

Units and the Derivative of $\sin x$

Any derivative measures the rate of change of one quantity per unit change of a second quantity. If you change the units of either quantity, the numerical value of the derivative changes. For example, the speed of a car is given by the derivative $\frac{dx}{dt}$, where x is the position of the car and t is time. If x is measured in kilometers and t is measured in hours, one might have, for example, $\frac{dx}{dt} = 60$ km/hr. But if you decide to measure t in minutes rather than hours, the same vehicle moving at the same speed will have $\frac{dx}{dt} = 60 \times \frac{1}{3600}$ km/min.

As a second example, consider the average rate of change of $\sin x$ as x moves from 0 to a right angle. If x is measured in radians,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1 - 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi}$$

because $\sin x$ changes from 0 to 1 as x changes from 0 radians to $\frac{\pi}{2}$ radians. But, if x is measured in degrees,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1 - 0}{90 - 0} = \frac{1}{90}$$

The two answers are different even though x moved from 0 to a right angle in both cases. The usual formula that you have memorized for the derivative of $\sin x$, namely $\frac{d}{dx} \sin x = \cos x$, is **valid only if x is measured in radians**.

Problem. Find $\frac{d}{dx} \sin x$, if x is in degrees, rather than radians.

Solution 1. I think that the safest and clearest way to approach this problem is to give different names to the sin of x degrees and the sin of x radians. Of course, in the real world, nobody does this. Use $\text{Sin}(x)$ to denote the sin of x degrees and $\sin(x)$ to denote the sin of x radians. Similarly, use $\text{Cos}(x)$ to denote the cos of x degrees and $\cos(x)$ to denote the cos of x radians. For example, $\text{Sin}(90) = \sin\left(\frac{\pi}{2}\right) = 1$ and $\text{Cos}(90) = \cos\left(\frac{\pi}{2}\right) = 0$. In general, $\text{Sin}(x) = \sin\left(\frac{\pi}{180}x\right)$. By definition,

$$\begin{aligned} \frac{d}{dx} \text{Sin}(x) &= \lim_{h \rightarrow 0} \frac{\text{Sin}(x+h) - \text{Sin}(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{180}(x+h)\right) - \sin\left(\frac{\pi}{180}x\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{180}x + \frac{\pi}{180}h\right) - \sin\left(\frac{\pi}{180}x\right)}{h} \end{aligned}$$

Set $H = \frac{\pi}{180}h$. Then

$$\begin{aligned} \frac{d}{dx} \text{Sin}(x) &= \lim_{H \rightarrow 0} \frac{\sin\left(\frac{\pi}{180}x + H\right) - \sin\left(\frac{\pi}{180}x\right)}{\frac{180}{\pi}H} = \frac{\pi}{180} \lim_{H \rightarrow 0} \frac{\sin\left(\frac{\pi}{180}x + H\right) - \sin\left(\frac{\pi}{180}x\right)}{H} \\ &= \frac{\pi}{180} \frac{d}{dt} \sin(t) \Big|_{t=\frac{\pi}{180}x} = \frac{\pi}{180} \cos\left(\frac{\pi}{180}x\right) = \frac{\pi}{180} \text{Cos}(x) \end{aligned}$$

Thus $\boxed{\frac{d}{dx} \text{Sin}(x) = \frac{\pi}{180} \text{Cos}(x)}$. Of course, people do not usually give separate names to $\text{Sin}(x)$ and $\sin(x)$. So, it is also correct to say that

$$\boxed{\frac{d}{dx} \sin(x) = \frac{\pi}{180} \cos(x), \text{ when } \sin(x) \text{ and } \cos(x) \text{ mean } \sin \text{ and } \cos \text{ of } x \text{ degrees}}$$

Solution 2. Using the same notation as in Solution 1, $\text{Sin}(x) = \sin\left(\frac{\pi}{180}x\right)$. So, by the chain rule, with $f(y) = \sin y$ and $g(x) = \frac{\pi}{180}x$,

$$\frac{d}{dx} \text{Sin}(x) = \frac{d}{dx} \sin\left(\frac{\pi}{180}x\right) = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = \cos\left(\frac{\pi}{180}x\right) \frac{\pi}{180}$$

This is the same answer as we got in Solution 1.