Long Division of Polynomials

Suppose that \( P(x) \) is a polynomial of degree \( p \) and suppose that you know that \( r \) is a root of that polynomial. In other words, suppose you know that \( P(r) = 0 \). Then it is always possible to factor \( (x - r) \) out of \( P(x) \). More precisely, it is always possible to find a polynomial \( Q(x) \) of degree \( p - 1 \) such that

\[
P(x) = (x - r)Q(x)
\]

In sufficiently simple cases, you can probably do this factoring by inspection. For example, \( P(x) = x^2 - 4 \) has \( r = 2 \) as a root because \( P(2) = 2^2 - 4 = 0 \). In this case, \( P(x) = (x - 2)(x + 2) \) so that \( Q(x) = (x + 2) \). As another example, \( P(x) = x^2 - 2x - 3 \) has \( r = -1 \) as a root because \( P(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0 \). In this case, \( P(x) = (x + 1)(x - 3) \) so that \( Q(x) = (x - 3) \).

Once you have found a root \( r \) of a polynomial, even if you cannot factor \( (x - r) \) out of the polynomial by inspection, you can find \( Q(x) \) by dividing \( P(x) \) by \( x - r \), using the long division algorithm you learned in public school, but with 10 replaced by \( x \).

**Example.** \( P(x) = x^3 - x^2 + 2 \).

Because \( P(-1) = (-1)^3 - (-1)^2 + 2 = -1 - 1 + 2 = 0 \), \( r = -1 \) is a root of this polynomial. So we divide \( x^3 - x^2 + 2 \). The first term, \( x^2 \), in the quotient is chosen so that when you multiply it by the denominator, \( x^2(x + 1) = x^3 + x^2 \), the leading term, \( x^3 \), matches the leading term in the numerator, \( x^3 - x^2 + 2 \), exactly.

\[
x + 1 \quad \overline{x^2} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2}
\]

When you subtract \( x^2(x + 1) = x^3 + x^2 \) from the numerator \( x^3 - x^2 + 2 \) you get the remainder \( -2x^2 + 2 \). Just like in public school, the 2 is not normally “brought down” until it is actually needed.

\[
x + 1 \quad \overline{x^2} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2 - 2x^2}
\]

The next term, \( -2x \), in the quotient is chosen so that when you multiply it by the denominator, \( -2x(x + 1) = -2x^2 - 2x \), the leading term \( -2x^2 \) matches the leading term in the remainder exactly.

\[
x + 1 \quad \overline{x^2 - 2x} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2 - 2x^2 - 2x}
\]

And so on.

\[
x + 1 \quad \overline{x^2 - 2x + 2} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2 - 2x^2 - 2x}
\]

\[
x + 1 \quad \overline{2x + 2} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2 - 2x^2 - 2x}
\]

\[
x + 1 \quad \overline{2x + 2} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2 - 2x^2 - 2x + 2}
\]

\[
x + 1 \quad \overline{0} \quad \frac{x^3 - x^2 + 2}{x^3 + x^2 - 2x^2 - 2x + 2}
\]

Note that we finally end up with a remainder \( 0 \). Since \(-1\) is a root of the numerator, \( x^3 - x^2 + 2 \), the denominator \( x - (-1) \) must divide the numerator exactly.

There is an alternative to long division that involves more writing. In the previous example, we know
that \( \frac{x^3-x^2+2}{x+1} \) must be a polynomial (since \(-1\) is a root of the numerator) of degree 2. So

\[
\frac{x^3-x^2+2}{x+1} = ax^2 + bx + c
\]

for some, as yet unknown, coefficients \( a, b \) and \( c \). Cross multiplying and simplifying

\[
x^3 - x^2 + 2 = (ax^2 + bx + c)(x + 1)
\]

\[
= ax^3 + (a + b)x^2 + (b + c)x + c
\]

Matching coefficients of the various powers of \( x \) on the left and right hand sides

- coefficient of \( x^3 \): \( a = 1 \)
- coefficient of \( x^2 \): \( a + b = -1 \)
- coefficient of \( x^1 \): \( b + c = 0 \)
- coefficient of \( x^0 \): \( c = 2 \)

tells us directly that \( a = 1 \) and \( c = 2 \). Subbing \( a = 1 \) into \( a + b = -1 \) tells us that \( 1 + b = -1 \) and hence \( b = -2 \).