_{1 \leq i,j \leq n} \left[ x_i(t)V_{i,j}(t) - y_i(t)H_{i,j}(t) \right] \) with \( i \neq j \) cancel. Each term with \( i = j \) is exactly zero. So \( \sum_{1 \leq i,j \leq n} \left[ x_i(t)V_{i,j}(t) - y_i(t)H_{i,j}(t) \right] = 0 \) and

\[
\sum_{i=1}^n m_i \left[ x_i(t)y_i''(t) - y_i(t)x_i''(t) \right] = \sum_{i=1}^n \left[ x_i(t)V_i(t) - y_i(t)H_i(t) \right]
\]

Define

\[
L(t) = \sum_{i=1}^n m_i \left[ x_i(t)y_i'(t) - y_i(t)x_i'(t) \right]
\]

\[
T(t) = \sum_{i=1}^n \left[ x_i(t)V_i(t) - y_i(t)H_i(t) \right]
\]

In this notation

\[
\frac{d}{dt} L(t) = T(t)
\]

(5)

- Equation (5) plays the role of Newton’s law of motion for rotational motion.
- \( T(t) \) is called the torque and plays the role of “rotational force”.
- \( L(t) \) is called the angular momentum (about the origin) and is a measure of the rate at which the piece of wood is rotating.
For example, if a particle of mass $m$ is traveling in a circle of radius $r$, centred on the origin, at $\omega$ radians per unit time, then

\[ x(t) = r \cos(\omega t), \quad y(t) = r \sin(\omega t) \]

and

\[
m[x(t)y'(t) - y(t)x'(t)] = m[r \cos(\omega t) r \omega \cos(\omega t) - r \sin(\omega t) (-r \omega \sin(\omega t))]
\]

\[ = mr^2 \omega \]

is proportional to $\omega$, which is the rate of rotation about the origin.

\[ x^2 + y^2 = r^2 \]

In any event, in order for the piece of wood to remain stationary, that is to have $x_i(t)$ and $y_i(t)$ be constant for all $1 \leq i \leq n$, we need to have

\[ X''(y) = Y''(t) = L(t) = 0 \]

and then equations (4) and (5) force

\[ H(t) = V(t) = T(t) = 0 \]

Now suppose that the piece of wood is a seesaw that is long and thin and is lying on the $x$–axis, supported on a fulcrum at $x = p$. Then every $y_i = 0$ and the torque simplifies to

\[ T(t) = \sum_{i=1}^{n} x_i(t)V_i(t). \]

The forces consist of

- gravity, $m_i g$, acting downwards on particle number $i$, for each $1 \leq i \leq n$ and the

- force $F$ imposed by the fulcrum that is pushing straight up on the particle at $x = p$.

So

- The net vertical force is $V(t) = F - \sum_{i=1}^{n} m_i g = F - M g$. If the seesaw is to remain stationary, this must be zero so that $F = Mg$. 
• The total torque (about the origin) is

\[ T = Fp - \sum_{i=1}^{n} m_i gx_i = Mg p - \sum_{i=1}^{n} m_i gx_i \]

If the seesaw is to remain stationary, this must also be zero and the fulcrum must be placed at

\[ .p = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]  \hspace{1cm} (6)

which is the centre of mass of the piece of wood.