

# Differentiation Rules

## Statement

Suppose that  $f'(x)$  and  $g'(x)$  both exist.

- a) If  $S(x) = af(x) + bg(x)$ , with  $a$  and  $b$  constants, then  $S'(x) = af'(x) + bg'(x)$ . That is,

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$$

- b) (Product Rule) If  $P(x) = f(x)g(x)$ , then  $P'(x) = f'(x)g(x) + f(x)g'(x)$ . That is,

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

- c) (Quotient Rule) If  $Q(x) = \frac{f(x)}{g(x)}$ , then  $Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ , provided  $g(x) \neq 0$ .

That is,

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Setting  $f(x)$  to the constant function  $f(x) = 1$  gives the special case

$$\frac{d}{dx} \left[ \frac{1}{g(x)} \right] = -\frac{g'(x)}{g(x)^2}$$

## Derivation

Define

$$F(x, h) = \frac{f(x+h) - f(x)}{h} \qquad G(x, h) = \frac{g(x+h) - g(x)}{h}$$

Observe that, cross multiplying by  $h$ ,

$$f(x+h) = f(x) + hF(x, h) \qquad g(x+h) = g(x) + hG(x, h)$$

By the definition of the derivative

$$\lim_{h \rightarrow 0} F(x, h) = f'(x) \qquad \lim_{h \rightarrow 0} G(x, h) = g'(x)$$

a)

$$\begin{aligned} S(x+h) - S(x) &= af(x+h) + bg(x+h) - af(x) - bg(x) \\ &= a[f(x) + hF(x,h)] + b[g(x) + hG(x,h)] - af(x) - bg(x) \\ &= ahF(x,h) + bhG(x,h) \end{aligned}$$

By definition

$$S'(x) = \lim_{h \rightarrow 0} \frac{S(x+h) - S(x)}{h} = \lim_{h \rightarrow 0} \frac{ahF(x,h) + bhG(x,h)}{h} = \lim_{h \rightarrow 0} [aF(x,h) + bG(x,h)] = af'(x) + bg'(x)$$

b)

$$\begin{aligned} P(x+h) - P(x) &= f(x+h)g(x+h) - f(x)g(x) \\ &= [f(x) + hF(x,h)][g(x) + hG(x,h)] - f(x)g(x) \\ &= hF(x,h)g(x) + f(x)hG(x,h) + h^2F(x,h)G(x,h) \end{aligned}$$

By definition

$$\begin{aligned} P'(x) &= \lim_{h \rightarrow 0} \frac{P(x+h) - P(x)}{h} = \lim_{h \rightarrow 0} \frac{hF(x,h)g(x) + hf(x)G(x,h) + h^2F(x,h)G(x,h)}{h} \\ &= \lim_{h \rightarrow 0} [F(x,h)g(x) + f(x)G(x,h) + hF(x,h)G(x,h)] \\ &= f'(x)g(x) + f(x)g'(x) + 0f'(x)g'(x) \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

c) By definition,

$$Q'(x) = \lim_{h \rightarrow 0} \frac{Q(x+h) - Q(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$

The numerator

$$\begin{aligned} f(x+h)g(x) - f(x)g(x+h) &= [f(x) + hF(x,h)]g(x) - f(x)[g(x) + hG(x,h)] \\ &= hF(x,h)g(x) - f(x)hG(x,h) \end{aligned}$$

Dividing by  $hg(x)g(x+h)$

$$\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} = \frac{hF(x,h)g(x) - hf(x)G(x,h)}{hg(x)g(x+h)} = \frac{F(x,h)}{g(x+h)} - \frac{f(x)G(x,h)}{g(x)g(x+h)}$$

Hence

$$\begin{aligned} Q'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)} = \lim_{h \rightarrow 0} \left[ \frac{F(x,h)}{g(x+h)} - \frac{f(x)G(x,h)}{g(x)g(x+h)} \right] = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

as desired.