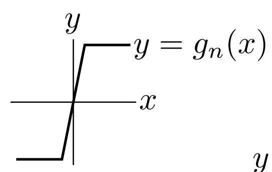
# Appendix S2: Problem Solutions for §2

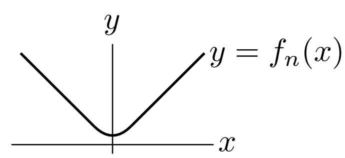
**Problem 2.1.1** Let $\Omega = (-1, 1)$ and $\ell = 1$. Think of $f(x) = |x|$ and $g(x) = \operatorname{sgn} x$ as functions in $L^2(\Omega)$. Find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C^1(\Omega)$ such that

- $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the norm $\|\ \|_{\ell,\Omega}$
- $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $f$.
- $\{\frac{df_n}{dx}\}_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $g$.

**Solution.** Let, for each $n \in \mathbb{N}$, $g_n$ be any continous function on $(-1, 1)$ that obeys

- $g_n(x) = -1$ for all $x < -\frac{1}{n}$
- $g_n(x) = 1$ for all $x > \frac{1}{n}$
- $g_n(-x) = -x$ for all $-1 < x < 1$
- $|g_n(x)| \le 1$ for all $-1 < x < 1$




Define

$$f_n(x) = 1 + \int_{-1}^{x} g_n(t)\ dt$$

Then $f_n \in C^1(\Omega)$ and $\frac{df_n}{dx} = g_n$. As $f(-1) = 1 = |-1|$ and $f(1) = 1 = |1|$ (note that $\int_{-1}^{1} g_n(t)\ dt = 0$ since $g_n$ is odd), we have

- $f_n(x) = |x|$ for all $x < -\frac{1}{n}$
- $f_n(x) = |x|$ for all $x > \frac{1}{n}$
- $-1 \le f_n(x) \le 3$ for all $-1 < x < 1$ since $-2 \le \int_{-1}^{x} g_n(t)\ dt \le 2$ (This is pretty sloppy, but still good enough.)

If $n < m$, then

- $f_n(x) - f_m(x)$ and $f_n(x) - f(x)$ vanish for all $x < -\frac{1}{n}$ and all $x > \frac{1}{n}$
- $|f_n(x) - f_m(x)| \le 4$ and $|f_n(x) - f(x)| \le 4$ for all $x$
- $\frac{df_n}{dx}(x) - \frac{df_m}{dx}(x)$ and $\frac{df_n}{dx}(x) - g(x)$ vanish for all $x < -\frac{1}{n}$ and all $x > \frac{1}{n}$
- $|\frac{df_n}{dx}(x) - \frac{df_m}{dx}(x)| \le 2$ and $|\frac{df_n}{dx}(x) - g(x)| \le 2$ for all $x$

Consequently

$$\big\|f_n - f_m\big\|_{L^2(\Omega)}, \big\|f_n - f_m\big\|_{L^2(\Omega)} \le \left[\int_{-1/n}^{1/n} 4^2\ dx\right]^{1/2} = 4\sqrt{\tfrac{2}{n}}$$

and

$$\big\|\tfrac{df_n}{dx} - \tfrac{df_m}{dx}\big\|_{L^2(\Omega)}, \big\|\tfrac{df_n}{dx} - g\big\|_{L^2(\Omega)} \le \left[\int_{-1/n}^{1/n} 2^2\ dx\right]^{1/2} = 2\sqrt{\tfrac{2}{n}}$$

All three desired conclusions follow.

**Problem 2.1.2** Prove that $C_0^\ell(\Omega) \subset H_0^\ell(\Omega)$.

**Solution.** Let $\delta(x) \in C_0^\infty(\mathbb{R}^n)$ have support in the ball of radius one centred on the origin and obey $\int_{\mathbb{R}^n} \delta(x)\ d^n x = 1$. Then, for any $m > 0$, $\delta_m(x) = m^n \delta(mx)$ has support in the ball of radius $\frac{1}{m}$ centred on the origin and still obeys $\int_{\mathbb{R}^n} \delta_m(x)\ d^n x = 1$. Let $\varphi(x) \in C_0^\ell(\Omega)$. Define $\varphi_m(x) = \int_{\mathbb{R}^n} \delta_m(x-y)\varphi(y)\ d^n y$. Because $\varphi$ is absolutely integrable and $\delta_m$ is $C_0^\infty$, $\varphi_m$ is $C^\infty$. Because the support of $\varphi$ is compact the distance from that support to the boundary of $\Omega$ is strictly positive. The support of $\varphi_m$ is contained in the set of all points of distance at most $\frac{1}{m}$ from the support of $\varphi$. So $\varphi_m \in C_0^\infty(\Omega)$ for all sufficiently large $m > 0$. It remains only to show that $\varphi_m$ converges in $H^\ell(\Omega)$ to $\varphi$. Let $|\alpha| \le \ell$. Then

$$\begin{aligned}\partial^\alpha \varphi_m(x) &= \int_{\mathbb{R}^n} \partial_x^\alpha \delta_m(x-y)\varphi(y)\ d^n y = \int_{\mathbb{R}^n} (-1)^{|\alpha|}[\partial_y^\alpha \delta_m(x-y)]\varphi(y)\ d^n y \\ &= \int_{\mathbb{R}^n} \delta_m(x-y)\partial_y^\alpha \varphi(y)\ d^n y\end{aligned}$$

by integration by parts. Since $\int_{\mathbb{R}^n} \delta_m(x-y)\ d^n y = 1$,

$$\begin{aligned}\partial^\alpha\varphi(x) - \partial^\alpha\varphi_m(x) &= \int_{\mathbb{R}^n} \delta_m(x-y)\big[\partial^\alpha\varphi(x) - \partial^\alpha\varphi(y)\big]\ d^n y \\ &= \int_{\mathbb{R}^n} \delta(z)\big[\partial^\alpha\varphi(x) - \partial^\alpha\varphi\big(x - \tfrac{z}{m}\big)\big]\ d^n z \text{ where } z = m(x-y)\end{aligned}$$

Since $\partial^\alpha\varphi$ is continuous and compactly supported, it is uniformly continuous. Hence, given any $\varepsilon > 0$, there is an $M$ such that $\big|\partial^\alpha\varphi(x) - \partial^\alpha\varphi\big(x - \frac{z}{m}\big)\big| < \varepsilon$ for all $x$, all $|z| \le 1$ and all $m > M$. Thus

$$\big|\partial^\alpha\varphi(x) - \partial^\alpha\varphi_m(x)\big| \le \int_{\mathbb{R}^n} \delta(z)\big|\partial^\alpha\varphi(x) - \partial^\alpha\varphi\big(x - \tfrac{z}{m}\big)\big|\ d^n z \le \varepsilon \int_{\mathbb{R}^n} \delta(z)\ d^n z = \varepsilon$$

for all $m > M$. As the set of points within distance one of the support of $\varphi$ has finite volume, $\varphi_m$ converges to $\varphi$ in $H^\ell(\Omega)$. ■

**Problem 2.1.3** Let $\ell \in \mathbb{N}_0$.

(a) Let $\varphi : \Omega \to \mathbb{C}$ and all of its derivatives of order at most $\ell$ be bounded and continuous. Prove that the map $u \in \{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \} \mapsto \varphi u$, where, as you would expect $(\varphi u)(x) = \varphi(x)u(x)$, has a unique extension to a bounded, linear map on $H^\ell(\Omega)$. Prove, in particular, that there is a constant $C_{\ell,n}$, depending only on $\ell$ and $n$, such that

$$\|\varphi u\|_{\ell,\Omega} \le C_{\ell,n} \sum_{k=0}^{\ell} \|\varphi\|_{C^k(\Omega)} \|u\|_{\ell-k,\Omega}$$

for all $u \in H^\ell(\Omega)$. Here $\|\varphi\|_{C^k(\Omega)} = \sup_{\substack{x\in\Omega \\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\le k}} \big|\partial^\alpha\varphi(x)\big|$.

(b) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$. Prove that the map $u \in \big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\} \mapsto \partial^\alpha u$ has a unique extension to a bounded, linear map from $H^\ell(\Omega)$ to $H^{\ell-|\alpha|}(\Omega)$.

**Solution.** (a) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and $u \in \big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\}$. By the product rule

$$\partial^\alpha(\varphi u) = \sum_{\substack{\beta\in\mathbb{N}_0^n \\ \beta\le\alpha}} \tbinom{\alpha}{\beta}(\partial^\beta\varphi)(\partial^{\alpha-\beta}u)$$

where "$\beta \le \alpha$" signifies that $\beta_i \le \alpha_i$ for all $1 \le i \le n$. Hence

$$\begin{aligned}
\|\partial^\alpha(\varphi u)\|_{L^2(\Omega)} &\le \sum_\beta \tbinom{\alpha}{\beta}\|(\partial^\beta\varphi)(\partial^{\alpha-\beta}u)\|_{L^2(\Omega)} \\
&\le \sum_\beta \tbinom{\alpha}{\beta} \sup_{x\in\Omega}\big|\partial^\beta\varphi(x)\big|\ \|\partial^{\alpha-\beta}u\|_{L^2(\Omega)} \\
&\le \sum_{\substack{\beta\in\mathbb{N}_0^n \\ \beta\le\alpha}} \tbinom{\alpha}{\beta}\ \|\varphi\|_{C^{|\beta|}(\Omega)}\ \|u\|_{|\alpha-\beta|,\Omega} \\
&\le C_\alpha \sum_{k=0}^{\ell} \|\varphi\|_{C^k(\Omega)}\ \|u\|_{\ell-k,\Omega} \qquad \text{where } C_\alpha = \max_{0\le k\le|\alpha|} \sum_{\substack{\beta\in\mathbb{N}_0^n \\ \beta\le\alpha\ \ |\beta|=k}} \tbinom{\alpha}{\beta}
\end{aligned}$$

The existence of a unique bounded extension now follows by the BLT theorem and the specified bound extends to all $u \in H^\ell(\Omega)$ by continuity.

(b) If $u \in C^\ell(\Omega)$ with $\|u\|_{\ell,\Omega} < \infty$, then

$$\begin{aligned}
\|\partial^\alpha u\|^2_{\ell-|\alpha|,\Omega} &= \sum_{\substack{\beta\in\mathbb{N}_0^{\ell-|\alpha|} \\ |\beta|\le\ell-|\alpha|}} \|\partial^\beta(\partial^\alpha)u\|^2_{L^2(\Omega)} \\
&= \sum_{\substack{\gamma\in\mathbb{N}_0^{\ell} \\ |\gamma|\le\ell,\gamma\ge\alpha}} \|\partial^\gamma u\|^2_{L^2(\Omega)} \text{ where } \gamma = \alpha+\beta \\
&\le \|u\|^2_{\ell,\Omega}
\end{aligned}$$

and the B.L.T. theorem generates the desired extension.

■

**Problem 2.1.4** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function with bounded derivative. Let $H^1_{\mathbb{R}}(\Omega)$ denote the set of real valued functions in $H^1(\Omega)$.

(a) Prove that if $u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, then $F \circ u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$.

(b) Prove that the map $u \mapsto F \circ u$, with domain $C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, is continuous with respect to the topology of $H^1_{\mathbb{R}}(\Omega)$. That is, prove that if $\varepsilon > 0$ and $v \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, then there is a $\delta > 0$ such that $\|F \circ u - F \circ v\|_{1,\Omega} < \varepsilon$ for all $u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$ obeying $\|u - v\|_{1,\Omega} < \delta$.

(c) Prove that the map $u \mapsto F \circ u$ has a continuous extension to $H^1_{\mathbb{R}}(\Omega)$.

**Solution.** (a) We check that if $u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, then $F \circ u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, so that the (nonlinear) map

$$\begin{aligned} \Phi : C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega) &\mapsto C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega) \\ u &\mapsto F \circ u \end{aligned}$$

is well–defined. That $F \circ u \in C^1(\Omega)$ is obvious.Let $M$ be the supremum of $|F'|$. Since

$$\begin{aligned} F \circ u(x) &= F(0) + \int_0^1 \tfrac{d}{dt} F\big(tu(x)\big)\ dt \\ &= F(0) + \int_0^1 F'\big(tu(x)\big)u(x)\ dt \\ \tfrac{\partial}{\partial x_i} F \circ u(x) &= F'\big(u(x)\big)\tfrac{\partial u}{\partial x_i}(x) \end{aligned}$$

we have

$$\begin{aligned} \big|F \circ u(x)\big| \le |F(0)| + M|u(x)| \quad &\Longrightarrow \quad \big\|F \circ u\big\|_{L^2(\Omega)} \le |F(0)|\sqrt{|\Omega|} + M\|u\|_{L^2(\Omega)} \\ \big|\tfrac{\partial}{\partial x_i} F \circ u(x)\big| \le M\big|\tfrac{\partial u}{\partial x_i}(x)\big| \quad &\Longrightarrow \quad \big\|\tfrac{\partial}{\partial x_i} F \circ u\big\|_{L^2(\Omega)} \le M\big\|\tfrac{\partial u}{\partial x_i}\big\|_{L^2(\Omega)} \end{aligned}$$

where $|\Omega|$ is the volume of $\Omega$. Thus $F \circ u \in H^1_{\mathbb{R}}(\Omega)$.

(b) Observe that

$$\begin{aligned} F \circ u(x) - F \circ v(x) &= \int_0^1 \tfrac{d}{dt} F\big(tu(x) + (1-t)v(x)\big)\ dt \\ &= \int_0^1 F'\big(tu(x) + (1-t)v(x)\big)\,[u(x) - v(x)]\ dt \\ \tfrac{\partial}{\partial x_\ell}\big[F \circ u(x) - F \circ v(x)\big] &= F'\big(u(x)\big)\big[\tfrac{\partial u}{\partial x_\ell}(x) - \tfrac{\partial v}{\partial x_\ell}(x)\big] \\ &\qquad + \big[F'\big(u(x)\big) - F'\big(v(x)\big)\big]\tfrac{\partial v}{\partial x_\ell}(x) \end{aligned}$$

If $\|u - v\|_{1,\Omega} < \delta$, then

$$\begin{aligned} \big\|F \circ u - F \circ v\big\|_{L^2(\Omega)} &\le M\|u - v\|_{L^2(\Omega)} \le M\delta \\ \big\|F'\big(u(x)\big)\big[\tfrac{\partial u}{\partial x_\ell}(x) - \tfrac{\partial v}{\partial x_\ell}(x)\big]\big\|_{L^2(\Omega)} &\le M\big\|\tfrac{\partial u}{\partial x_\ell} - \tfrac{\partial v}{\partial x_\ell}\big\|_{L^2(\Omega)} \le M\delta \end{aligned}$$

So our main problem is to bound the $L^2(\Omega)$–norm of $\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big]\frac{\partial v}{\partial x_\ell}(x)$. Since $\frac{\partial v}{\partial x_\ell}(x) \in L^2(\Omega)$, the Lebesgue dominated convergence theorem implies that

$$\lim_{P\to\infty} \int_{\left|\frac{\partial v}{\partial x_\ell}\right|>P} \left|\tfrac{\partial v}{\partial x_\ell}(x)\right|^2\ d^n x = 0$$

Pick $P$ such that

$$\int_{\left|\frac{\partial v}{\partial x_\ell}\right|>P} \left|\tfrac{\partial v}{\partial x_\ell}(x)\right|^2\ d^n x \le \tfrac{\varepsilon^2}{48M^2 n}$$

and define

$$B_1 = \left\{\ x \in \Omega\ \middle|\ \left|\tfrac{\partial v}{\partial x_\ell}\right| > P\ \right\}$$

Since $|F'|$ is bounded by $M$

$$\text{(S2.1a)}\qquad \int_{B_1} \left|\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big]\tfrac{\partial v}{\partial x_\ell}(x)\right|^2\ d^n x \le 4M^2 \int_{B_1} \left|\tfrac{\partial v}{\partial x_\ell}(x)\right|^2\ d^n x \le \tfrac{\varepsilon^2}{12n}$$

We will eventually pick $\delta < 1$. Hence $\|u\|_{1,\Omega} \le \|v\|_{1,\Omega} + 1 \overset{\text{def}}{=} r$. As $F'$ is continuous on the compact interval $|t| \le \frac{7rMP\sqrt{n}}{\varepsilon}$, there is an $\eta > 0$ such that $\big|F'(t') - F'(t)\big| \le \frac{\varepsilon}{P\sqrt{12n\mu(\Omega)}}$ for all $t, t' \in [-\frac{7rMP\sqrt{n}}{\varepsilon}, \frac{7rMP\sqrt{n}}{\varepsilon}]$ obeying $|t - t'| \le \eta$. Define

$$\begin{aligned}
B_2 &= \left\{\ x \in \Omega \setminus B_1\ \middle|\ |u(x)| \ge \tfrac{7rMP\sqrt{n}}{\varepsilon}\ \right\}\\
B_3 &= \left\{\ x \in \Omega \setminus B_1\ \middle|\ |v(x)| \ge \tfrac{7rMP\sqrt{n}}{\varepsilon}\ \right\}\\
B_4 &= \left\{\ x \in \Omega \setminus B_1\ \middle|\ |u(x) - v(x)| \ge \eta\ \right\}\\
B_5 &= \Omega \setminus (B_1 \cup B_2 \cup B_3 \cup B_4)
\end{aligned}$$

Then

$$r^2 \ge \|u\|_{1,\Omega}^2 \ge \int_{B_2} |u(x)|^2 d^n x \ge \tfrac{49r^2M^2P^2n}{\varepsilon^2}\mu(B_2) \implies \mu(B_2) \le \tfrac{\varepsilon^2}{49M^2P^2n}$$

Hence

$$\text{(S2.1b)}\qquad \int_{B_2} \left|\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big]\tfrac{\partial v}{\partial x_\ell}(x)\right|^2\ d^n x \le 4M^2P^2\mu(B_2) \le \tfrac{\varepsilon^2}{12n}$$

Similarly,

$$\text{(S2.1c)}\qquad \int_{B_3} \left|\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big]\tfrac{\partial v}{\partial x_\ell}(x)\right|^2\ d^n x \le 4M^2P^2\mu(B_3) \le \tfrac{\varepsilon^2}{12n}$$

As

$$\delta^2 \ge \|u - v\|_{1,\Omega}^2 \ge \int_{B_4} |u(x) - v(x)|^2 d^n x \ge \eta^2\mu(B_4) \implies \mu(B_4) \le \tfrac{\delta^2}{\eta^2}$$

we have

$$\text{(S2.1d)} \qquad \int_{B_4} \big|\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big] \tfrac{\partial v}{\partial x_\ell}(x)\big|^2 \ d^n x \le 4M^2P^2\mu(B_4) \le \tfrac{\varepsilon^2}{12n}$$

provided we choose $\delta^2 < \frac{\eta^2\varepsilon^2}{49M^2P^2n}$. Finally

$$\text{(S2.1e)} \qquad \int_{B_5} \big|\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big] \tfrac{\partial v}{\partial x_\ell}(x)\big|^2 \ d^n x \le \tfrac{\varepsilon^2}{12P^2\mu(\Omega)n} P^2\mu(\Omega) \le \tfrac{\varepsilon^2}{12n}$$

Adding (S2.1a–e) gives

$$\int_{\Omega} \big|\big[F'\big(u(x)\big) - F'\big(v(x)\big)\big] \tfrac{\partial v}{\partial x_\ell}(x)\big|^2 \ d^n x \le \tfrac{5\varepsilon^2}{12n}$$

Hence

$$\begin{aligned} \|u - v\|_{1,\Omega}^2 &= \big\|F \circ u - F \circ v\big\|_{L^2(\Omega)}^2 + \sum_{\ell=1}^n \big\|\tfrac{\partial}{\partial x_\ell}\big[F \circ u(x) - F \circ v(x)\big]\big\|_{L^2(\Omega)}^2 \\ &\le M^2\delta^2 + 2nM^2\delta^2 + 2n\tfrac{5\varepsilon^2}{12n} < \varepsilon^2 \end{aligned}$$

provided we choose $\delta$ smaller than $\min\big\{1, \frac{\eta\varepsilon}{7MP\sqrt{n}}, \frac{\varepsilon}{5M\sqrt{n}}\big\}$.

(c) We now verify that if $\{u_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, then the sequence $\{F \circ u_i\}_{i\in\mathbb{N}}$ is also Cauchy in $H^1_{\mathbb{R}}(\Omega)$. By the completeness of $H^1_{\mathbb{R}}(\Omega)$ there is are $v, v_1, \cdots, v_n \in H^1_{\mathbb{R}}(\Omega)$ such that $u_i$ converges to $v$ and $\frac{\partial u_i}{\partial x_\ell}(x)$ converges to $v_\ell(x)$ in $L^2(\Omega)$ for all $1 \le \ell \le n$. Observe that

$$\begin{aligned} F \circ u_i(x) - F \circ v(x) &= \int_0^1 \tfrac{d}{dt} F\big(tu_i(x) + (1-t)v(x)\big) \ dt \\ &= \int_0^1 F'\big(tu_i(x) + (1-t)v(x)\big) \, [u_i(x) - v(x)] \ dt \end{aligned}$$

$$\begin{aligned} \tfrac{\partial}{\partial x_\ell}\big[F \circ u_i(x)\big] - F'\big(v(x)\big)v_\ell(x) &= F'\big(u_i(x)\big)\big[\tfrac{\partial u_i}{\partial x_\ell}(x) - v_\ell(x)\big] \\ &\quad + \big[F'\big(u_i(x)\big) - F'\big(v(x)\big)\big]v_\ell(x) \end{aligned}$$

Clearly

$$\begin{aligned} \big\|F \circ u_i - F \circ v\big\|_{L^2(\Omega)} &\le M\|u_i - v\|_{L^2(\Omega)} \\ \big\|F'\big(u_i(x)\big)\big[\tfrac{\partial u_i}{\partial x_\ell}(x) - v_\ell(x)\big]\big\|_{L^2(\Omega)} &\le M\|\tfrac{\partial u_i}{\partial x_\ell} - v_\ell\|_{L^2(\Omega)} \end{aligned}$$

converge to zero as $i \to \infty$. The argument that $\big[F'\big(u_i(x)\big) - F'\big(v(x)\big)\big]v_\ell(x)$ also converges to zero in $L^2(\Omega)$ is much the same as the corresponding argument in part (b). Hence $F \circ u_i$ converges to $F \circ v$ in $H^1_{\mathbb{R}}(\Omega)$ and we may define $\Phi(v) = F \circ v$. The proof that $\Phi$ is continuous is similar to the proof of part (b) with $\frac{\partial v}{\partial x_\ell}$ replaced by $v_\ell$. ■

**Problem 2.1.5** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Prove that if $u \in H^1_{\mathbb{R}}(\Omega)$, then $|u|, \max\{0,u\} \in H^1_{\mathbb{R}}(\Omega)$.

**Solution.** Let $F_i : \mathbb{R} \to \mathbb{R}$ be a sequence of $C^1$ functions all of whose derivatives are bounded in magnitude by some fixed $M > 0$. Suppose that this sequence converges uniformly to a function $F$ and that the derivatives $F_i'(t)$ converge pointwise to a function $G(t)$. We prove that $F_i \circ u$ converges in $H^1_{\mathbb{R}}(\Omega)$. Since $F_i(u(x))$ converges uniformly in $x$ and $\Omega$ has finite measure, $F_i(u(x))$ also converges in $L^2(\Omega)$ to $F(u(x))$. Recall from Remark 2.1.2, that, for each $u \in H^1(\Omega)$, $\frac{\partial u}{\partial x_\ell}$ is defined and is an element of $L^2(\Omega)$. The integrand of

$$\big\| \big[F_i'\big(u(x)\big) - G\big(u(x)\big)\big] \tfrac{\partial u}{\partial x_\ell}(x) \big\|^2_{L^2(\Omega)} = \int_\Omega \Big| \big[F_i'\big(u(x)\big) - G\big(u(x)\big)\big] \tfrac{\partial u}{\partial x_\ell}(x) \Big|^2 \, d^n x$$

is bounded, uniformly in $i$, by the $L^2$ function $4M^2 \Big| \frac{\partial u}{\partial x_\ell}(x) \Big|^2$ and converges pointwise to zero as $i \to \infty$. Hence, by the Lebesgue dominated convergence theorem,

$$\lim_{i\to\infty} \big\| \big[F_i'\big(u(x)\big) - G\big(u(x)\big)\big] \tfrac{\partial u}{\partial x_\ell}(x) \big\|_{L^2(\Omega)} = 0$$

Hence $F_i(u(x))$ converges in $H^1(\Omega)$ to $F(u(x))$ and $F\big(u(x)\big) \in H^1(\Omega)$.

Applying this with $F_i(t) = \sqrt{t^2 + \frac{1}{2^i}}$ yields that $|u| \in H^1_{\mathbb{R}}(\Omega)$ when $u \in H^1_{\mathbb{R}}(\Omega)$. Since

$$\max\{0,u\} = \tfrac{1}{2}\big(u + |u|\big)$$

we have that $\max\{0,u\} \in H^1_{\mathbb{R}}(\Omega)$ too. ■

**Problem 2.1.6** Let $\Omega = (0, \infty)$.

(a) Prove that there is a constant $C$ such that if $u \in C^1(\Omega)$ and $\|u\|_{1,\Omega} < \infty$, then $\lim\limits_{x\to 0} u(x)$ exists and obeys

$$\big| \lim_{x\to 0} u(x) \big| \le C \|u\|_{1,\Omega}$$

(b) Prove that there is a unique bounded linear map $B : H^1(\Omega) \to \mathbb{C}$ such that $Bu = \lim\limits_{x\to 0} u(x)$ for all $u \in C^1(\Omega)$ with $\|u\|_{1,\Omega} < \infty$.

(c) Prove that $H^1_0(\Omega)$ is of codimension one in $H^1(\Omega)$. This means that there exists a $u_0 \in H^1(\Omega)$ such that each $u \in H^1(\Omega)$ has a unique representation of the form $u = \alpha u_0 + u_1$ with $\alpha \in \mathbb{C}$ and $u_1 \in H^1_0(\Omega)$.

(d) Let $I = (a,b)$ be a finite open interval in $\mathbb{R}$. What is the codimension of $H_0^1(I)$ in $H^1(I)$?

**Solution.** (a) By Cauchy–Schwarz, if $0 < x < y$

$$\text{(S2.2)} \quad \big|u(y) - u(x)\big| = \left|\int_x^y 1 \times u'(t)\ dt\right| \le \sqrt{\int_x^y 1\ dt}\sqrt{\int_x^y |u'(t)|^2\ dt} \le \sqrt{y-x}\|u\|_{1,\Omega}$$

So $\{u(1/n)\}$ is a Cauchy sequence and converges to some number $A$ as $n \to \infty$. Taking the limit as $x \to 0$ in (S2.2) gives that $\big|u(y) - A\big| \le \sqrt{y}\|u\|_{1,\Omega}$. Hence $\lim_{x\to 0} u(x)$ exists and also equals $A$. Furthermore

$$\left|A - \int_0^1 u(y)\ dy\right| = \left|\int_0^1 [u(y) - A]\ dy\right| \le \|u\|_{1,\Omega}\int_0^1 \sqrt{y}\ dy = \tfrac{2}{3}\|u\|_{1,\Omega}$$

Since, by Cauchy–Schwarz again, $\left|\int_0^1 u(y)\ dy\right| \le \|u\|_{L^2} \le \|u\|_{1,\Omega}$, we have

$$|\lim_{x\to 0} u(x)| \le \tfrac{5}{3}\|u\|_{1,\Omega}$$

■

(b) By part (a), the map $u \mapsto \lim\limits_{x\to 0} u(x)$ is a well–defined linear transformation from the dense subset $\{\ u \in C^1(\Omega)\ \big|\ \|u\|_{1,\Omega} < \infty\ \}$ of $H^1(\Omega)$ to $\mathbb{C}$ that obeys $\big|\lim_{x\to 0} u(x)\big| \le C\|u\|_{1,\Omega}$. It now suffices to apply the B.L.T. theorem. ■

(c) Let $B$ be the linear functional of part (b). We claim that

$$H_0^1(\Omega) = \{\ u \in H^1(\Omega)\ \big|\ Bu = 0\ \}$$

Once this claim is proven, we are done because, if $u \in H^1(\Omega)$ then $\alpha = Bu$, $u_0(x) = e^{-x}$, $u_1(x) = u(x) - (Bu)e^{-x}$ provides the desired decomposition. The uniqueness of the decomposition is consequence of the observation that $\alpha e^{-x} + u_1(x) = \tilde{\alpha}e^{-x} + \tilde{u}_1(x)$ implies $(\alpha - \tilde{\alpha})e^{-x} = \tilde{u}_1(x) - u_1(x)$. Applying $B$ to both sides forces $\alpha = \tilde{\alpha}$ which in turn forces $\tilde{u}_1(x) = u_1(x)$.

**Proof of claim**. Applying $B$ to any element of $C_0^\infty(\Omega)$ gives zero. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and $B$ is continuous, applying $B$ to any element of $H_0^1(\Omega)$ gives zero.

Now for the converse. Suppose $u \in H^1(\Omega)$ and $Bu = 0$. We must prove $u \in H_0^1(\Omega)$. By definition, there is a sequence of functions $u_i \in \{\ u \in C^1(\Omega)\ \big|\ \|u\|_{1,\Omega} < \infty\ \}$ that converge in $H^1(\Omega)$ to $u$. Since $B$ is continuous, $\lim_{i\to\infty} Bu_i = Bu = 0$. Replacing $u_i(x)$ by $u_i(x) - (Bu_i)e^{-x}$, we may assume without loss of generality that $\lim_{x\to 0} u_i(x) = 0$ for all i. Since $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, it suffices to prove that if $v \in C^1(\Omega)$, $\|v\|_{1,\Omega} < \infty$ and $\lim_{x\to 0} v(x) = 0$, then $v \in H_0^1(\Omega)$.

Let $\chi(x)$ be a $C^\infty$ function that takes values in $[0,1]$, is zero for $x < \frac{1}{2}$ and one for $x > 2$ and let $\tilde\chi = 1 - \chi$. Then $v_m(x) = \chi(mx)v(x)\tilde\chi(x/m) \in C_0^1(\Omega) \subset H_0^1(\Omega)$. For $m \geq 4$,

$$\begin{aligned} v(x) - v_m(x) &= \big(1 - \chi(mx)\big)v(x) + \chi(mx)v(x)\big(1 - \tilde\chi(x/m)\big) \\ &= \big(1 - \chi(mx)\big)v(x) + v(x)\big(1 - \tilde\chi(x/m)\big) \end{aligned}$$

So, to show that $v_m$ converges to $v$ in $\in H_0^1(\Omega)$, it suffices to prove that $\big(1-\chi(mx)\big)v(x) = \tilde\chi(mx)v(x)$ and $v(x)\big(1 - \tilde\chi(x/m)\big) = v(x)\chi(x/m)$ converge to zero in $H_0^1(\Omega)$. Let $V_m = \sqrt{\int_0^{2/m} |v'(x)|^2\ dx}$. As in (S2.2), $|v(x)| \leq \sqrt{x}\, V_m$ for all $0 < x < \frac{2}{m}$. Since $\|v\|_{1,\Omega}^2$ is finite, the Lebesgue dominated convergence theorem yields $\lim_{m\to\infty} V_m = 0$. That $\tilde\chi(mx)v(x)$ converges to zero follows from

$$\int |\tilde\chi(mx)v(x)|^2\ dx \leq \int_0^{2/m} |v(x)|^2\ dx \leq \int_0^{2/m} xV_m^2\ dx = \tfrac{2}{m^2}V_m^2$$

and

$$\begin{aligned} \int |\tfrac{d}{dx}[\tilde\chi(mx)v(x)]|^2\ dx &= \int |m\tilde\chi'(mx)v(x) + \tilde\chi(mx)v'(x)|^2\ dx \\ &\leq \text{const}\, m^2 \int_0^{2/m} |v(x)|^2\ dx + 2\int_0^{2/m} |v'(x)|^2\ dx \\ &\leq \text{const}\, m^2 \int_0^{2/m} xV_m^2\ dx + 2V_m^2 \leq \text{const}\, V_m^2 \end{aligned}$$

That $v(x)\chi(x/m)$ converges to zero follows from the Lebesgue dominated convergence theorem,

$$\int |v(x)\chi(x/m)|^2\ dx \leq \int_{m/2}^\infty |v(x)|^2\ dx$$

and

$$\begin{aligned} \int |\tfrac{d}{dx}[v(x)\chi(x/m)]|^2\ dx &= \int |v'(x)\chi(x/m) + \tfrac{1}{m}v(x)\chi'(x/m)|^2\ dx \\ &\leq 2\int_{m/2}^\infty |v'(x)|^2\ dx + \text{const}\, \tfrac{1}{m^2}\int_{m/2}^\infty |v(x)|^2\ dx \end{aligned}$$

(d) Two. Define $B_1 u = \lim_{x\to a+} u(x)$, $B_2 u = \lim_{x\to b-} u(x)$, $u_0(x) = \frac{b-x}{b-a}$, $u_1(x) = \frac{x-a}{b-a}$. Then $u = \alpha u_0 + \beta u_1 + u_2$ with $\alpha = B_1 u$, $\beta = B_2 u$ and $u_2 = u - \alpha u_0 - \beta u_1 \in H_0^1(I)$.

■

**Problem 2.1.7** The goal of this problem is to prove the Paley-Wiener theorem, which says that a function $f$ is $C^\infty$ and supported in the closed ball $\bar{B}_R = \{\, x \in \mathbb{R}^n \bigm| |x| \le R \,\}$ if and only if $\hat{f}(k)$ extends to a holomorphic function on $\mathbb{C}^n$ which obeys

$$\big|\hat{f}(k)\big| \le \tfrac{C_N}{1+|k|^{2N}} e^{R|\operatorname{Im} k|} \tag{S2.3}$$

for all $N \in \mathbb{N}$.

(a) Let $f \in C_0^\infty(\mathbb{R}^n)$ be supported in $\bar{B}_R$. Prove that $\hat{f}(k)$ extends to a holomorphic function on $\mathbb{C}^n$ and that, for each $N \in \mathbb{N}$, there is a constant $C_N$ such that (S2.3) holds.

(b) Assume that the Fourier transform $\hat{f}(k)$ of a function $f(x)$ extends to a holomorphic function on $\mathbb{C}^n$ and that, for each $N \in \mathbb{N}$, there is a constant $C_N$ such that (S2.3) holds. Let $p \in \mathbb{R}^n$. Prove that

$$f(x) = e^{-p\cdot x} \int e^{ik\cdot x} \hat{f}(k+ip)\ \tfrac{d^nk}{(2\pi)^n}$$

(c) Prove that, under the hypotheses of part (b), $f(x)$ is supported in $\bar{B}_R$.

**Solution.** (a) For any $k \in \mathbb{C}^n$ and any multiindex $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned}\big|k^\alpha \hat{f}(k)\big| &= \left| \int_{\bar{B}_R} k^\alpha e^{-ik\cdot x} f(x)\ d^nx \right| = \left| \int_{\bar{B}_R} e^{-ik\cdot x} \tfrac{\partial^\alpha f}{\partial x^\alpha}(x)\ d^nx \right| \\ &\le e^{R|\operatorname{Im} k|} \int_{\bar{B}_R} \big|\tfrac{\partial^\alpha f}{\partial x^\alpha}(x)\big|\ d^nx\end{aligned}$$

so that $\hat{f}(k)$ is defined for all $k \in \mathbb{C}^n$ and obeys (S2.3). For each $1 \le j \le n$, $x_j f(x)$ is absolutely integrable. Hence $\hat{f}(k)$ is (complex) differentiable with respect to $k_j$ and obeys

$$\tfrac{\partial}{\partial k_j} \hat{f}(k) = -i \int_{\bar{B}_R} e^{-ik\cdot x} x_j f(x)\ d^nx$$

Thus $\hat{f}(k)$ is entire.

(b) By the Cauchy integral formula, all of the derivatives of $\hat{f}(k)$ also obey bounds of the form (S2.3). In particular, the restriction of $\hat{f}(k)$ to $k \in \mathbb{R}^n$ is in $\mathcal{S}(\mathbb{R}^n)$ and

$$f(x) = \int_{\mathbb{R}^n} e^{ik\cdot x} \hat{f}(k)\ \tfrac{d^nk}{(2\pi)^n}$$

Make the change of variables $k \to k + ip$:

$$f(x) = \int_{\mathbb{R}^n - ip} e^{i(k+ip)\cdot x} \hat{f}(k+ip)\ \tfrac{d^nk}{(2\pi)^n}$$

Since $f$ is entire and, for each fixed $p \in \mathbb{R}^n$

$$\left|\hat{f}(k+ip)\right| \le \text{const}\, e^{R|p|} \prod_{j=1}^{n} \tfrac{1}{1+|k_j|^2}$$

we may move the domain of integration back to $\mathbb{R}^n$, as desired.

(c) From part (b)

$$\begin{aligned} |f(x)| &= e^{-p\cdot x}\left|\int e^{ik\cdot x}\hat{f}(k+ip)\ \tfrac{d^nk}{(2\pi)^n}\right| \\ &\le e^{-p\cdot x}\int \tfrac{C_N}{1+|k+ip|^{2N}} e^{R|p|}\ \tfrac{d^nk}{(2\pi)^n} \\ &\le e^{-p\cdot x}e^{R|p|}\int \tfrac{C_N}{1+|k|^{2N}}\ \tfrac{d^nk}{(2\pi)^n} \\ &\le \text{const}_n\ e^{-p\cdot x}e^{R|p|} \end{aligned}$$

where we have chosen, for example, $N = n$. Note that $\text{const}_n$ is independent of $p$. Choosing $p = tx$ with $t > 0$ we have that

$$|f(x)| \le \text{const}_n\, e^{-t|x|^2}e^{tR|x|} = \text{const}_n\, e^{-t|x|(|x|-R)}$$

for all $t > 0$. If $|x| > R$, this is possible only when $f(x) = 0$. ■

**Problem 2.1.8** Let $s \in \mathbb{R}$. Prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

**Solution.** We first prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Let $\varphi(k) \in C_0^\infty(\mathbb{R}^n)$ take values in $[0,1]$ and be identically one for $|k| \le 1$. Let $u \in L^2(\mathbb{R}^n)$ and set $v_j(k) = \varphi(k/j)\hat{u}(k)$ and $u_j = \check{v}_j$. By Cauchy–Schwarz,

$$\int \left|v_j(k)\right|\, d^nk = \int \left|\varphi(k/j)\right|\,\left|\hat{u}(k)\right|\, d^nk \le \left[\int \left|\varphi(k/j)\right|^2\, d^nk\right]^{1/2} (2\pi)^{n/2}\|u\|_{L^2(\mathbb{R}^n)} < \infty$$

Hence $v_j(k)$ is both $L^1$ and of compact support, so that $u_j \in C^\infty(\mathbb{R}^n)$. Since $|v_j(k)| \le |\hat{u}(k)|$, we also have that $u_j \in L^2(\mathbb{R}^n)$. By the Lebesgue dominated convergence theorem

$$\begin{aligned} \lim_{j\to\infty}\|u-u_j\|^2_{L^2(\mathbb{R}^n)} &= \lim_{j\to\infty}\int \left|1-\varphi(k/j)\right|^2\left|\hat{u}(k)\right|^2\ \tfrac{d^nk}{(2\pi)^n} \\ &\le \limsup_{j\to\infty}\int_{|k|>j}\left|\hat{u}(k)\right|^2\ \tfrac{d^nk}{(2\pi)^n} \\ &= 0 \end{aligned}$$

This proves that $C^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Let $u\in C^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then $u_j(x)=\varphi(x/j)u(x)\in C_0^\infty(\mathbb{R}^n)$ converges in $L^2(\mathbb{R}^n)$ to $u$ since, again by the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{j\to\infty}\|u-u_j\|^2_{L^2(\mathbb{R}^n)} &= \lim_{j\to\infty}\int \big|1-\varphi(x/j)\big|^2\big|u(x)\big|^2\ d^nx\\ &\le \limsup_{j\to\infty}\int_{|x|>j}\big|u(x)\big|^2\ d^nx\\ &=0\end{aligned}$$

Next, we prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. Let $u\in H^s(\mathbb{R}^n)$. Choose a sequence of functions $f_j(k)\in C_0^\infty(\mathbb{R}^n)$ that converge in $L^2(\mathbb{R}^n)$ to $(1+|k^2|)^{s/2}\hat u(k)$. Let $u_j(x)$ be the inverse Fourier transform of $(1+|k^2|)^{-s/2}f_j(k)$. Since $(1+|k^2|)^{-s/2}f_j(k)\in C_0^\infty(\mathbb{R}^n)\subset\mathcal{S}(\mathbb{R}^n)$, $u_j(x)\in\mathcal{S}(\mathbb{R}^n)$. Since

$$\|u-u_j\|_s^2=\int(1+|k|^2)^s\big|\hat u(k)-(1+|k^2|)^{-s/2}f_j(k)\big|^2\ \tfrac{d^nk}{(2\pi)^n}=\int\big|(1+|k^2|)^{s/2}\hat u(k)-f_j(k)\big|^2\ \tfrac{d^nk}{(2\pi)^n}$$

$u_j$ converges to $u$ in $H^s(\mathbb{R}^n)$.

It remains only to prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$. Let $u\in\mathcal{S}(\mathbb{R}^n)$. Pick any natural number $\ell$ with $2\ell\ge s$. Set $u_j(x)=\varphi(x/j)u(x)\in C_0^\infty(\mathbb{R}^n)$. Then

$$\|u-u_j\|_s\le\|(1-\Delta)^\ell(u-u_j)\|_{L^2(\mathbb{R}^n)}$$

All derivatives of $u$ are in $L^2$ and all derivatives of $\varphi(x/j)$ are bounded, uniformly in $j$. Furthermore all nontrivial derivatives of $\varphi(x/j)$ are supported in $|x|>j$. So, the right hand side converges to zero as $j\to\infty$, by the Lebesgue dominated convergence theorem, yet again. ■

**Problem 2.1.9** Let $r<s<t$. Prove that, for each $\varepsilon>0$, there is a $C>0$, depending only on $r,s,t$ and $\varepsilon$ such that $\|f\|_s\le\varepsilon\|f\|_t+C\|f\|_r$ for all $f\in H^t(\mathbb{R}^n)$.

**Solution.** We first find a $C$ such that

$$\big(1+|k|^2\big)^s\le\varepsilon^2\big(1+|k|^2\big)^t+C^2\big(1+|k|^2\big)^r$$

for all $k\in\mathbb{R}^n$, or equivalently, setting $x=\big(1+|k|^2\big)^{t-r}$ and $\alpha=\frac{s-r}{t-r}$, such that

$$\varepsilon^2x^{1-\alpha}+C^2\tfrac{1}{x^\alpha}\ge1$$

for all $x \geq 1$. Observe that

$$\tfrac{d}{dx}\big(\varepsilon^2 x^{1-\alpha} + C^2 \tfrac{1}{x^\alpha}\big) = \tfrac{1}{x^\alpha}\Big[\varepsilon^2(1-\alpha) - C^2\alpha\tfrac{1}{x}\Big]$$

The minimum value of $\varepsilon^2 x^{1-\alpha} + C^2\frac{1}{x^\alpha}$ either occurs when $x = 1$, in which case it suffices to choose $C = 1$, or occurs when

$$\varepsilon^2(1-\alpha) - C^2\alpha\tfrac{1}{x} = 0 \iff x = \tfrac{C^2}{\varepsilon^2}\tfrac{\alpha}{1-\alpha}$$

In the latter case, the minimum value is

$$\begin{aligned}
\varepsilon^2\big(\tfrac{C^2}{\varepsilon^2}\tfrac{\alpha}{1-\alpha}\big)^{1-\alpha} + C^2\big(\tfrac{C^2}{\varepsilon^2}\tfrac{\alpha}{1-\alpha}\big)^{-\alpha} &= \varepsilon^{2\alpha}C^{2(1-\alpha)}\big(\tfrac{\alpha}{1-\alpha}\big)^{-\alpha}\Big[\tfrac{\alpha}{1-\alpha}+1\Big] \\
&= \varepsilon^{2\alpha}C^{2(1-\alpha)}\big(\tfrac{\alpha}{1-\alpha}\big)^{-\alpha}\tfrac{1}{1-\alpha} \\
&= \varepsilon^{2\alpha}C^{2(1-\alpha)}\tfrac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}
\end{aligned}$$

So an acceptable choice of $C$ is

$$C = \max\left\{1, \Big[\tfrac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\varepsilon^{2\alpha}}\Big]^{1/[2(1-\alpha)]}\right\}$$

With this choice of $C$

$$|f|_s^2 \leq \varepsilon^2|f|_t^2 + C|f|_r^2 \leq \big(\varepsilon|f|_t + C|f|_r\big)^2$$

■

**Problem 2.1.10** Let $s \in \mathbb{R}$.

(a) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. By Lemma 2.3.5, the map $u \in \mathcal{S}(\mathbb{R}^n) \mapsto fu \in \mathcal{S}(\mathbb{R}^n)$ has a unique extension to a bounded, linear map on $H^s(\mathbb{R}^n)$. Prove that

$$\partial^\alpha(fu) = \sum_{\substack{\beta\in\mathbb{N}_0^n \\ \beta\leq\alpha}} \tbinom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta}u)$$

for all $u \in H^s(\mathbb{R}^n)$. Here $\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for all $1 \leq i \leq n$, $(\alpha-\beta)_i = \alpha_i - \beta_i$ for all $1 \leq i \leq n$ and $\binom{\alpha}{\beta} = \prod_{i=1}^n \frac{\alpha_i!}{\beta_i!(\alpha_i-\beta_i)!}$.

(b) Let $u, v \in H^s(\mathbb{R}^n)$ and let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Make up a definition for "$u = v$ on $\mathcal{O}$".

(c) Prove that differentiation is local in the sense that if $u, v \in H^s(\mathbb{R}^n)$ and $\mathcal{O}$ is any open subset of $\mathbb{R}^n$ with $u = v$ on $\mathcal{O}$, then $\partial^\alpha u = \partial^\alpha v$ on $\mathcal{O}$ for all $\alpha \in \mathbb{N}_0^n$.

**Solution.** (a) By the product rule, $\partial^\alpha(fu) = \sum_{\substack{\beta\in\mathbb{N}_0^n\\ \beta\le\alpha}} \binom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta}u)$ for all $u, f \in \mathcal{S}(\mathbb{R}^n)$. By Lemma 2.3.5 and part (b) of Lemma 2.1.5, the maps

$$u \in H^s(\mathbb{R}^n) \mapsto fu \in H^s(\mathbb{R}^n) \mapsto \partial^\alpha(fu) \in H^{s-|\alpha|}(\mathbb{R}^n)$$

and

$$\begin{aligned} u \in H^s(\mathbb{R}^n) &\mapsto \partial^{\alpha-\beta}u \in H^{s-|\alpha-\beta|}(\mathbb{R}^n) \mapsto \partial^{\alpha-\beta}u \in H^{s-|\alpha|}(\mathbb{R}^n)\\ &\mapsto (\partial^\beta f)(\partial^{\alpha-\beta}u) \in H^{s-|\alpha|}(\mathbb{R}^n) \end{aligned}$$

are bounded maps from $H^s(\mathbb{R}^n)$ to $H^{s-|\alpha|}(\mathbb{R}^n)$. By Problem 2.1.8, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. So the product rule extends by continuity to all $u \in H^s(\mathbb{R}^n)$.

(b) We define $u$ to be equal to $v$ on $\mathcal{O}$ if and only if $fu = fv$, as elements of $H^s(\mathbb{R}^n)$, for all of $f \in C_0^\infty(\mathcal{O})$.

(c) The proof is by induction on $|\alpha|$. It is trivially true for $|\alpha| = 0$. So suppose that $|\alpha| > 0$ and $\partial^\gamma u = \partial^\gamma v$ on $\mathcal{O}$ for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| < |\alpha|$. If $f \in C_0^\infty(\mathcal{O})$, then $(\partial^\beta f)(\partial^\gamma u) = (\partial^\beta f)(\partial^\gamma v)$ for all $\beta \in \mathbb{N}_0^n$ and all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| < |\alpha|$ so that

$$\begin{aligned} f\partial^\alpha u &= \partial^\alpha(fu) - \sum_{\substack{\beta\in\mathbb{N}_0^n\\ 0\ne\beta\le\alpha}} \binom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta}u)\\ &= \partial^\alpha(fv) - \sum_{\substack{\beta\in\mathbb{N}_0^n\\ 0\ne\beta\le\alpha}} \binom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta}v)\\ &= f\partial^\alpha v \end{aligned}$$

■

**Problem 2.1.11** Let $\alpha \in \mathbb{N}_0^n$ and $s > |\alpha| + \frac{n}{2}$. Prove that if $u \in H^s(\mathbb{R}^n)$, then $\partial^\alpha u$ is continuous and there is a constant $C$, depending only on $s$, $n$ and $|\alpha|$, such that

$$\sup_{x\in\mathbb{R}^n} \big|\partial^\alpha u(x)\big| \le C\|u\|_s$$

**Solution.** The Fourier transform of $\partial^\alpha u$ is $i^{|\alpha|}k^\alpha \hat{u}(k)$. We first prove that this Fourier transform is $L^1$. By Cauchy–Schwarz

$$\begin{aligned} \int \big|k^\alpha \hat{u}(k)\big|\, d^nk &= \int \big(1+|k|^2\big)^{-s/2}\big|k^\alpha\big|\, \big(1+|k|^2\big)^{s/2}\big|\hat{u}(k)\big|\, d^nk\\ &\le \Big[\int \big(1+|k|^2\big)^{-s}\big|k^{2\alpha}\big|\, d^nk\Big]^{1/2} (2\pi)^{n/2}\|u\|_s \end{aligned}$$

This is finite whenever the integral converges, which is the case when $-2s + 2|\alpha| < -n$. The integrand of

$$\partial^\alpha u(x) = \int e^{ik\cdot x}\ i^{|\alpha|} k^\alpha \hat{u}(k) \tfrac{d^n k}{(2\pi)^n}$$

is uniformly bounded by the $L^1$ function $\frac{1}{(2\pi)^n}\big|k^\alpha \hat{u}(k)\big|$ and is continuous in $x$. So the Lebesgue dominated convergence theorem implies that $\partial^\alpha u$ is continuous and obeys the prescribed bound with

$$C = \left[ \int \left(1 + |k|^2\right)^{-s} \big|k^{2\alpha}\big|\ \tfrac{d^n k}{(2\pi)^n} \right]^{1/2}$$

■

**Problem 2.1.12** Let $s$ and $s'$ be real numbers and $0 \le \mu \le 1$. Prove that

$$|\!|u|\!|_{\mu s + (1-\mu)s'} \le |\!|u|\!|_s^\mu\ |\!|u|\!|_{s'}^{1-\mu}$$

for all $u \in H^{\max\{s,s'\}}(\mathbb{R}^n)$.

**Solution.** The cases $\mu = 0$ and $\mu = 1$ are trivial, so assume that $0 < \mu < 1$. By Hölder's inequality, with $p = \frac{1}{\mu}$ and $q = \frac{1}{1-\mu}$,

$$\begin{aligned}
|\!|u|\!|^2_{\mu s+(1-\mu)s'} &= \int_{\mathbb{R}^n} \left(1 + |k|^2\right)^{\mu s + (1-\mu)s'} |\hat{u}(k)|^2\ \tfrac{d^n k}{(2\pi)^n} \\
&= \int_{\mathbb{R}^n} \left[\left(1 + |k|^2\right)^s |\hat{u}(k)|^2\right]^\mu \left[\left(1 + |k|^2\right)^{s'} |\hat{u}(k)|^2\right]^{1-\mu}\ \tfrac{d^n k}{(2\pi)^n} \\
&\le \left[\int_{\mathbb{R}^n} \left(1 + |k|^2\right)^s |\hat{u}(k)|^2\ \tfrac{d^n k}{(2\pi)^n}\right]^\mu \left[\int_{\mathbb{R}^n} \left(1 + |k|^2\right)^{s'} |\hat{u}(k)|^2\ \tfrac{d^n k}{(2\pi)^n}\right]^{1-\mu} \\
&= |\!|u|\!|_s^{2\mu}\ |\!|u|\!|_{s'}^{2(1-\mu)}
\end{aligned}$$

■

**Problem 2.1.13** Let $\ell \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^n$ be open and $u \in H^\ell(\Omega)$.

(a) Let $K \subset \Omega$ be compact. Suppose that $u(x) = 0$ for all $x \in \Omega \setminus K$. Prove that $u \in H_0^\ell(\Omega)$.

(b) Let $K \subset \Omega$. Suppose that $u(x) = 0$ for all $x \in \Omega \setminus K$ and that, for each $R > 0$, $K \cap \{|x| \le R\}$ is compact. Prove that $u \in H_0^\ell(\Omega)$.

**Solution.** (a) Extend $u(x)$ to all $x \in \mathbb{R}^n$ by setting $u(x) = 0$ for all $x \in \mathbb{R}^n \setminus \Omega$. Since the original $u$ vanishes in a neighbourhood of $\partial\Omega$, the new $u$ is in $H^\ell(\mathbb{R}^n)$. By Problem 2.1.8, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^\ell(\mathbb{R}^n)$. Thus there is a sequence $u_j \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} |u - u_j|_\ell = 0$$

Since $K$ is compact and $\Omega$ is open, there is a function $\varphi \in C_0^\infty(\Omega)$ that is identically one on $K$. Thus $\varphi u = u$, $\varphi u_j \in C_0^\infty(\Omega)$ and

$$\lim_{j\to\infty} \|u - \varphi u_j\|_{\ell,\Omega} = \lim_{j\to\infty} \|\varphi u - \varphi u_j\|_{\ell,\Omega} = \lim_{j\to\infty} |\varphi(u - u_j)|_\ell = 0$$

since multiplication by $\varphi$ is a bounded operator on $H^\ell(\mathbb{R}^n)$, by part (a) of Problem 2.1.3. Since $H_0^\ell(\Omega)$ is a closed subspace of $H^\ell(\mathbb{R}^n)$, we have that $u \in H_0^\ell(\Omega)$. ■

(b) Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ take values in $[0,1]$ and be identically one for $|x| \le 1$. By part (a) of Problem 2.1.3, we have that $\varphi(x/j)u(x) \in H^\ell(\Omega)$. By part (a) of this question, $\varphi(x/j)u(x) \in H_0^\ell(\Omega)$. Since $H_0^\ell(\Omega)$ is a closed subspace of $H^\ell(\mathbb{R}^n)$, if suffices to prove that $\varphi(x/j)u(x)$ converges to $u$ in $H^\ell(\Omega)$. Equivalently, it suffices to prove that, for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$, $\partial^\alpha[u(x) - \varphi(x/j)u(x)]$ converges to zero in $L^2(\Omega)$. But

$$\partial^\alpha[u(x) - \varphi(x/j)u(x)] = [1 - \varphi(x/j)]\partial^\alpha u(x) + \sum_{\substack{\beta\in\mathbb{N}_0^n\\ 0\ne\beta\le\alpha}} \tfrac{1}{j^{|\beta|}}(\partial^\beta\varphi)(x/j)\partial^{\alpha-\beta}u$$

Now

$$\lim_{j\to\infty} \big\|[1 - \varphi(x/j)]\partial^\alpha u(x)\big\|_{L^2(\Omega)} \le \lim_{j\to\infty} \big\|\partial^\alpha u\big\|_{L^2(\{x\in\Omega \ \mid\ |x|\ge j\})} = 0$$

by the Lebesgue dominated convergence theorem and, for all $\beta \in \mathbb{N}_0^n$ with $0 \ne \beta \le \alpha$,

$$\lim_{j\to\infty} \big\|\tfrac{1}{j^{|\beta|}}(\partial^\beta\varphi)(x/j)\partial^{\alpha-\beta}u\big\|_{L^2(\Omega)} \le \sup_{x\in\mathbb{R}^n} \big|(\partial^\beta\varphi)(x)\big| \lim_{j\to\infty} \tfrac{1}{j^{|\beta|}}\big\|\partial^{\alpha-\beta}u\big\|_{L^2(\Omega)} = 0$$

■

**Problem 2.1.14** Let $s < t$ and $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Since $|u|_s \le |u|_t$, $H_0^t(\Omega) \subset H_0^s(\Omega)$. The goal of this problem is to prove Rellich's theorem, which states that any bounded subset of $H_0^t(\Omega)$ is precompact when viewed as a subset of $H_0^s(\Omega)$.

(a) Let $\psi$ be a real valued function in $C_0^\infty(\mathbb{R}^n)$ that is identically one on $\Omega$. Prove that if $u \in C_0^\infty(\Omega)$ then, for all $\alpha \in \mathbb{N}_0^n$,

$$\big|\partial_k^\alpha \hat{u}(k)\big| \le |u|_t |\psi_k^\alpha|_{-t}$$

where $\psi_k^\alpha(x) = x^\alpha e^{ik\cdot x}\psi(x)$.

(b) Prove that if $u \in H_0^t(\Omega)$ then, for all $\alpha \in \mathbb{N}_0^n$, $\partial_k^\alpha \hat{u}(k)$ exists, is continuous and obeys the bound of part (a).

(c) Let $r > 0$. Prove that if $\{u_i\}_{i\in\mathbb{N}} \subset \big\{\ u \in H_0^t(\Omega)\ \big|\ |u|_t \le r\ \big\}$, then there is a subsequence $u_{i_j}$ which converges in $H_0^s(\Omega)$. *Hints:* (1) Let $R > 0$. Use the Arzelà–Ascoli theorem to prove the existence of a subsequence that converges uniformly on $|k| \le R$. (2) Bound separately the two terms in

$$\big|u_{i_j} - u_{i_\ell}\big|_s^2 = \int_{|k|\le R} (1+|k|^2)^s \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2\ \tfrac{d^n k}{(2\pi)^n} + \int_{|k|>R} (1+|k|^2)^{-(t-s)}(1+|k|^2)^t \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2\ \tfrac{d^n k}{(2\pi)^n}$$

**Solution.** (a) Since $\psi$ is identically one on the support of $u$

$$\hat{u}(k) = \int e^{-ik\cdot x} u(x)\ d^n x = \int e^{-ik\cdot x} u(x)\psi(x)\ d^n x$$

Hence, by Parseval,

$$\begin{aligned}\partial_k^\alpha \hat{u}(k) &= \int (-ix)^\alpha e^{-ik\cdot x} u(x)\psi(x)\ d^n x = (-i)^{|\alpha|}\int u(x)\overline{\psi_k^\alpha(x)}\ d^n x \\ &= (-i)^{|\alpha|}\int \hat{u}(p)\overline{\hat{\psi}_k^\alpha(p)}\ \tfrac{d^n p}{(2\pi)^n}\end{aligned}$$

Inserting $1 = (1+|p|^2)^{t/2}(1+|p|^2)^{-t/2}$ and applying Cauchy–Schwarz yields the desired bound.

(b) Let $u_i \in C_0^\infty(\Omega)$ converge in $H_0^t(\Omega)$ to $u$. Then $|u_i|_s$ is bounded in $i$ and $|u_i - u_j|_t$ converges to zero as $\min\{i,j\} \to \infty$. Hence, for every $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha \hat{u}_i(k)$ is a sequence of bounded continuous functions that converges uniformly on $\mathbb{R}^n$ by the bound of part (a). The limit $\lim\limits_{i\to\infty} \hat{u}_i(k)$ exists pointwise and is $C^\infty$. In Lemma 2.1.5, $\hat{u}(k)$ was defined as the $(1+|k|^2)^s$–weighted $L^2$ limit of the $\hat{u}_i(k)$'s. Every $L^2$ convergent sequence has an almost everywhere pointwise convergent subsequence. So $\hat{u}(k) = \lim\limits_{i\to\infty}\hat{u}_i(k)$. The bound is obvious.

(c) Let $R > 0$. Since $\hat{\psi}_k^\alpha(p) = \hat{\psi}_0^\alpha(p-k)$, Peetre's inequality, Lemma 2.3.4, implies that

$$\begin{aligned}|\psi_k^\alpha|_{-t}^2 &= \int (1+|p|^2)^{-t}\big|\hat{\psi}_0^\alpha(p-k)\big|^2\ \tfrac{d^n p}{(2\pi)^n} \\ &= \int (1+|p+k|^2)^{-t}\big|\hat{\psi}_0^\alpha(p)\big|^2\ \tfrac{d^n p}{(2\pi)^n} \\ &\le 2^{|t|}(1+|k|^2)^{|t|}\int (1+|p|^2)^{-t}\big|\hat{\psi}_0^\alpha(p)\big|^2\ \tfrac{d^n p}{(2\pi)^n}\end{aligned}$$

is bounded on $|k| \le R$. Hence, by the inequality of part (b), the sequence $\hat{u}_i(k)$ is bounded and equicontinuous on $\{|k| \le R\}$. By the Arzelà–Ascoli theorem [F, Theorem 4.43], there is a subsequence $\hat{u}_{i_j}(k)$ that converges uniformly on $|k| \le R$. For this subsequence

$$\begin{aligned}
\|u_{i_j} - u_{i_\ell}\|_s^2 &= \int (1+|k|^2)^s \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 \, \tfrac{d^n k}{(2\pi)^n} \\
&= \int_{|k|\le R} (1+|k|^2)^s \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 \, \tfrac{d^n k}{(2\pi)^n} \\
&\qquad + \int_{|k|>R} (1+|k|^2)^{-(t-s)} (1+|k|^2)^t \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 \, \tfrac{d^n k}{(2\pi)^n} \\
&\le (1+R^2)^{|s|} \Gamma_n R^n \sup_{|k|\le R} \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 \\
&\qquad + 2(1+R^2)^{-(t-s)} \big[\|u_{i_j}\|_t^2 + \|u_{i_\ell}\|_t^2\big] \\
&\le (1+R^2)^{|s|} \Gamma_n R^n \sup_{|k|\le R} \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 + 4r^2 (1+R^2)^{-(t-s)}
\end{aligned}$$

where $\Gamma_n$ is volume of the unit ball in $\mathbb{R}^n$. Let $\varepsilon > 0$. First pick $R$ such that $4r^2(1+R^2)^{-(t-s)} < \frac{\varepsilon^2}{2}$. This is possible since $t > s$. Next pick $M$ such that $j, \ell > M$ implies that $\sup_{|k|\le R} \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 < \frac{\varepsilon^2}{2\Gamma_n}(1+R^2)^{-|s|} R^{-n}$. Then $j, \ell > M$ implies

$$\|u_{i_j} - u_{i_\ell}\|_s < \varepsilon$$

■

**Problem 2.1.15**

(a) Let $s \in \mathbb{R}$ and $v \in H^{-s}(\mathbb{R}^n)$. Prove that

$$\|v\|_{-s,n} = \sup_{\substack{u \in H^s(\mathbb{R}^n) \\ \|u\|_{s,n} \le 1}} \Big| \int \hat{u}(k) \overline{\hat{v}(k)} \, \tfrac{d^n k}{(2\pi)^n} \Big|$$

(b) Let $v \in L^2(\mathbb{R}^n)$ and $s \ge 0$. Prove that

$$\|v\|_{-s,n} = \sup_{\substack{u \in H^s(\mathbb{R}^n) \\ \|u\|_{s,n} \le 1}} \big| \langle u, v \rangle_{L^2(\mathbb{R}^n)} \big|$$

(c) Let $t < s$ and $v \in H^t(\mathbb{R}^n)$. Prove that if

$$M = \sup \Big\{ \Big| \int \hat{u}(k) \overline{\hat{v}(k)} \, \tfrac{d^n k}{(2\pi)^n} \Big| \;\Big|\; \hat{u} \in C_0^\infty(\mathbb{R}^n),\ \|u\|_{-s,n} \le 1 \Big\} < \infty$$

then $v \in H^s(\mathbb{R}^n)$ and $\|v\|_{s,n} = M$.

**Solution.** (a) By part (a) of Proposition 2.1.8,

$$\Big| \int \hat{u}(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n} \Big| \le \|u\|_s \, \|v\|_{-s}$$

Hence

$$\sup_{\substack{u \in H^s(\mathbb{R}^n) \\ \|u\|_s \le 1}} \Big| \int \hat{u}(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n} \Big| \le \|v\|_{-s}$$

For the other inequality, it suffices to consider $v \ne 0$. Choose $u$ so that

$$\hat{u}(k) = \big(1 + |k|^2\big)^{-s} \tfrac{\hat{v}(k)}{\|v\|_{-s}}$$

As

$$\|u\|_s^2 = \int \big(1 + |k|^2\big)^s |\hat{u}(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} = \tfrac{1}{\|v\|_{-s}^2} \int \big(1 + |k|^2\big)^{-s} |\hat{v}(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} = 1$$

and

$$\int \hat{u}(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n} = \tfrac{1}{\|v\|_{-s}} \int \big(1 + |k|^2\big)^{-s} |\hat{v}(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} = \|v\|_{-s}$$

we have

$$\sup_{\substack{u \in H^s(\mathbb{R}^n) \\ \|u\|_s \le 1}} \Big| \int \hat{u}(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n} \Big| \ge \|v\|_{-s}$$

(b) When $s \ge 0$, $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. If $v \in L^2(\mathbb{R}^n)$ as well,

$$\langle u, v\rangle_{L^2(\mathbb{R}^n)} = \int \hat{u}(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n}$$

and part (b) follows from part (a).

(c) Let $\psi \in C_0^\infty(\mathbb{R}^n)$ take values in $[0,1]$, be identically one on $\big\{\ k \in \mathbb{R}^n \ \big|\ |k| < 1\ \big\}$, identically zero on $\big\{\ k \in \mathbb{R}^n \ \big|\ |k| > 2\ \big\}$ and monotone decreasing in $|k|$. The function, $v_R$, whose Fourier transform is $\hat{v}_R(k) = \psi(k/R)\hat{v}(k)$ obeys

$$\|v_R\|_s^2 = \int \big(1 + |k|^2\big)^s |\psi(k/R)\hat{v}(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le \big(1 + 4R^2\big)^{s-t} \int \big(1 + |k|^2\big)^t |\hat{v}(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} < \infty$$

and hence is in $H^s(\mathbb{R}^n)$. If $\|u\|_{-s} \le 1$, the function, $u_R$, whose Fourier transform is $\hat{u}_R(k) = \psi(k/R)\hat{u}(k)$ also obeys $\|u_R\|_{-s} \le \|u\|_{-s} \le 1$. Hence

$$\Big| \int \hat{u}(k)\overline{\hat{v}_R(k)} \ \tfrac{d^n k}{(2\pi)^n} \Big| = \Big| \int \hat{u}_R(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n} \Big| \le M$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ with $|u|_{-s} \le 1$. Since $v_R \in H^s(\mathbb{R}^n)$, the map $u \mapsto \int \hat{u}(k)\overline{\hat{v}_R(k)}\ \frac{d^nk}{(2\pi)^n}$ is continuous on $H^{-s}(\mathbb{R}^n)$ by Proposition 2.1.8. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{-s}(\mathbb{R}^n)$ we have

$$\Big|\int \hat{u}(k)\overline{\hat{v}_R(k)}\ \tfrac{d^nk}{(2\pi)^n}\Big| \le M$$

for all $u \in H^{-s}(\mathbb{R}^n)$ with $|u|_{-s} \le 1$. By part (a), $|v_R|_s \le M$ for all $R > 0$. By the monotone convergence theorem

$$\int \big(1+|k|^2\big)^s |\hat{v}(k)|^2\ \tfrac{d^nk}{(2\pi)^n} = \lim_{R\to\infty} \int \big(1+|k|^2\big)^s |\psi(k/R)\hat{v}(k)|^2\ \tfrac{d^nk}{(2\pi)^n} \le M$$

■

**Problem 2.1.16** Prove Lemma 2.1.14. That is:

Let $\ell \in \mathbb{Z}$.

(a) Let $\varphi : \Omega \to \mathbb{C}$ and all of its derivatives of order at most $|\ell|$ be bounded and continuous. Prove that the map $v \in \big\{\ v \in C^{|\ell|}(\Omega)\ \big|\ \|v\|_{\max\{0,\ell\},\Omega} < \infty\ \big\} \mapsto \varphi v$ has a unique extension to a bounded, linear map on $H^\ell(\Omega)$ and that there is a constant $C_{|\ell|,n}$, depending only on $|\ell|$ and $n$, such that

$$\|\varphi u\|_{\ell,\Omega} \le C_{|\ell|,n} \|\varphi\|_{C^{|\ell|}(\Omega)} \|u\|_{\ell,\Omega}$$

for all $u \in H^\ell(\Omega)$. Here $\|\varphi\|_{C^\ell(\Omega)} = \sup_{\substack{x\in\Omega \\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\le\ell}} \big|\partial^\alpha \varphi(x)\big|$.

(b) Let $\alpha \in \mathbb{N}_0^n$. Prove that the map $v \in \big\{\ v \in C^{\max\{\ell,|\alpha|\}}(\Omega)\ \big|\ \|v\|_{\max\{\ell,|\alpha|\},\Omega} < \infty\ \big\} \mapsto \partial^\alpha v$ has a unique extension to a bounded, linear map from $H^\ell(\Omega)$ to $H^{\ell-|\alpha|}(\Omega)$. In the case $0 < \ell < |\alpha|$, assume that $\big\{\ v \in C^{|\alpha|}(\Omega)\ \big|\ \|v\|_{|\alpha|,\Omega} < \infty\ \big\}$ is dense in $H^\ell(\Omega)$.

**Solution.** (a) The case $\ell \ge 0$ has already been dealt with part (a) of Problem 2.1.3, so let $\ell < 0$. Observe that if $u \in C_0^{|\ell|}(\Omega)$, then $\varphi u \in C_0^{|\ell|}(\Omega) \subset H_0^{|\ell|}(\Omega)$, by Problem 2.1.2. As $H_0^{|\ell|}(\Omega)$ is a closed subspace of $H^{|\ell|}(\Omega)$, part (a) of Problem 2.1.3 implies that multiplication by $\varphi$ maps $H_0^{|\ell|}(\Omega)$ into itself. If $\mathcal{L} \in H^\ell(\Omega) = H_0^{|\ell|}(\Omega)^*$ then,

$$\big|\mathcal{L}(\varphi u)\big| \le \|\mathcal{L}\|_{\ell,\Omega} \|\varphi u\|_{|\ell|,\Omega} \le C_{|\ell|,n} \|\varphi\|_{C^{|\ell|}(\Omega)} \|\mathcal{L}\|_{\ell,\Omega} \|u\|_{|\ell|,\Omega}$$

for all $u \in H_0^{|\ell|}(\Omega)$, by part (a) of Problem 2.1.3. Hence

$$(M_\varphi \mathcal{L})u = \mathcal{L}(\varphi u)$$

defines another element of $H^\ell(\Omega) = H_0^{|\ell|}(\Omega)^*$ and furthermore

$$\|M_\varphi \mathcal{L}\|_{\ell,\Omega} \le C_{|\ell|,n} \|\varphi\|_{C^{|\ell|}(\Omega)} \|\mathcal{L}\|_{\ell,\Omega}$$

If $v \in L^2(\Omega)$ and $\mathcal{L}_v$ is the element of $H_0^{|\ell|}(\Omega)^*$ that is defined in part (b) of Theorem 2.1.11 and is identified with $v$ in Remark 2.1.12, then

$$(M_\varphi \mathcal{L}_v)u = \mathcal{L}_v(\varphi u) = \langle \varphi u, \bar{v} \rangle_{L^2(\Omega)} = \langle u, \overline{\varphi v} \rangle_{L^2(\Omega)} = \mathcal{L}_{\varphi v} u$$

Thus $M_\varphi$ is a bounded, linear map on $H^\ell(\Omega)$ that extends the prescribed $v \mapsto \varphi v$ map. By part (b) of Theorem 2.1.11, $C_0^\infty(\Omega) \subset \{\, v \in C^{|\ell|}(\Omega) \bigm| \|v\|_{0,\Omega} < \infty \,\}$ is dense in $H^\ell(\Omega)$. So the extension is unique.

(b) We can always express $\partial^\alpha$ as a product of derivatives of order one, so it suffices to consider $|\alpha| = 1$. The case $\ell \ge |\alpha|$ has already been dealt with part (b) of Problem 2.1.3, so let $\ell \le 0$. Observe that

- if $u \in C_0^{|\ell|+1}(\Omega)$, then $\partial^\alpha u \in C_0^{|\ell|}(\Omega) \subset H_0^{|\ell|}(\Omega)$, by Problem 2.1.2. As $H_0^{|\ell|}(\Omega)$ is a closed subspace of $H^{|\ell|}(\Omega)$, part (b) of Problem 2.1.3 implies that $\partial^\alpha$ maps $H_0^{|\ell|+1}(\Omega)$ into $H_0^{|\ell|}(\Omega)$.
- Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, $H^0(\Omega) = H_0^0(\Omega)$. By the Riesz representation theorem, there is a natural identification between $H^\ell(\Omega)$ and $H_0^{|\ell|}(\Omega)^*$ for $\ell = 0$ too.

If $\mathcal{L} \in H^\ell(\Omega) = H_0^{|\ell|}(\Omega)^*$ then,

$$\left|\mathcal{L}(\partial^\alpha u)\right| \le \|\mathcal{L}\|_{\ell,\Omega} \|\partial^\alpha u\|_{|\ell|,\Omega} \le \|\mathcal{L}\|_{\ell,\Omega} \|u\|_{|\ell|+1,\Omega}$$

for all $u \in H_0^{|\ell|+1}(\Omega)$, by part (b) of Problem 2.1.3. Hence

$$(D_\alpha \mathcal{L})u = -\mathcal{L}(\partial^\alpha u)$$

defines an element of $H^{\ell-1}(\Omega) = H_0^{|\ell|+1}(\Omega)^*$ and furthermore $\|D_\alpha \mathcal{L}\|_{\ell-1,\Omega} \le \|\mathcal{L}\|_{\ell,\Omega}$. If $v \in C^1(\Omega)$ with $v, \partial^\alpha v \in L^2(\Omega)$ and $\mathcal{L}_v$ is the element of $H_0^{|\ell|}(\Omega)^*$ that is defined in part (b) of Theorem 2.1.11 and is identified with $v$ in Remark 2.1.12, then

$$(D_\alpha \mathcal{L}_v)u = -\mathcal{L}_v(\partial^\alpha u) = -\langle \partial^\alpha u, \bar{v} \rangle_{L^2(\Omega)} = \left\langle u, \overline{\partial^\alpha v} \right\rangle_{L^2(\Omega)} = \mathcal{L}_{\partial^\alpha v} u$$

for all $u \in C_0^\infty(\Omega)$, by the divergence theorem. Thus $D_\alpha$ is a bounded, linear map from $H^\ell(\Omega)$ to $H^{\ell-1}(\Omega)$ that extends the prescribed $v \mapsto \partial^\alpha v$ map. Since $C_0^\infty(\Omega)$, which is contained in the domain of the prescribed map, is dense in $L^2(\Omega)$ and $L^2(\Omega)$ is, in turn, dense in $H^\ell(\Omega)$, by part (b) of Theorem 2.1.11, the extension is unique. ■

**Problem 2.1.17** This problem illustrates the need for the multiplier $\varphi$ of part (a) of Lemma 2.1.14 to be relatively smooth if the product $\varphi v$ is to be well–defined when $v$ is quite unsmooth. In this problem, $\varphi$ will be a characteristic function with a discontinuity at 0 and there will be two different $v$'s. One will be a Dirac delta function and the other the derivative of a Dirac delta function. Both will be supported on the point 0. Let $\Omega = (-1,1)$. Let $w \in C_0^\infty(\mathbb{R})$ be supported in $\big[-\frac{1}{2},\frac{1}{2}\big]$. Assume that $w$ is even and obeys $\int_{\mathbb{R}} w(x)\,dx = 1$ and $w'(x) \ge 0$ for $x < 0$. Define, for $\varepsilon > 0$ and $u \in C_0^\infty(\Omega)$,

$$\delta(u) = u(0) \qquad \delta_\varepsilon(u) = \int_{-1}^{1} \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big) u(x)\,dx \qquad \delta_{\varepsilon,+}(u) = \int_{-1}^{1} \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big) u(x)\,dx$$

$$\delta'(u) = -u'(0) \quad \delta'_\varepsilon(u) = \int_{-1}^{1} \tfrac{1}{\varepsilon^2} w'\big(\tfrac{x}{\varepsilon}\big) u(x)\,dx$$

(a) Prove that $\delta$, $\delta_\varepsilon$ and $\delta_{\varepsilon,+}$ all have unique continuous extensions to elements of $H^{-1}(\Omega)$. Prove that $\delta'$ and $\delta'_\varepsilon$ have unique continuous extensions to elements of $H^{-2}(\Omega)$.

(b) Prove that

$$\lim_{\varepsilon\to 0+} \delta_\varepsilon = \delta \qquad \text{and} \qquad \lim_{\varepsilon\to 0+} \delta_{\varepsilon,+} = \delta$$

in $H^{-2}(\Omega)$ and

$$\lim_{\varepsilon\to 0+} \delta'_\varepsilon = \delta'$$

in $H^{-3}(\Omega)$.

(c) Let $\varphi$ be the characteristic function of $(0,1)$. Prove that

$$\lim_{\varepsilon\to 0+} \varphi\delta_\varepsilon = \tfrac{1}{2}\delta \qquad \text{and} \qquad \lim_{\varepsilon\to 0+} \varphi\delta_{\varepsilon,+} = \delta$$

in $H^{-2}(\Omega)$ and that $\lim\limits_{\varepsilon\to 0+} \varphi\delta'_\varepsilon$ diverges in $H^{-\ell}(\Omega)$ for all $\ell \in \mathbb{N}$.

**Solution.** a) Recall, from Problem 2.1.11, that, for all $u \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R})$, there is a constant $C$ such that

$$\sup_x \big|u(x)\big| \le C\|u\|_{1,\Omega} \qquad \sup_x \big|u'(x)\big| \le C\|u\|_{2,\Omega} \qquad \sup_x \big|u''(x)\big| \le C\|u\|_{3,\Omega}$$

The existence and uniqueness of the desired extensions follows from the denseness of $C_0^\infty(\Omega)$

in $H_0^\ell(\Omega)$ and the bounds

$$\begin{aligned}
|\delta(u)| &= |u(0)| \le C\|u\|_{1,\Omega} \\
|\delta'(u)| &= |u'(0)| \le C\|u\|_{2,\Omega} \\
\big|\delta_\varepsilon(u)\big| &\le \int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,|u(x)|\,dx \le \int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,C\|u\|_{1,\Omega}\,dx = C\|u\|_{1,\Omega}\int w(y)\,dy = C\|u\|_{1,\Omega} \\
\big|\delta_{\varepsilon,+}(u)\big| &\le \int \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big)\,|u(x)|\,dx \le \int \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big)\,C\|u\|_{1,\Omega}\,dx = C\|u\|_{1,\Omega} \\
\big|\delta'_\varepsilon(u)\big| &= \left|\int \tfrac{1}{\varepsilon^2} w'\big(\tfrac{x}{\varepsilon}\big)u(x)\,dx\right| = \left|\int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)u'(x)\,dx\right| \le C\|u\|_{2,\Omega}\int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,dx = C\|u\|_{2,\Omega}
\end{aligned}$$

We have used the change of variables $y = \frac{x}{\varepsilon}$ in lines 3 and 5, the change of variables $y = \frac{x-\varepsilon}{\varepsilon}$ in line 4 and integration by parts in line 5.

(b) Since $\int w(x)\,dx = 1$, we have

$$\int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,u(0)\,dx = u(0)\int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,dx = u(0)\int w(y)\,dy = u(0)$$

For $0 < \varepsilon < 1$, $w\big(\frac{x}{\varepsilon}\big)$ is supported on $\big[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\big]$ and

$$\begin{aligned}
\big|\delta_\varepsilon(u) - \delta(u)\big| = \big|\delta_\varepsilon(u) - u(0)\big| &= \left|\int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,[u(x) - u(0)]\,dx\right| \\
&\le \int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,|x|\sup_x |u'(x)|\,dx \le C\|u\|_{2,\Omega}\int \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big)\,|x|\,dx \\
&= \varepsilon\, C\|u\|_{2,\Omega}\int |y| w(y)\,dy \le \varepsilon\, C\|u\|_{2,\Omega}\int w(y)\,dy = \varepsilon\, C\|u\|_{2,\Omega}
\end{aligned}$$

which implies that $\delta_\varepsilon$ converges to $\delta$ in $H^{-2}(\Omega)$ as $\varepsilon \to 0$. Similarly

$$\begin{aligned}
\big|\delta_{\varepsilon,+}(u) - \delta(u)\big| = \big|\delta_{\varepsilon,+}(u) - u(0)\big| &\le \big|\delta_{\varepsilon,+}(u) - u(\varepsilon)\big| + \big|u(\varepsilon) - u(0)\big| \\
&= \left|\int \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big)\,[u(x) - u(\varepsilon)]\,dx\right| + \big|u(\varepsilon) - u(0)\big| \\
&\le \int \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big)\,|x-\varepsilon|\sup_x |u'(x)|\,dx + |\varepsilon|\sup_x |u'(x)| \\
&\le C\|u\|_{2,\Omega}\int \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big)\,|x-\varepsilon|\,dx + \varepsilon\, C\|u\|_{2,\Omega} \\
&= \varepsilon\, C\|u\|_{2,\Omega}\int |y| w(y)\,dy + \varepsilon\, C\|u\|_{2,\Omega} \le 2\varepsilon C\,\|u\|_{2,\Omega}
\end{aligned}$$

implies that $\delta_{\varepsilon,+}$ converges to $\delta$ in $H^{-2}(\Omega)$. By integration by parts, $\delta'_\varepsilon(u) = -\delta_\varepsilon(u')$. So

$$\big|\delta'_\varepsilon(u) - \delta'(u)\big| = \big|-\delta_\varepsilon(u') + u'(0)\big| \le \varepsilon\, C\|u'\|_{2,\Omega} \le \varepsilon\, C\|u\|_{3,\Omega}$$

(c) Since $w$ is even, $\int_0^\infty w(x)\,dx = \frac{1}{2}\int w(x)\,dx = \frac{1}{2}$, so that

$$\int_0^\infty \tfrac{1}{\varepsilon}w\big(\tfrac{x}{\varepsilon}\big)\,u(0)\,dx = u(0)\int_0^\infty \tfrac{1}{\varepsilon}w\big(\tfrac{x}{\varepsilon}\big)\,dx = u(0)\int_0^\infty w(y)\,dy = \tfrac{1}{2}\,u(0)$$

Hence, for $0<\varepsilon<1$,

$$\begin{aligned}\big|(\varphi\delta_\varepsilon)(u) - \tfrac{1}{2}\delta(u)\big| = \big|(\varphi\delta_\varepsilon)(u) - \tfrac{1}{2}u(0)\big| &= \left|\int_0^\infty \tfrac{1}{\varepsilon}w\big(\tfrac{x}{\varepsilon}\big)\,[u(x)-u(0)]\,dx\right| \\ &\le \int \tfrac{1}{\varepsilon}w\big(\tfrac{x}{\varepsilon}\big)\,|x|\sup_x |u'(x)|\,dx \le \varepsilon\, C\|u\|_{2,\Omega}\end{aligned}$$

and $\varphi\delta_\varepsilon$ converges to $\frac{1}{2}\delta$ in $H^{-2}(\Omega)$. Since $w\big(\frac{x-\varepsilon}{\varepsilon}\big)$ is supported on $\big[\frac{\varepsilon}{2},\frac{3\varepsilon}{2}\big]$, $\varphi\delta_{\varepsilon,+} = \delta_{\varepsilon,+}$ for $0<\varepsilon<\frac{1}{2}$. So $\varphi\delta_{\varepsilon,+}$ converges to $\frac{1}{2}\delta$ in $H^{-2}(\Omega)$.

Fix any $u \in C_0^\infty(\Omega) \subset H^{-\ell}(\Omega)$ that is identically one on $\big[-\frac{1}{2},\frac{1}{2}\big]$. Since $w'\big(\frac{x}{\varepsilon}\big)$ is supported on $\big[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\big]$, we have, for $0<\varepsilon<1$,

$$\begin{aligned}(\varphi\delta'_\varepsilon)(u) &= \int_0^1 \tfrac{1}{\varepsilon^2}w'\big(\tfrac{x}{\varepsilon}\big)u(x)\,dx = \int_0^1 \tfrac{1}{\varepsilon^2}w'\big(\tfrac{x}{\varepsilon}\big)\,dx = \int_0^1 \tfrac{1}{\varepsilon}\tfrac{d}{dx}w\big(\tfrac{x}{\varepsilon}\big)\,dx = \tfrac{1}{\varepsilon}\big[w\big(\tfrac{1}{\varepsilon}\big) - w(0)\big] \\ &= -\tfrac{w(0)}{\varepsilon}\end{aligned}$$

This diverges as $\varepsilon \to 0$.

**Problem 2.1.18** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $k \in \mathbb{N}$. Prove that $\Omega$ has $C^k$ boundary if and only if, for each $p \in \partial\Omega$, there is an open neighbourhood $\mathcal{U}(p)$ of $p$ and a $C^k$ function $\phi_p : \mathcal{U}(p) \to \mathbb{R}$ such that $\nabla\phi_p(p) \ne 0$, $\mathcal{U}(p)\cap\partial\Omega = \phi_p^{-1}(0)$ and $\mathcal{U}(p)\cap\Omega = \phi_p^{-1}\big((0,\infty)\big)$.

**Solution.** Assume that $\Omega$ has $C^k$ boundary and let $p\in\partial\Omega$. Let $\mathcal{U}(p)$ and $\psi_p$ be the neighbourhood of $p$ and the diffeomorphism, respectively, provided by Definition 2.1.15. Let $\phi_p$ be the $n^{\text{th}}$ component of $\psi_p$. The gradient of $\phi_p$ at $p$ is one row of the Jacobian matrix of $\psi_p$ and hence cannot vanish, as $\psi_p$ is a diffeomorphism. Furthermore,

$$\begin{aligned}\mathcal{U}(p)\cap\partial\Omega &= \psi_p^{-1}\big(\{\ x\in\mathbb{R}^n \ \big|\ x_n = 0\ \}\big) = \phi_p^{-1}(0) \\ \mathcal{U}(p)\cap\Omega &= \psi_p^{-1}\big(\{\ x\in\mathbb{R}^n \ \big|\ x_n > 0\ \}\big) = \phi_p^{-1}\big((0,\infty)\big)\end{aligned}$$

Let $p\in\partial\Omega$ and assume that there is an open neighbourhood $\mathcal{V}(p)$ of $p$ and a $C^k$ function $\phi_p : \mathcal{V}(p)\to\mathbb{R}$ such that $\nabla\phi_p(p)\ne 0$ and $\mathcal{V}(p)\cap\partial\Omega = \phi_p^{-1}(0)$. Since $\nabla\phi_p(p)\ne 0$, there is a $1\le j\le n$ such that $\frac{\partial\phi_p}{\partial x_j}(p)\ne 0$. By the implicit function theorem, there is a neighbourhood $\mathcal{U}(p)\subset\mathcal{V}(p)$ of $p$ and a $C^k$ function $\zeta_p : \mathcal{U}(p)\to\mathbb{R}$, which is independent of $x_j$ such that

$$\mathcal{U}(p)\cap\partial\Omega = \{\ x\in\mathcal{U}(p)\ \big|\ x_j = \zeta_p(x)\ \}$$

The function $\zeta_p$ is independent of the variable $x_j$. Set, $\tilde\psi(x) = \big(\tilde\psi_1(x), \cdots, \tilde\psi_n(x)\big)$ where, for $1 \le i \le n$,

$$\tilde\psi_i(x) = \begin{cases} x_i - p_i & \text{if } i \ne j \\ x_j - \zeta_p(x) & \text{if } i = j \end{cases}$$

Then $\tilde\psi$ is a $C^k$ diffeomorphism from $\mathcal{U}(p)$ to the image under $\tilde\psi$ of $\mathcal{U}(p)$, which contains $\tilde\psi(p) = 0$. Its inverse map has $i^{\text{th}}$ component

$$\tilde\psi_i^{-1}(y) = \begin{cases} y_i + p_i & \text{if } i \ne j \\ y_j + \zeta_p(y+p) & \text{if } i = j \end{cases}$$

By permuting the components of $\tilde\psi$ we may assume that $j = n$. By shrinking $\mathcal{U}(p)$, we may assume that the range of $\tilde\psi$ is a ball of some radius $r > 0$ centred on $\tilde\psi(p) = 0$. Possibly replacing the $n^{\text{th}}$ component of $\tilde\psi$ by its negative, we have

$$\tilde\psi\big(\mathcal{U}(p) \cap \Omega\big) = \big\{\ x \in \mathbb{R}^n\ \big|\ |x| < r,\ x_n > 0\ \big\}$$
$$\tilde\psi\big(\mathcal{U}(p) \cap \partial\Omega\big) = \big\{\ x \in \mathbb{R}^n\ \big|\ |x| < r,\ x_n = 0\ \big\}$$

It now suffices to compose $\tilde\psi$ with a suitable diffeomorphism from the ball of radius $r$ to $\mathbb{R}^n$. For example, let $\rho : [0, r) \to [0, \infty)$ be a $C^\infty$ function that is identically one on $[0, r/2)$ and increases monotonically to $\infty$ as its argument increases to $r$. Then

$$\psi_p(x) = \rho\big(\,|\tilde\psi(x)|\,\big)\ \tilde\psi(x)$$

does the job. ■

**Problem 2.1.19** Prove that $\|\cdot\|_{s,\partial\Omega}$, given in Definition 2.1.18, is a norm. Prove further that

$$\langle f, g\rangle_{s,\partial\Omega} = \sum_{i=1}^{N} \big\langle (\chi_i f) \circ \psi_{p_i}^{-1}, (\chi_i g) \circ \psi_{p_i}^{-1} \big\rangle_{s,n-1}$$

is an inner product.

**Solution.** Define, for $f, g \in C^\infty(\partial\Omega)$ and $i \le 1 \le N$,

$$A_i = \big|(\chi_i f) \circ \psi_{p_i}^{-1}\big|_{s,n-1} \qquad B_i = \big|(\chi_i g) \circ \psi_{p_i}^{-1}\big|_{s,n-1}$$

Then clearly $\|f\|_{s,\partial\Omega}^2 = \sum_{i=1}^N A_i^2$ is positive and vanishes only if $A_i = 0$ for all $1 \le i \le N$. But in the latter case $\chi_i f$ is the zero function for all $i$. As the $\chi_i$'s form a partition of unity for $\partial\Omega$, this implies that $f$ is also the zero function. This verifies the positivity and nondegeneracy of both $\|\ \cdot\ \|_{s,\partial\Omega}$ and $\langle\ \cdot\ ,\ \cdot\ \rangle_{s,\partial\Omega}$. The sesquilinearity of $\langle\ \cdot\ ,\ \cdot\ \rangle_{s,\partial\Omega}$ is obvious. So is $\langle g,f\rangle_{s,\partial\Omega} = \overline{\langle f,g\rangle_{s,\partial\Omega}}$. The remaining two norm axioms are automatic. Alternatively, they can be easily proven directly:

$$\|\alpha f\|_{s,\partial\Omega}^2 = \sum_{i=1}^N \left|(\chi_i\alpha f)\circ\psi_{p_i}^{-1}\right|_{s,n-1}^2 = \sum_{i=1}^N |\alpha|^2\left|(\chi_i f)\circ\psi_{p_i}^{-1}\right|_{s,n-1}^2 = |\alpha|^2\ \|f\|_{s,\partial\Omega}^2$$

and

$$\begin{aligned}\|f+g\|_{s,\partial\Omega} &= \sqrt{\sum_{i=1}^N \left|(\chi_i(f+g))\circ\psi_{p_i}^{-1}\right|_{s,n-1}^2} \le \sqrt{\sum_{i=1}^N (A_i+B_i)^2} \\ &= \left\|(A_i+B_i)_{1\le i\le N}\right\|_{\mathbb{R}^N} \le \left\|(A_i)_{1\le i\le N}\right\|_{\mathbb{R}^N} + \left\|(B_i)_{1\le i\le N}\right\|_{\mathbb{R}^N} \\ &= \|f\|_{s,\partial\Omega} + \|g\|_{s,\partial\Omega}\end{aligned}$$

■

**Problem 2.1.20** Let $\varphi \in C_0^\infty(\mathbb{R}^m)$ and let $\psi : \mathbb{R}^m \to \mathbb{R}^m$ be a $C^\infty$ diffeomorphism. Then, for each $\ell \in \mathbb{N}_0$, the map

$$\begin{aligned} C_0^\infty(\mathbb{R}^m) &\to C_0^\infty(\mathbb{R}^m) \\ f &\mapsto (\varphi f)\circ\psi^{-1}\end{aligned}$$

extends to a bounded linear map on $H^\ell(\mathbb{R}^m)$.

**Solution.** Let $\alpha \in \mathbb{N}_0^m$ with $|\alpha| \le \ell$. We first consider $|\alpha| = 0$. Making the change of variables $x = \psi(y)$ and using $\left|\frac{\partial x}{\partial y}\right|$ to denote the associated Jacobian determinant

$$\int \left|\varphi\big(\psi^{-1}(x)\big)f\big(\psi^{-1}(x)\big)\right|^2\ d^m x = \int \left|\varphi(y)f(y)\right|^2 \left|\tfrac{\partial x}{\partial y}(y)\right|\ d^m y$$

Since $\psi^{-1}$ and $\varphi$ are $C^\infty$ and $\varphi$ is of compact support, $C \equiv \sup_y \left|\varphi(y)\right|^2 \left|\frac{\partial x}{\partial y}(y)\right|$ is finite, so that

$$\int \left|\varphi\big(\psi^{-1}(x)\big)f\big(\psi^{-1}(x)\big)\right|^2\ d^m x \le C\int \left|f(y)\right|^2\ d^m y$$

Since $C_0^\infty(\mathbb{R}^m)$ is dense in $H^0(\mathbb{R}^m) = L^2(\mathbb{R}^m)$, the B.L.T. theorem now finishes the proof when $\ell = 0$.

Next consider general $\alpha$. By the product and chain rules

$$\tfrac{\partial}{\partial x_i}\Big[\varphi\big(\psi^{-1}(x)\big)f\big(\psi^{-1}(x)\big)\Big] = \sum_{j=1}^m \Big[\tfrac{\partial\varphi}{\partial x_j}(y)f(y) + \varphi(y)\tfrac{\partial f}{\partial x_j}(y)\Big]_{y=\psi^{-1}(x)} \tfrac{\partial\psi_j^{-1}}{\partial\ x_i}(x)$$

By further repeated application of the product and chain rules

$$\partial^\alpha\Big[\varphi\big(\psi^{-1}(x)\big)f\big(\psi^{-1}(x)\big)\Big] = \sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}} g_\beta(x)(\partial^\beta f)\big(\psi^{-1}(x)\big)$$

with each $g_\beta$ bounded and of compact support, since $\psi^{-1}$ and $\varphi$ are $C^\infty$ and $\varphi$ is of compact support. Again making the change of variables $x=\psi(y)$,

$$\begin{aligned}\int\Big|\partial^\alpha\Big[\varphi\big(\psi^{-1}(x)\big)f\big(\psi^{-1}(x)\big)\Big]\Big|^2\,d^mx &= \int\Big|\sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}} g_\beta(x)(\partial^\beta f)\big(\psi^{-1}(x)\big)\Big|^2\,d^mx\\ &= \int\Big|\sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}} g_\beta\big(\psi(y)\big)(\partial^\beta f)(y)\Big|^2\Big|\tfrac{\partial x}{\partial y}(y)\Big|\,d^my\\ &\le \int\Big[\sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}}\big|g_\beta\big(\psi(y)\big)\big|^2\Big]\Big[\sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}}\big|(\partial^\beta f)(y)\big|^2\Big]\Big|\tfrac{\partial x}{\partial y}(y)\Big|\,d^my\end{aligned}$$

by Cauchy–Schwarz. Once again,

$$C_{|\alpha|}\equiv\sup_y\Big|\tfrac{\partial x}{\partial y}(y)\Big|\Big[\sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}}\big|g_\beta\big(\psi(y)\big)\big|^2\Big]$$

is finite since $\psi^{-1}$ is $C^\infty$ and the $g_\beta$'s are bounded and of compact support. Hence

$$\int\Big|\partial^\alpha\Big[\varphi\big(\psi^{-1}(x)\big)f\big(\psi^{-1}(x)\big)\Big]\Big|^2\,d^mx \le C_{|\alpha|}\int\Big[\sum_{\substack{\beta\in\mathbb{N}_0^m\\ |\beta|\le|\alpha|}}\big|(\partial^\beta f)(y)\big|^2\Big]\,d^my = C_{|\alpha|}|f|_{|\alpha|}^2$$

The claim now follows by the B.L.T. theorem. ■

**Problem 2.1.21** Prove that $\|\ \cdot\ \|_{0,\partial\Omega}$ is equivalent to

$$\|f\|_{L^2(\partial\Omega)}=\sqrt{\int_{\partial\Omega}|f(x)|^2\,d\sigma(x)}$$

where $d\sigma(x)$ is the surface measure on $\partial\Omega$.

**Solution.** Let $\big(\mathcal{U}(p),\psi_p\big)$ be a coordinate system as in Notation 2.1.16 and let $\chi_i\in C_0^\infty\big(\mathcal{U}(p_i)\big)$, $1\le i\le N$ be a partition of unity as in Definition 2.1.18. Then

$$\begin{aligned}\|f\|_{L^2(\partial\Omega)}^2 &= \sum_{i,j}\int_{\partial\Omega}\chi_i f\ \overline{\chi_j f}\ d\sigma(x) \qquad \le \sum_{i,j}\|\chi_i f\|_{L^2(\partial\Omega)}\|\chi_j f\|_{L^2(\partial\Omega)}\\ &\le \sum_{i,j}\tfrac12\big[\|\chi_i f\|_{L^2(\partial\Omega)}^2+\|\chi_j f\|_{L^2(\partial\Omega)}^2\big] = N\sum_i\|\chi_i f\|_{L^2(\partial\Omega)}^2\end{aligned}$$

and

$$\|f\|^2_{L^2(\partial\Omega)} = \sum_i \int_{\partial\Omega} \chi_i |f|^2 \, d\sigma(x) \geq \sum_i \int_{\partial\Omega} \chi_i^2 |f|^2 \, d\sigma(x) = \sum_i \|\chi_i f\|^2_{L^2(\partial\Omega)}$$

So it remains only to show that $\int_{\partial\Omega} \chi_i^2 |f|^2 \, d\sigma(x)$ is bounded above and below by constants times

$$\left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|^2_{0,n-1} = \int_{\mathbb{R}^{n-1}} \left|(\chi_i f) \circ \psi_{p_i}^{-1}(x_1, \cdots, x_{n-1}, 0)\right|^2 \, dx_1 \cdots dx_{n-1}$$

As in Problem 2.1.18, we may always arrange that, in the support of $\chi_i$,

- $\partial\Omega$ is given by the equation $x_n = \zeta(x)$, with $\zeta$ a $C^\infty$ function that is independent of $x_n$ and
- $\psi_{p_i}(x) = \big(x_1, \cdots, x_{n-1}, x_n - \zeta(x)\big)$

In this setting $d\sigma(x) = \left|\nabla\zeta(x) - \hat{e}_n\right| \, dx_1 \cdots dx_{n-1}$ where $\hat{e}_n = (0, \cdots, 0, 1)$. As $\nabla\zeta$ is $C^\infty$ and has no $\hat{e}_n$ component, $\left|\nabla\zeta(x) - \hat{e}_n\right|$ is bounded below by one and above by some constant so that

$$\int_{\partial\Omega} \chi_i^2 |f|^2 \, d\sigma(x) = \int_{\mathbb{R}^{n-1}} \left|(\chi_i f)\big(x_1, \cdots, x_{n-1}, \zeta(x)\big)\right|^2 \left|\nabla\zeta(x) - \hat{e}_n\right| dx_1 \cdots dx_{n-1}$$

is clearly bounded above and below by constants times

$$\left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|^2_{0,n-1} = \int_{\mathbb{R}^{n-1}} \left|(\chi_i f)\big(x_1, \cdots, x_{n-1}, \zeta(x)\big)\right|^2 \, dx_1 \cdots dx_{n-1}$$

■

**Problem 2.1.22** Let $s \in \mathbb{R}$. Prove that there are constants $C_s$ and $c_s$, depending only on n, $\Omega$ and $s$, such that

$$\left| \langle f, \bar{g} \rangle_{L^2(\partial\Omega)} \right| \leq C_s \, \|f\|_{-s,\partial\Omega} \|g\|_{s,\partial\Omega}$$

for all $f, g \in C^\infty(\partial\Omega)$ and

$$\|f\|_{-s,\partial\Omega} \leq c_s \sup \left\{ \, \left| \langle f, \bar{g} \rangle_{L^2(\partial\Omega)} \right| \, \Big| \, g \in C^\infty(\partial\Omega), \ \|g\|_{s,\partial\Omega} \leq 1 \, \right\}$$

for all $f \in C^\infty(\partial\Omega)$. Here, as you would expect,

$$\langle f, g \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} f(x) \overline{g(x)} \, d\sigma(x)$$

where $d\sigma(x)$ is the surface measure on $\partial\Omega$.

**Solution.** (a) Let $\big(\mathcal{U}(p),\psi_p\big)$ be a coordinate system as in Notation 2.1.16 and let $\chi_i\in C_0^\infty\big(\mathcal{U}(p_i)\big)$, $1\le i\le N$ be a partition of unity as in Definition 2.1.18. For each $1\le i\le N$, let $\xi_i\in C^\infty(\partial\Omega\cap\mathcal{U}(p_i))$ be the function $\big|\nabla\zeta-\hat e_n\big|$ constructed in Problem 2.1.21 for patch number $i$ and also choose some $\tilde\chi_i\in C_0^\infty\big(\mathcal{U}(p_i)\big)$ that is identically one on the support of $\chi_i$. Then

$$\begin{aligned}\langle f,\bar g\rangle_{L^2(\partial\Omega)}&=\sum_{i,j}\langle\chi_i f,\chi_j\bar g\rangle_{L^2(\partial\Omega)}=\sum_{i,j}\langle\chi_i f,\tilde\chi_i\chi_j\bar g\rangle_{L^2(\partial\Omega)}\\&=\sum_{i,j}\big\langle(\chi_i f)\circ\psi_{p_i}^{-1},(\tilde\chi_i\chi_j\xi_i\bar g)\circ\psi_{p_i}^{-1}\big\rangle_{L^2(\mathbb{R}^{n-1})}\end{aligned}$$

By Proposition 2.1.8,

$$\begin{aligned}\big|\big\langle(\chi_i f)\circ\psi_{p_i}^{-1},(\tilde\chi_i\chi_j\xi_i\bar g)\circ\psi_{p_i}^{-1}\big\rangle_{L^2(\mathbb{R}^{n-1})}\big|&\le\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{-s,n-1}\big|(\tilde\chi_i\chi_j\xi_i\bar g)\circ\psi_{p_i}^{-1}\big|_{s,n-1}\\&\le\|f\|_{-s,\partial\Omega}\big|(\tilde\chi_i\chi_j\xi_i g)\circ\psi_{p_i}^{-1}\big|_{s,n-1}\\&=\|f\|_{-s,\partial\Omega}\big|(\tilde\chi_i\chi_j\xi_i g)\circ\psi_{p_j}^{-1}\circ(\psi_{p_j}\circ\psi_{p_i}^{-1})\big|_{s,n-1}\\&\le C'\|f\|_{-s,\partial\Omega}\big|(\chi_j g)\circ\psi_{p_j}^{-1}\big|_{s,n-1}\\&\le C'\|f\|_{-s,\partial\Omega}\|g\|_{s,\partial\Omega}\end{aligned}$$

For the second last inequality, we used Lemma 2.1.19, with $\varphi=(\tilde\chi_i\tilde\chi_j\xi_i)\circ\psi_{p_j}^{-1}$ and $\psi$ a $C^\infty$ diffeomorphism of $\mathbb{R}^{n-1}$ that coincides with $\psi_{p_i}\circ\psi_{p_j}^{-1}$ on $\psi_{p_j}\big(\mathcal{U}(p_i)\cap\mathcal{U}(p_j)\big)$. The desired bound now follows with $C_s=N^2C'$.

(b) Recall that

$$\|f\|_{-s,\partial\Omega}^2=\sum_{i=1}^N\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{-s,n-1}^2$$

Let, for each $1\le i\le N$, $g_i\in\mathcal{S}(\mathbb{R}^{n-1})$ be the function whose Fourier transform is $\big(1+|k|^2\big)^{-s}$ times the Fourier transform of $(\chi_i f)\circ\psi_{p_i}^{-1}$. Then

$$|g_i|_{s,n-1}=\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{-s,n-1}\le\|f\|_{-s,\partial\Omega}$$

and

$$\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{-s,n-1}^2=\big\langle(\chi_i f)\circ\psi_{p_i}^{-1},g_i\big\rangle_{L^2(\mathbb{R}^{n-1})}=\big\langle\chi_i f,\xi_i^{-1}(g_i\circ\psi_{p_i})\big\rangle_{L^2(\partial\Omega)}$$

where $\xi_i$ was defined in part (a). Thus

$$\|f\|_{-s,\partial\Omega}^2=\big\langle f,\bar G\big\rangle_{L^2(\partial\Omega)}\qquad\text{where}\qquad G=\sum_{i=1}^N\chi_i\xi_i^{-1}(\bar g_i\circ\psi_{p_i})$$

so that

$$\sup\left\{ \left| \langle f, \bar g\rangle_{L^2(\partial\Omega)} \right| \;\middle|\; g \in C^\infty(\partial\Omega),\ \|g\|_{s,\partial\Omega} \le 1 \right\} \ge \tfrac{\|f\|^2_{-s,\partial\Omega}}{\|G\|_{s,\partial\Omega}}$$

The proof is completed by verifying that

$$\begin{aligned}
\|G\|_{s,\partial\Omega} &\le N \max_{1\le i\le N} \|\chi_i \xi_i^{-1}(\bar g_i \circ \psi_{p_i})\|_{s,\partial\Omega} \\
&\le N^{3/2} \max_{1\le i,j\le N} \left| \left(\chi_j \chi_i \xi_i^{-1}(\bar g_i \circ \psi_{p_i})\right) \circ \psi_{p_j}^{-1} \right|_{s,n-1} \\
&\le c_s \max_{1\le i\le N} |\bar g_i|_{s,n-1} \\
&\le c_s \|f\|_{-s,\partial\Omega}
\end{aligned}$$

For the second last inequality, we used Lemma 2.1.19, with $\varphi = (\chi_i\chi_j\xi_i^{-1}) \circ \psi_{p_i}^{-1}$ and $\psi$ a $C^\infty$ diffeomorphism of $\mathbb{R}^{n-1}$ that coincides with $\psi_{p_j} \circ \psi_{p_i}^{-1}$ on $\psi_{p_i}\big(\mathcal{U}(p_i) \cap \mathcal{U}(p_j)\big)$. ■

**Problem 2.1.23** Let $0 \le s \in \mathbb{R}$. By Problem 2.1.22, if $f \in C^\infty(\partial\Omega)$ then

$$g \in C^\infty(\partial\Omega) \mapsto \mathcal{L}_f g = \langle f, \bar g\rangle_{L^2(\partial\Omega)}$$

extends to a bounded linear functional on $H^s(\partial\Omega)$ with norm bounded by $C_s\|f\|_{-s,\partial\Omega}$. Prove that the map $f \mapsto \mathcal{L}_f$ has a unique continuous extension to an isomorphism

$$\mathcal{L} : H^{-s}(\partial\Omega) \to H^s(\partial\Omega)^*$$

and that $\left\{ \mathcal{L}_f \;\middle|\; f \in C^\infty(\partial\Omega) \right\}$ is dense in $H^s(\partial\Omega)^*$.

**Solution.** By definition, $C^\infty(\partial\Omega)$ is dense in $H^s(\partial\Omega)$. So the existence of a unique continuous extension is an immediate consequence of the BLT theorem. Let $\|\cdot\|_{-s,\partial\Omega,*}$ denote the norm on $H^s(\partial\Omega)^*$. By Problem 2.1.22,

$$c_s^{-1}\|f\|_{-s,\partial\Omega} \le \|\mathcal{L}_f\|_{-s,\partial\Omega,*} \le C_s\|f\|_{-s,\partial\Omega}$$

for all $f \in C^\infty(\partial\Omega)$. These inequalities extend by continuity to all $f \in H^s(\partial\Omega)$. So $\mathcal{L}$ is an isometry onto a closed linear subspace of $H^s(\partial\Omega)^*$.

Now we prove the denseness. Since $H^s(\partial\Omega)$ is a Hilbert space, with the inner product given in Problem 2.1.19, the Riesz representation theorem says that, for each $\mathcal{L} \in H^s(\partial\Omega)^*$, there is a vector $u_{\mathcal{L}} \in H^s(\partial\Omega)$ such that $\mathcal{L}w = \langle w, u_{\mathcal{L}}\rangle_{s,\partial\Omega}$ for all $w \in H^s(\partial\Omega)$. The map $\mathcal{L} \mapsto u_{\mathcal{L}}$ is an isometry from $H^s(\partial\Omega)^*$ to $H^s(\partial\Omega)$. So if $\left\{ \mathcal{L}_f \;\middle|\; f \in C^\infty(\partial\Omega) \right\}$ is not dense in $H^s(\partial\Omega)^*$, then $\left\{ u_{\mathcal{L}_f} \;\middle|\; f \in C^\infty(\partial\Omega) \right\}$ is not dense in $H^s(\partial\Omega)$ and there is a nonzero vector $w \in H^s(\partial\Omega)$ that is orthogonal to $\left\{ u_{\mathcal{L}_f} \;\middle|\; f \in C^\infty(\partial\Omega) \right\}$. But

$$\begin{aligned} &w \perp \left\{ u_{\mathcal{L}_f} \;\middle|\; f \in C^\infty(\partial\Omega) \right\} \\ &\qquad\qquad \implies 0 = \left\langle w, u_{\mathcal{L}_f} \right\rangle_{s,\partial\Omega} = \mathcal{L}_f(w) = \mathcal{L}_w(f) \text{ for all } f \in C^\infty(\partial\Omega) \end{aligned}$$

Since $C^\infty(\partial\Omega)$ is dense in $H^s(\partial\Omega)$, $\mathcal{L}_w = 0$. So $w = 0$ as an element of $H^{-s}(\partial\Omega)$ and hence also as an element of $H^s(\partial\Omega)$. This contradicts the assumption that $\left\{ \mathcal{L}_f \;\middle|\; f \in C^\infty(\partial\Omega) \right\}$ is not dense in $H^s(\partial\Omega)^*$. This also completes the proof that $\mathcal{L}$ is surjective. ■

**Problem 2.1.24** Let $\ell \in \mathbb{N}_0$ and $s > \ell + \frac{n}{2}$. Prove that if $u \in H^s(\partial\Omega)$, then $u \in C^\ell(\partial\Omega)$ and there is a constant $C$, depending only on $\Omega$ and $|\ell|$, such that

$$\|u\|_{C^\ell(\partial\Omega)} \le C\|u\|_{s,\partial\Omega}$$

**Solution.** Using the notation of Definition 2.1.18, we have, by Problem 2.1.11,

$$\left\|(\chi_i u) \circ \psi_{p_i}^{-1}\right\|_{C^\ell(\mathbb{R}^{n-1})}^2 \le C^2 \left|(\chi_i u) \circ \psi_{p_i}^{-1}\right|_{s,n-1}^2 \le C^2 \|u\|_{s,\partial\Omega}^2$$

for each $1 \le i \le N$. The claim follows. ■

**Problem 2.1.25** Let $s$ and $s'$ be real numbers and $0 \le \mu \le 1$. Let $N$ be the number of neighbourhoods in the cover of $\partial\Omega$ used in Definition 2.1.18. Prove that

$$\|f\|_{\mu s + (1-\mu)s',\partial\Omega} \le N \|f\|_{s,\partial\Omega}^\mu \; \|u\|_{s',\partial\Omega}^{1-\mu}$$

for all $f \in H^{\max\{s,s'\}}(\partial\Omega)$.

**Solution.** Using the notation of Definition 2.1.18, we have, by Problem 2.1.12,

$$\begin{aligned} \|f\|_{\mu s+(1-\mu)s',\partial\Omega}^2 &= \sum_{i=1}^N \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{\mu s+(1-\mu)s',n-1}^2 \\ &\le \sum_{i=1}^N \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{s,n-1}^{2\mu} \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{s',n-1}^{2(1-\mu)} \\ &\le \left[\sum_{i=1}^N \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{s,n-1}^{2\mu}\right] \left[\sum_{i=1}^N \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{s',n-1}^{2(1-\mu)}\right] \\ &\le N \left[\sum_{i=1}^N \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{s,n-1}^2\right]^\mu \left[\sum_{i=1}^N \left|(\chi_i f) \circ \psi_{p_i}^{-1}\right|_{s',n-1}^2\right]^{1-\mu} \\ &= N \|f\|_{s,\partial\Omega}^{2\mu} \|f\|_{s',\partial\Omega}^{2(1-\mu)} \end{aligned}$$

For the third inequality, we used that, for any $\nu, A_1, \cdots, A_N \geq 0$,

$$[A_1 + \cdots + A_N]^\nu \leq [N \max\{A_1, \cdots, A_N\}]^\nu \leq N^\nu \max\{A_1, \cdots, A_N\}^\nu \leq N^\nu [A_1^\nu + \cdots + A_N^\nu]$$

■

**Problem 2.2.1** Let $k \in \mathbb{N}$ and $s \in \mathbb{R}$ obey $k < n$ and $s > \frac{k}{2}$. Identify $\mathbb{R}^{n-k}$ with $\{\, x \in \mathbb{R}^n \bigm| x_{n-k+1} = \cdots = x_n = 0 \,\}$ and write $x \in \mathbb{R}^n$ as $x = (x', y)$ with $x' \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. Prove that the linear transformation $r : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-k})$ defined by

$$(ru)(x') = u(x', 0)$$

has a unique extension to a bounded linear map

$$R : H^s(\mathbb{R}^n) \to H^{s-\frac{k}{2}}(\mathbb{R}^{n-k})$$

**Solution.** We shall prove that there is a constant $C$ (depending only on $k$ and $s$) such that

$$\|ru\|_{s-\frac{k}{2}, n-k} \leq C \|u\|_{s,n}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. The Lemma will then follow by the B.L.T. theorem.

If $p \in \mathbb{R}^n$, write $p = (p', q)$ with $p' = (p_1, \cdots, p_{n-k})$ and $q \in \mathbb{R}^k$. The definition of the Fourier transform gives

$$(\widehat{ru})(p') = \int \hat{u}(p', q) \ \tfrac{d^k q}{(2\pi)^k}$$

Thus, by Cauchy–Schwarz,

$$\begin{aligned}
\big|(\widehat{ru})(p')\big|^2 &= \Big| \int \hat{u}(p', q)\ \big(1 + p'^2 + q^2\big)^{s/2} \big(1 + p'^2 + q^2\big)^{-s/2}\ \tfrac{d^k q}{(2\pi)^k} \Big|^2 \\
&\leq \Big[ \int \tfrac{1}{(1+p'^2+q^2)^s}\ \tfrac{d^k q}{(2\pi)^k} \Big] \Big[ \int \big|\hat{u}(p', q)\big|^2\ \big(1 + p'^2 + q^2\big)^s\ \tfrac{d^k q}{(2\pi)^k} \Big] \\
&= \Big[ \tfrac{1}{(1+p'^2)^{s-k/2}} \int \tfrac{1}{(1+Q^2)^s}\ \tfrac{d^k Q}{(2\pi)^k} \Big] \Big[ \int \big|\hat{u}(p', q)\big|^2 \big(1 + p'^2 + q^2\big)^s\ \tfrac{d^k q}{(2\pi)^k} \Big] \\
&\hspace{8cm} \text{where } q = Q\sqrt{1 + p'^2} \\
&= \tfrac{C^2}{(1+p'^2)^{s-k/2}} \int \big|\hat{u}(p', q)\big|^2\ \big(1 + p'^2 + q^2\big)^s\ \tfrac{d^k q}{(2\pi)^k}
\end{aligned}$$

where the constant $C^2 = \int \frac{1}{(1+Q^2)^s} \frac{d^kQ}{(2\pi)^k}$ depends only on $s, k$ and is finite because $s > \frac{k}{2}$. Hence

$$\begin{aligned} |ru|^2_{s-\frac{k}{2},n-1} &= \int (1 + p'^2)^{s-\frac{k}{2}} |\widehat{ru}(p')|^2 \frac{d^{n-k}p'}{(2\pi)^{n-k}} \\ &\le C^2 \int \left|\hat{u}(p', q)\right|^2 \left(1 + p'^2 + q^2\right)^s \frac{d^np}{(2\pi)^n} \\ &= C^2 |u|^2_{s,n} \end{aligned}$$

■

**Problem 2.2.2** Let $\frac{1}{2} < s \le \ell \in \mathbb{N}$ and let $u \in C^\ell(\mathbb{R}^n)$. Suppose that each derivative of $u$ of order at most $\ell$ is bounded by a constant times $(1 + |x|)^{-\alpha}$ for some $\alpha > \frac{n}{2}$. Then $u \in H^\ell(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ and $u(x', 0) \in H^\ell(\mathbb{R}^{n-1}) \subset H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Prove that $(Ru)(x') = u(x', 0)$, where $R : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is the map of Lemma 2.2.5.

**Solution.** Let $C^\ell_d(\mathbb{R}^n)$ be the set of $u$'s specified in the statement of the problem. Define $\tilde{r} : C^\ell_d(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ by

$$(\tilde{r}u)(x') = u(x', 0)$$

The identical calculation as in Lemma 2.2.5 shows that $|\tilde{r}u|^2_{s-\frac{1}{2},n-1} \le C|u|^2_{s,n}$. Hence $\tilde{r}$ has a bounded linear extension, $\tilde{R}$, to $H^s(\mathbb{R}^n)$. Since $\tilde{R}$ also extends $r$ and the extension of $r$ is unique, $\tilde{R} = R$. ■

**Problem 2.2.3** Let $s > \frac{1}{2}$. Define, for each $t \in \mathbb{R}$, $R_t : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ to be the unique bounded linear extension of the linear transformation $r_t : C^\infty_0(\mathbb{R}^n) \to C^\infty_0(\mathbb{R}^{n-1})$ defined by

$$(ru)(x') = u(x', t)$$

Prove that $R_t$ is strongly continuous in $t$. That is, prove that

$$\lim_{t \to t_0} |R_t f - R_{t_0} f|^2_{s-\frac{1}{2},n-1} = 0$$

for each $t_0 \in \mathbb{R}$ and each $f \in H^s(\mathbb{R}^n)$.

**Solution.** As in the proof of Lemma 2.2.5,

$$(\widehat{r_t u})(k') = \int e^{ik_n t} \hat{u}(k', k_n) \frac{dk_n}{2\pi}$$

and

$$\left|r_t u - r_{t_0} u\right|^2_{s-\frac{1}{2},n-1} \le C^2 \int \left|\left[e^{ik_n t} - e^{ik_n t_0}\right]\hat{u}(k',k_n)\right|^2 \left(1+k'^2+k_n^2\right)^s \tfrac{d^n k}{(2\pi)^n}$$

Since both sides are continuous in $u$, for each fixed $t$ and $t_0$, we also have that

$$\left|R_t f - R_{t_0} f\right|^2_{s-\frac{1}{2},n-1} \le C^2 \int \left|\left[e^{ik_n t} - e^{ik_n t_0}\right]\hat{f}(k',k_n)\right|^2 \left(1+k'^2+k_n^2\right)^s \tfrac{d^n k}{(2\pi)^n}$$

for all $f \in H^s(\mathbb{R}^n)$. The right hand side converges to zero as $t$ tends to $t_0$ by the Lebesgue dominated convergence theorem.

■

**Problem 2.2.4** Let $\ell \in \mathbb{N}_0$, $\varepsilon, \varepsilon', R > 0$ and $u \in H^\ell(\mathbb{R}^n_+)$. Suppose that $u(x) = 0$ for all $|x| \ge R$. Show that there is a function $v \in C_0^\infty(\mathbb{R}^n)$ that obeys $v(x) = 0$ for all $|x| \ge R+\varepsilon'$ and $\|u - P_+ v\|_{\ell,\mathbb{R}^n_+} < \varepsilon$, where $P_+$ is the restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_+$.

**Solution.** Let $\varphi(x)$ be a $C^\infty$ function on $\mathbb{R}^n$ that is identically one for $|x| \le R$ and 0 for $|x| \ge R+\varepsilon'$. By Lemma 2.2.7, there is a sequence of functions $v_j \in C_0^\infty(\mathbb{R}^n)$ such that $u = \lim_{j\to\infty} P_+ v_j$ in $H^\ell(\mathbb{R}^n_+)$. Multiplication by $\varphi$ is a bounded operator in $H^\ell(\mathbb{R}^n_+)$ and $\varphi(x)u(x) = u(x)$ for all $x$ in $\mathbb{R}^n_+$. Hence $u = \varphi u = \lim_{j\to\infty} \varphi P_+ v_j = \lim_{j\to\infty} P_+ \varphi v_j$ in $H^\ell(\mathbb{R}^n_+)$. It suffices to choose $v = \varphi v_j$, for $j$ sufficiently large. ■

**Problem 2.2.5** Let $\{x_i\}_{i\in\mathbb{N}}$ be any sequence of complex numbers. Define the Vandermonde determinant

$$V_n = \det\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

(a) Use row and column operations to prove that $V_n = V_{n-1}\prod_{1\le i<n}\big(x_n - x_i\big)$.

(b) Prove that $V_n = \prod_{1\le i<j\le n}\big(x_j - x_i\big)$.

**Solution.** (a) First subtract the last column from each of the other columns.

$$V_n = \det\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ x_1 - x_n & x_2 - x_n & \cdots & x_{n-1} - x_n & x_n \\ x_1^2 - x_n^2 & x_2^2 - x_n^2 & \cdots & x_{n-1}^2 - x_n^2 & x_n^2 \\ \vdots & \vdots & \ddots & \vdots & \\ x_1^{n-1} - x_n^{n-1} & x_2^{n-1} - x_n^{n-1} & \cdots & x_{n-1}^{n-1} - x_n^{n-1} & x_n^{n-1} \end{vmatrix}$$

Next expand along the first row.

$$V_n = (-1)^{n-1} \det \begin{vmatrix} x_1 - x_n & x_2 - x_n & \cdots & x_{n-1} - x_n \\ x_1^2 - x_n^2 & x_2^2 - x_n^2 & \cdots & x_{n-1}^2 - x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} - x_n^{n-1} & x_2^{n-1} - x_n^{n-1} & \cdots & x_{n-1}^{n-1} - x_n^{n-1} \end{vmatrix}$$

Next factor $(x_j - x_n)$ out of column $j$, for each of $j = 1, \cdots, n-1$.

$$\begin{aligned} V_n &= (-1)^{n-1} \prod_{1 \le i < n} \big(x_i - x_n\big) \det \begin{vmatrix} \cdots & 1 & \cdots \\ \cdots & x_j + x_n & \cdots \\ \cdots & x_j^2 + x_j x_n + x_n^2 & \cdots \\ \ddots & \vdots & \ddots \\ \cdots & x_j^{n-2} + x_j^{n-3} x_n + \cdots + x_j x_n^{n-3} + x_n^{n-2} & \cdots \end{vmatrix} \\ &= \prod_{1 \le i < n} \big(x_n - x_i\big) \det \begin{vmatrix} \cdots & 1 & \cdots \\ \cdots & x_j + x_n & \cdots \\ \cdots & x_j^2 + x_j x_n + x_n^2 & \cdots \\ \ddots & \vdots & \ddots \\ \cdots & x_j^{n-2} + x_j^{n-3} x_n + \cdots + x_j x_n^{n-3} + x_n^{n-2} & \cdots \end{vmatrix} \end{aligned}$$

Finally, subtract $x_n$ times the first row from the second row and then subtract $x_n^2$ times the first row and $x_n$ times the second row form the third row and so on.

$$\begin{aligned} V_n &= \prod_{1 \le i < n} \big(x_n - x_i\big) \det \begin{vmatrix} \cdots & 1 & \cdots \\ \cdots & x_j & \cdots \\ \cdots & x_j^2 + x_j x_n + x_n^2 & \cdots \\ \ddots & \vdots & \ddots \\ \cdots & x_j^{n-2} + x_j^{n-3} x_n + \cdots + x_j x_n^{n-3} + x_n^{n-2} & \cdots \end{vmatrix} \\ &= \prod_{1 \le i < n} \big(x_n - x_i\big) \det \begin{vmatrix} \cdots & 1 & \cdots \\ \cdots & x_j & \cdots \\ \cdots & x_j^2 & \cdots \\ \ddots & \vdots & \ddots \\ \cdots & x_j^{n-2} + x_j^{n-3} x_n + \cdots + x_j x_n^{n-3} + x_n^{n-2} & \cdots \end{vmatrix} \\ &= \cdots = \prod_{1 \le i < n} \big(x_n - x_i\big) \det \begin{vmatrix} \cdots & 1 & \cdots \\ \cdots & x_j & \cdots \\ \cdots & x_j^2 & \cdots \\ \ddots & \vdots & \ddots \\ \cdots & x_j^{n-2} & \cdots \end{vmatrix} \\ &= V_{n-1} \prod_{1 \le i < n} \big(x_n - x_i\big) \end{aligned}$$

(b) For $n = 2$

$$V_2 = \det \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

Now just proceed by induction, using part (a). ■

**Problem 2.2.6** Let $\ell \in \mathbb{N}_0$ and $\ell' \in \mathbb{Z}$ with $\ell' \le \ell$. Denote by $E_\ell$ the operator of Lemma 2.2.8. Prove that there is a constant $C$ such that

$$|\!|\!| E_\ell u |\!|\!|_{\ell',n} \le C \| u \|_{\ell', \mathbb{R}^n_+}$$

for all $u \in H^\ell(\mathbb{R}^n_+)$.

**Solution.** We have already shown, in the proof of Lemma 2.2.8, that there is a constant $C$ such that

$$\| \partial^\alpha E_\ell u \|_{L^2(\mathbb{R}^n)} \le C \| \partial^\alpha u \|_{L^2(\mathbb{R}^n_+)}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and $u \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$. Squaring both sides, summing over $\alpha$ with $|\alpha| \le \ell'$ and then taking the square root gives

$$|\!|\!| E_\ell u |\!|\!|_{\ell',n} \le C \| u \|_{\ell', \mathbb{R}^n_+}$$

for all $u \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$ and $0 \le \ell' \le \ell$. By Lemma 2.2.7, $C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$ is dense in $H^\ell(\mathbb{R}^n_+)$. This proves the desired bound for all $0 \le \ell' \le \ell$.

Now consider $\ell' < 0$. By part (b) of Problem 2.1.15,

$$|\!|\!| E_\ell u |\!|\!|_{\ell',n} = \sup_{\substack{v \in H^{-\ell'}(\mathbb{R}^n) \\ |\!|\!| v |\!|\!|_{-\ell',n} \le 1}} \Big| \langle v, E_\ell u \rangle_{L^2(\mathbb{R}^n)} \Big|$$

Write $x = (x', x_n) \in \mathbb{R}^n$ with $x' = (x_1, \cdots, x_{n-1})$. By (2.2.1), if $u \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$, then

$$\begin{aligned} \langle v, E_\ell u \rangle_{L^2(\mathbb{R}^n)} &= \sum_{j=1}^{\ell+1} \beta_j \Big\{ \int_{x_n < 0} v(x) \overline{u\big(x', -\tfrac{x_n}{j}\big)} \, dx + \int_{x_n > 0} v(x) \overline{u(x)} \, dx \Big\} \\ &= \sum_{j=1}^{\ell+1} \beta_j \Big\{ j \int_{x_n > 0} v(x', -j x_n) \overline{u(x)} \, dx + \int_{x_n > 0} v(x) \overline{u(x)} \, dx \Big\} \\ &= \sum_{j=1}^{\ell+1} \beta_j \Big\{ j \, \langle Q_j v, u \rangle_{L^2(\mathbb{R}^n_+)} + \langle P_+ v, u \rangle_{L^2(\mathbb{R}^n_+)} \Big\} \end{aligned}$$

where $P_+$ restricts to $\mathbb{R}^n_+$ and $(Q_jv)(x) = v(x', -jx_n)$. As in (2.2.2), if $v \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{x_n>0} \left|\tfrac{\partial^\alpha}{\partial x^\alpha} v\big(x', -jx_n\big)\right|^2 \, d^n x = \tfrac{1}{j} j^{2\alpha_n} \int_{x_n<0} \left|\partial^\alpha v(x)\right|^2 \, d^n x \le j^{2\alpha_n - 1} |v|^2_{|\alpha|,n}$$

so that

$$\|Q_j v\|^2_{-\ell',\mathbb{R}^n_+} \le n^{-\ell'+1} j^{-2\ell'-1} |v|^2_{-\ell',n}$$

for all $v \in C_0^\infty(\mathbb{R}^n)$ and hence for all $v \in H^{-\ell'}(\mathbb{R}^n)$. By Remark 2.1.12,

$$\begin{aligned} \big|\, \langle v, E_\ell u\rangle_{L^2(\mathbb{R}^n)} \,\big| &\le \sum_{j=1}^{\ell+1} |\beta_j| \Big\{ j \|Q_j v\|_{-\ell',\mathbb{R}^n_+} \|u\|_{\ell',\mathbb{R}^n_+} + \|P_+ v\|_{-\ell',\mathbb{R}^n_+} \|u\|_{\ell',\mathbb{R}^n_+} \Big\} \\ &\le \sum_{j=1}^{\ell+1} |\beta_j| \Big\{ j\sqrt{n^{-\ell'+1} j^{-2\ell'-1}} + 1 \Big\} |v|_{-\ell',n} \|u\|_{\ell',\mathbb{R}^n_+} \end{aligned}$$

Setting $C = \sum_{j=1}^{\ell+1} |\beta_j| \Big\{ j\sqrt{n^{-\ell'+1} j^{-2\ell'-1}} + 1 \Big\}$, we have $|E_\ell u|_{\ell',n} \le C\|u\|_{\ell',\mathbb{R}^n_+}$ for all $u \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$ and hence for all $u \in H^\ell(\mathbb{R}^n_+)$. ■

**Problem 2.2.7** Let $\ell', \ell \in \mathbb{N}_0$ with $\ell' \le \ell$ and $\alpha \in \mathbb{N}_0^n$ with $\alpha_n = 0$. Denote by $E_\ell$ the operator of Lemma 2.2.8. By Problem 2.2.6, $E_\ell$ has a unique bounded extension to $H^{\ell'}(\mathbb{R}^n_+)$ and it in turn has a unique bounded extension to $H^{\ell'-|\alpha|}(\mathbb{R}^n_+)$. We persist in denoting both of them $E_\ell$. Recall, from Lemma 2.1.14, that $\partial^\alpha$ is a bounded operator from $H^{\ell'}(\mathbb{R}^n_+)$ to $H^{\ell'-|\alpha|}(\mathbb{R}^n_+)$ and from $H^{\ell'}(\mathbb{R}^n)$ to $H^{\ell'-|\alpha|}(\mathbb{R}^n)$. Let $u \in H^{\ell'}(\mathbb{R}^n_+)$. Prove $\partial^\alpha E_\ell u = E_\ell \partial^\alpha u$ and that $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$ if and only if $\partial^\alpha E_\ell u \in H^\ell(\mathbb{R}^n)$.

**Solution.** If $u$ is the restriction to $\mathbb{R}^n_+$ of a function in $C_0^\infty(\mathbb{R}^n)$, then since $\alpha_n = 0$, $\partial^\alpha E_\ell u = E_\ell \partial^\alpha u$ by (2.2.1). Such functions are dense in $H^{\ell'}(\mathbb{R}^n_+)$ by Lemma 2.2.7. Since the operators

$$\begin{aligned} E_\ell &: H^{\ell'}(\mathbb{R}^n_+) \to H^{\ell'}(\mathbb{R}^n) & E_\ell &: H^{\ell'-|\alpha|}(\mathbb{R}^n_+) \to H^{\ell'-|\alpha|}(\mathbb{R}^n) \\ \partial^\alpha &: H^{\ell'}(\mathbb{R}^n_+) \to H^{\ell'-|\alpha|}(\mathbb{R}^n_+) & \partial^\alpha &: H^{\ell'}(\mathbb{R}^n) \to H^{\ell'-|\alpha|}(\mathbb{R}^n) \end{aligned}$$

are all bounded, $\partial^\alpha E_\ell u = E_\ell \partial^\alpha u$ for all $u \in H^{\ell'}(\mathbb{R}^n_+)$.

If $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$, then $E_\ell \partial^\alpha u$ and hence $\partial^\alpha E_\ell u$ are in $H^\ell(\mathbb{R}^n)$. Conversely, if $\partial^\alpha E_\ell u \in H^\ell(\mathbb{R}^n)$, then $E_\ell \partial^\alpha u \in H^\ell(\mathbb{R}^n)$. As $\partial^\alpha u$ is the restriction to $\mathbb{R}^n_+$ of $E_\ell \partial^\alpha u$, $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$. ■

**Problem 2.2.8**

(a) Let $\{a_\ell\}_{\ell\in\mathbb{N}_0}$ be any sequence of real numbers. Prove that there is a function $f \in C^\infty(\mathbb{R})$ such that $f^{(m)}(0) = a_m$ for all $m \in \mathbb{N}_0$.

(b) Let $\{a_\ell(x')\}_{\ell\in\mathbb{N}_0}$ be any sequence of $C^\infty$ functions on $\mathbb{R}^{n-1}$. Prove that there is a function $f \in C^\infty(\mathbb{R}^n)$ such that $\frac{\partial^m f}{\partial x_n^m}(x', 0) = a_m(x')$ for all $m \in \mathbb{N}_0$ and $x' \in \mathbb{R}^{n-1}$.

(c) Prove that if $f \in C^\infty(\overline{\mathbb{R}^n_+})$, then there exists $F \in C^\infty(\mathbb{R}^n)$ such that $f(x) = F(x)$ for all $\mathbb{R}^n_+$.

**Solution.** (a) Let $\{A_\ell\}_{\ell\in\mathbb{N}_0}$ be any sequence of real numbers with $A_\ell \geq |a_\ell|$ for all $\ell \in \mathbb{N}_0$. Let $\varphi \in C_0^\infty(\mathbb{R})$ obey $\varphi(x) = 1$ for all $|x| < \frac{1}{2}$ and $\varphi(x) = 0$ for all $|x| \geq 1$ and set, for each $n \in \mathbb{N}_0$,

$$f_\ell(x) = \tfrac{1}{\ell!} a_\ell x^\ell \varphi(A_\ell x)$$

Then $f_\ell \in C^\infty(\mathbb{R})$ and, since all derivatives of $\varphi(A_\ell x)$ vanish at $x = 0$, except for the $0^{\text{th}}$ which takes the value 1,

$$f_\ell^{(m)}(0) = \begin{cases} a_m & \text{if } m = \ell \\ 0 & \text{if } m \neq \ell \end{cases}$$

Furthermore, for all $\ell > m$ and all $x \in \mathbb{R}$

$$\begin{aligned}
\big|f_\ell^{(m)}(x)\big| &= \Big| \sum_{j=0}^{m} \tfrac{1}{(\ell-m+j)!} \tbinom{m}{j} a_\ell x^{\ell-m+j} A_\ell^j \varphi^{(j)}(A_\ell x) \Big| \\
&\leq \tfrac{1}{(\ell-m)!} \sum_{j=0}^{m} \tbinom{m}{j} |x|^{\ell-m+j} A_\ell^{j+1} \big|\varphi^{(j)}(A_\ell x)\big| \\
&\leq \tfrac{1}{(\ell-m)!} \sum_{j=0}^{m} \tbinom{m}{j} |x|^{\ell-m-1} \big|\varphi^{(j)}(A_\ell x)\big| \\
&\leq C_m \tfrac{1}{(\ell-m)!} |x|^{\ell-m-1}
\end{aligned}$$

where $C_m = \sum_{0\leq j\leq m} \binom{m}{j} \|\varphi^{(j)}\|_\infty$. Hence, for every fixed $m \in \mathbb{N}_0$ the series

$$\sum_{\ell=0}^{\infty} f_\ell^{(m)}(x) = \sum_{\ell=0}^{m} f_\ell^{(m)}(x) + \sum_{\ell=m+1}^{\infty} f_\ell^{(m)}(x)$$

converges absolutely and uniformly on all compact subsets of $\mathbb{R}$. As a result, the series $\sum_{\ell=0}^\infty f_\ell(x)$ converges to $f \in C^\infty(\mathbb{R})$ and, for all $m \in \mathbb{N}_0$

$$f^{(m)}(x) = \sum_{\ell=0}^{\infty} f_\ell^{(m)}(x)$$

In particular,

$$f^{(m)}(0) = \sum_{\ell=0}^{\infty} f_\ell^{(m)}(0) = a_m$$

(b) The proof is similar to that of part (a), but this time we choose $\{A_\ell\}_{\ell\in\mathbb{N}_0}$ to be a sequence of real numbers with

$$A_\ell \geq \sup_{\substack{\alpha'\in\mathbb{N}_0^{n-1},\ x'\in\mathbb{R}^{n-1} \\ |\alpha'|\leq\ell,\ |x'|\leq\ell}} |\partial^{\alpha'} a_\ell(x')|$$

for all $\ell \in \mathbb{N}_0$ and set, for $\ell \in \mathbb{N}_0$, $x' \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$,

$$f_\ell(x', x) = \tfrac{1}{\ell!} a_\ell(x') x^\ell \varphi(A_\ell x)$$

Then, if $\alpha = \big(\alpha', m\big)$ with $\alpha' \in \mathbb{N}_0^{n-1}$, $m \in \mathbb{N}_0$ and $\ell \geq \max\{m+1, |\alpha'|, |x'|\}$

$$\begin{aligned}
\big|\partial^\alpha f_\ell(x', x)\big| &= \left|\sum_{j=0}^{m} \tfrac{1}{(\ell-m+j)!}\tbinom{m}{j}\ \partial^{\alpha'} a_\ell(x')\ x^{\ell-m+j} A_\ell^j \varphi^{(j)}(A_\ell x)\right| \\
&\leq \tfrac{1}{(\ell-m)!} \sum_{j=0}^{m} \tbinom{m}{j} |x|^{\ell-m+j} A_\ell^{j+1} \big|\varphi^{(j)}(A_\ell x)\big| \\
&\leq \tfrac{1}{(\ell-m)!} \sum_{j=0}^{m} \tbinom{m}{j} |x|^{\ell-m-1} \big|\varphi^{(j)}(A_\ell x)\big| \\
&\leq C_m \tfrac{1}{(\ell-m)!} |x|^{\ell-m-1}
\end{aligned}$$

Once again, for every fixed $\alpha \in \mathbb{N}_0^n$ the series $\sum_{\ell=0}^\infty \partial^\alpha f_\ell(x', x)$ converges absolutely and uniformly on all compact subsets of $\mathbb{R}^n$. As a result, the series $\sum_{\ell=0}^\infty f_\ell(x', x)$ converges to $f \in C^\infty(\mathbb{R}^n)$ and, for all $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f(x', x) = \sum_{\ell=0}^{\infty} \partial^\alpha f_\ell(x', x)$$

so that, for all $m \in \mathbb{N}_0$,

$$\tfrac{\partial^m f}{\partial x_n^m}(x', 0) = \sum_{\ell=0}^{\infty} \tfrac{\partial^m f_\ell}{\partial x_n^m}(x', 0) = a_m(x')$$

(c) Since $f \in C^\infty(\overline{\mathbb{R}_+^n})$, $a_m(x') = \lim_{x\to 0+} \frac{\partial^m f}{\partial x_n^m}(x', x)$ exists and is in $C^\infty(\mathbb{R}^{n-1})$ for all $m \in \mathbb{N}_0$. By part (b), there is a function $g \in C^\infty(\mathbb{R}^n)$ such that $\frac{\partial^m g}{\partial x_n^m}(x', 0) = a_m(x')$ for all $m \in \mathbb{N}_0$ and $x' \in \mathbb{R}^{n-1}$. Then

$$F(x) = \begin{cases} f(x) & \text{if } x_n > 0 \\ g(x) & \text{if } x_n \leq 0 \end{cases}$$

does the job. ■

**Problem 2.2.9** Let $\ell \in \mathbb{N}$ and let $\Omega = \left\{ x \in \mathbb{R}^n \;\middle|\; |x| < 2,\ |x| \neq 1 \right\}$. Prove that $C^\infty(\overline{\Omega})$ is NOT dense in $H^\ell(\Omega)$.

**Solution.** Let

$$u(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

and suppose that $\{u_i\}_{i\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ converges in $H^\ell(\Omega)$ to $u$. Set

$$\Omega_+ = \left\{ x \in \Omega \;\middle|\; |x| > 1 \right\} \qquad \Omega_- = \left\{ x \in \Omega \;\middle|\; |x| < 1 \right\} \qquad \Omega_1 = \left\{ x \in \mathbb{R}^n \;\middle|\; |x| = 1 \right\}$$

and let $R_+$ and $R_-$ be the operators of Theorem 2.2.2 applied to $\Omega_+$ and $\Omega_-$ respectively. Since they are bounded

$$\lim_{i\to\infty} R_+ u_i = R_+ u = 1 \qquad \lim_{i\to\infty} R_- u_i = R_- u = 0$$

In particular, since $\Omega_1$ is contained in both $\partial\Omega_+$ and $\partial\Omega_-$,

$$\lim_{i\to\infty} u_i \restriction \Omega_1 = 1 \qquad \lim_{i\to\infty} u_i \restriction \Omega_1 = 0$$

These conclusions are contradictory. ■

**Problem 2.2.10** Let $\alpha \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$ obey $\ell > |\alpha| + \frac{n}{2}$. Prove that if $u \in H^\ell(\Omega)$, then $\partial^\alpha u$ is continuous on $\overline{\Omega}$ and there is a constant $C$, depending only on $\Omega$, $\ell$ and $|\alpha|$, such that

$$\sup_{x\in\overline{\Omega}} \left|\partial^\alpha u(x)\right| \le C\|u\|_{\ell,\Omega}$$

**Solution.** Let $\Omega'$ be the set of all points in $\mathbb{R}^n$ whose distance from $\Omega$ is less than one and let $\mathcal{O}$ be the interior of $\mathbb{R}^n \setminus \mathcal{O}'$. Let $E : H^\ell(\Omega) \to H^\ell(\mathbb{R}^n)$ be the bounded linear operator of Lemma 2.2.12. Then $Eu \in H^\ell(\mathbb{R}^n)$ and, by Problem 2.1.11, $\partial^\alpha Eu$ may be chosen continuous and there is a constant $C'$, depending only on $\ell$, $n$ and $|\alpha|$, such that

$$\sup_{x\in\mathbb{R}^n} \left|\partial^\alpha Eu(x)\right| \le C' \left|Eu\right|_{\ell,n} \le C\|u\|_{\ell,\Omega}$$

As we may choose a representative for $u$ that coincides with $Eu$ on $\overline{\Omega}$, the proof is complete. ■

**Problem 2.2.11** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Define the restriction map

$$r : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$$
$$u \mapsto u \restriction \partial\Omega$$

Prove that there does NOT exist a bounded map

$$R : L^2(\Omega) \to L^2(\partial\Omega)$$

such that $Ru = ru$ for all $u \in C^\infty(\overline{\Omega})$.

**Solution.** Suppose that $R$ existed. Let $u$ be the function that is identically one on $\overline{\Omega}$. Then $ru$ is identically one on $\partial\Omega$. But $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$. So there is a sequence of functions $u_i \in C_0^\infty(\Omega)$ that converges in $L^2(\Omega)$ to $u$. For example, one can choose $u_i$ to be a $C^\infty$ function that takes values in $[0,1]$, is identically one on the part of $\Omega$ that is farther than $\frac{1}{2^i}$ from $\partial\Omega$ and is identically zero on the part of $\Omega$ that is of distance at most $\frac{1}{2^{2i}}$ from $\partial\Omega$. Then $Ru_i = ru_i = 0$. Since $R$ is continuous $Ru = \lim_{i\to\infty} Ru_i = 0$. $\Rightarrow\!\Leftarrow$ ■

**Problem 2.3.1** Let $t < s$ and let $A$ be a bounded linear map on $H^t(\mathbb{R}^m)$. Prove that if there is a constant $M$ such that

$$\Big| \int_{\mathbb{R}^m} \widehat{Af}(k)\overline{\hat{g}}(k) \ \tfrac{d^m k}{(2\pi)^m} \Big| \le M \,|f|_s \,|g|_{-s}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^m)$ then $A$ is a bounded linear map $H^s(\mathbb{R}^m)$ with norm at most $M$.

**Solution.** By Problem 2.1.15, if $f \in \mathcal{S}(\mathbb{R}^m)$ then $Af \in H^s(\mathbb{R}^m)$ and $|Af|_s \le M|f|_s$.

Now let $f \in H^s(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^m)$ is dense in $H^s(\mathbb{R}^m)$ there is a sequence of functions $f_i \in \mathcal{S}(\mathbb{R}^m)$ that converge in $H^s(\mathbb{R}^m)$ to $f$. As the sequence $\{f_i\}_{i\in\mathbb{N}}$ is Cauchy in $H^s(\mathbb{R}^m)$, the sequence $\{Af_i\}_{i\in\mathbb{N}}$ is also Cauchy in $H^s(\mathbb{R}^m)$ and hence converges to some $g \in H^s(\mathbb{R}^m)$ that obeys $|g|_s \le M|f|_s$. The sequences $\{f_i\}_{i\in\mathbb{N}}$ and $\{Af_i\}_{i\in\mathbb{N}}$ also converge to $f$ and $g$, respectively, in $H^t(\mathbb{R}^m)$. Since $A$ is continuous in $H^t(\mathbb{R}^m)$, $Af = g$. Thus $Af \in H^s(\mathbb{R}^m)$ and $|Af|_s \le M|f|_s$. ■

**Problem 2.3.2** Let $s \in \mathbb{R}$ and $A$ be a bounded linear map on $H^s(\mathbb{R}^m)$. Let $f, g \in \mathcal{S}(\mathbb{R}^m)$ and set, for $z \in \mathbb{C}$, $\sigma(z) = (1-z)s_0 + zs_1$. Define

$$F(z) = \int_{\mathbb{R}^m} \big(1+|k|^2\big)^{\frac{\sigma(z)-s}{2}} \widehat{Af_z}(k)\overline{\hat{g}}(k) \ \tfrac{d^m k}{(2\pi)^m}$$

where $f_z \in \mathcal{S}(\mathbb{R}^m)$ is determined by

$$\hat{f}_z(k) = \big(1+|k|^2\big)^{\frac{s-\sigma(z)}{2}} \hat{f}(k)$$

(a) Prove that $\frac{f_{z'}-f_z}{z'-z}$ converges in $H^s(\mathbb{R}^m)$ as $z' \to z$.

(b) Prove that $F(z)$ is an entire function of $z$.

**Solution.** (a) Observe that

$$\begin{aligned}\big(1+|k|^2\big)^{\frac{s-\sigma(z')}{2}} - \big(1+|k|^2\big)^{\frac{s-\sigma(z)}{2}} &= \big(1+|k|^2\big)^{\frac{s-\sigma(z)}{2}} \Big[\big(1+|k|^2\big)^{\frac{\sigma(z)-\sigma(z')}{2}} - 1\Big] \\ &= \big(1+|k|^2\big)^{\frac{s-\sigma(z)}{2}} \Big[\big(1+|k|^2\big)^{\frac{(z-z')(s_1-s_0)}{2}} - 1\Big]\end{aligned}$$

By integration by parts and the fundamental theorem of calculus

$$G(1) - G(0) - G'(0) = \int_0^1 (1-t)G''(t)\,dt$$

In particular, with $G(t) = \frac{1}{\zeta}e^{ta\zeta}$,

$$\Big|\tfrac{e^{a\zeta}-1}{\zeta} - a\Big| = \Big|\int_0^1 (1-t)\tfrac{1}{\zeta}\tfrac{d^2}{dt^2}e^{ta\zeta}\ dt\ \Big| = \Big|a^2\zeta\int_0^1 (1-t)e^{ta\zeta}\ dt\ \Big| \le \tfrac{1}{2}|a^2\zeta|e^{|a\zeta|}$$

Applying this with $\zeta = z'-z$ and $a = \frac{s_1-s_0}{2}\ln\big(1+|k|^2\big)$ and writing

$$\hat{f}'_z(k) = \tfrac{s_1-s_0}{2}\ln\big(1+|k|^2\big)\ \big(1+|k|^2\big)^{\frac{s-\sigma(z)}{2}} \hat{f}(k)$$

we have

$$\Big|\tfrac{\hat{f}_{z'}(k)-\hat{f}_z(k)}{z'-z} - \hat{f}'_z(k)\Big| \le \tfrac{1}{2}a^2|z'-z|e^{a|z'-z|}\big(1+|k|^2\big)^{\frac{s-\mathrm{Re}\,\sigma(z)}{2}}\big|\hat{f}(k)\big|$$

For $z, z'$ in any compact subset of $\mathbb{C}$, the right hand side is bounded by a constant times some integer power of $\big(1+|k|^2\big)$ times $\big|\hat{f}(k)\big|$ times $|z'-z|$. As $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\int \big(1+|k|^2\big)^M\ \big|\hat{f}(k)\big|^2\ d^mk < \infty$$

for all $M \in \mathbb{N}$. Hence $\big\|\frac{f_{z'}-f_z}{z'-z} - f'_z\big\|_s \le \mathrm{const}\,|z'-z|$ and

$$\lim_{z'\to z}\Big\|\tfrac{f_{z'}-f_z}{z'-z} - f'_z\Big\|_s = 0$$

(b) Write $\hat{g}_z(k) = \big(1+|k|^2\big)^{\frac{\sigma(z)-s}{2}}\hat{g}(k)$. Then

$$\tfrac{F(z')-F(z)}{z'-z} = \int_{\mathbb{R}^m} \tfrac{\widehat{Af_{z'}}(k)-\widehat{Af_z}(k)}{z'-z}\overline{\hat{g}_{\bar{z}'}(k)}\ \tfrac{d^mk}{(2\pi)^m} + \int_{\mathbb{R}^m} \widehat{Af_z}(k)\overline{\tfrac{\hat{g}_{\bar{z}'}(k)-\hat{g}_{\bar{z}}(k)}{\bar{z}'-\bar{z}}}\ \tfrac{d^mk}{(2\pi)^m}$$

By part (a), $\frac{f_{z'}-f_z}{z'-z}$ converges in $H^s(\mathbb{R}^m)$. By hypothesis, $A$ is a bounded linear map on $H^s(\mathbb{R}^m)$. Hence $\frac{Af_{z'}-Af_z}{z'-z}$ converges in $H^s(\mathbb{R}^m)$ as $z' \to z$. As in part (a), $\frac{g_{\bar{z}'}-g_{\bar{z}}}{\bar{z}'-\bar{z}}$ and $g_{z'}$ converge in $H^{-s}(\mathbb{R}^m)$ as $z' \to z$. The claim now follows by the bound of part (a) of Proposition 2.1.8. ■

**Problem 2.3.3** Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $1 \le p < q < r \le \infty$. Prove that if $f \in L^p(\Omega) \cap L^r(\Omega)$, then $f \in L^q(\Omega)$ and

$$\|f\|_{L^q(\Omega)} \le \|f\|_{L^p(\Omega)}^{1-t} \|f\|_{L^r(\Omega)}^t$$

where $0 < t < 1$ is determined by $\frac{1}{q} = \frac{1-t}{p} + \frac{t}{r}$ .

**Solution.** If $r = \infty$, so that $t = 1 - \frac{p}{q}$, we just factor

$$|f(x)|^q = |f(x)|^{q-p} |f(x)|^p \le \|f\|_{L^\infty(\Omega)}^{tq} |f(x)|^p$$

integrate both sides and take the $q^{\text{th}}$ root.

If $r$ is finite, we use Hölder with $P = \frac{p}{(1-t)q}$ and

$$Q = \left[1 - \frac{(1-t)q}{p}\right]^{-1} = \left[1 - q\left(\frac{1}{q} - \frac{t}{r}\right)\right]^{-1} = \frac{r}{tq}$$

This gives

$$\int_\Omega |f(x)|^q \ d^n x = \int_\Omega |f(x)|^{(1-t)q} |f(x)|^{tq} \ d^n x \le \|f\|_{L^p(\Omega)}^{(1-t)q} \|f\|_{L^r(\Omega)}^{tq}$$

Taking the $q^{\text{th}}$ root gives the desired bound. ■

**Problem 2.3.4** The goal of this problem is to prove the Riesz–Thorin interpolation theorem, which is as follows. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and, for $0 \le t \le 1$, define $p_t$, $q_t$ by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

Suppose that $T$ is a linear transformation from $L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ to $L^{q_0}(\Omega) \cap L^{q_1}(\Omega)$ which satisfies

$$\|Tf\|_{L^{q_0}(\Omega)} \le M_0 \|f\|_{L^{p_0}(\Omega)} \qquad \|Tf\|_{L^{q_1}(\Omega)} \le M_1 \|f\|_{L^{p_1}(\Omega)}$$

for all $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$. Then, for each $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ and $0 < t < 1$, $Tf \in L^{q_t}(\Omega)$ and

$$\|Tf\|_{L^{q_t}(\Omega)} \le M_0^{1-t} M_1^t \|f\|_{L^{p_t}(\Omega)}$$

(a) Prove the Riesz–Thorin interpolation theorem in the special case that $p_0 = p_1$.

(b) Denote by

$$\Sigma(\Omega) = \Big\{ \sum_{j=1}^{m} a_j \chi_{E_j} \ \Big| \ m \in \mathbb{N},\ a_1, \cdots, a_m \in \mathbb{C}, \ E_1, \cdots, E_m \subset \Omega \text{ measureable and of finite measure} \Big\}$$

the set of simple functions on $\Omega$. Prove that if $f \in \Sigma(\Omega)$, then

$$\|Tf\|_{L^{q_t}(\Omega)} = \sup \Big\{ \Big| \int_\Omega (Tf)(x) g(x)\ d^n x \Big| \ \Big| \ g \in \Sigma(\Omega),\ \|g\|_{L^{q'_t}(\Omega)} \le 1 \Big\}$$

where $q'_t$, the conjugate index to $q_t$, is determined by $\frac{1}{q_t} + \frac{1}{q'_t} = 1$.

(c) Let $p_0 \ne p_1$ and $0 < t < 1$. Prove that

$$\Big| \int_\Omega (Tf)(x) g(x)\ d^n x \Big| \le M_0^{1-t} M_1^t$$

for all $f, g \in \Sigma(\Omega)$ with $\|f\|_{L^{p_t}(\Omega)}, \|g\|_{L^{q'_t}(\Omega)} \le 1$.

(d) Prove the Riesz–Thorin interpolation theorem.

**Solution.** (a) By problem 2.3.3,

$$\|Tf\|_{L^{q_t}(\Omega)} \le \|Tf\|_{L^{q_0}(\Omega)}^{1-t} \|Tf\|_{L^{q_1}(\Omega)}^{t} \le M_0^{1-t} \|f\|_{L^{p_0}(\Omega)}^{1-t} M_1^t \|f\|_{L^{p_1}(\Omega)}^{t} = M_0^{1-t} M_1^t \|f\|_{L^{p_t}(\Omega)}$$

(b) Let $S$ denote the supremum in the statement of the problem. Note that every function in $\Sigma(\Omega)$ is $L^p(\Omega)$ for every $1 \le p \le \infty$. Hence $Tf \in L^{q_0}(\Omega) \cap L^{q_1}(\Omega)$. As $q_t$ is between $q_0$ and $q_1$, $Tf \in L^{q_t}(\Omega)$ by problem 2.3.3. By Hölder,

$$\Big| \int_\Omega (Tf)(x) g(x)\ d^n x \Big| \le \|Tf\|_{L^{q_t}(\Omega)} \|g\|_{L^{q'_t}(\Omega)}$$

so that $S \le \|Tf\|_{L^{q_t}(\Omega)}$.

First consider $q'_t < \infty$. Choose $\theta(x)$ so that $(Tf)(x) = e^{i\theta(x)} |Tf(x)|$ and define

$$F(x) = e^{-i\theta(x)} \frac{|Tf(x)|^{q_t - 1}}{\|Tf\|_{L^{q_t}(\Omega)}^{q_t - 1}}$$

Then $F \in L^{q'_t}(\Omega)$ and

$$\|F\|_{L^{q'_t}(\Omega)} = \frac{\|Tf\|_{L^{q_t}(\Omega)}^{q_t/q'_t}}{\|Tf\|_{L^{q_t}(\Omega)}^{q_t - 1}} = 1$$

The set of simple functions is dense in $L^{q_t'}(\Omega)$. Consequently there is a sequence of simple functions $\varphi_j \in \{\ \varphi \in \Sigma(\Omega)\ \big|\ \|\varphi\|_{L^{q_t'}(\Omega)} \le 1\ \}$ that converges in $L^{q_t'}(\Omega)$ to $F$. Since $\varphi \mapsto \int_\Omega (Tf)(x)\varphi(x)\ d^n x$ is continuous on $L^{q_t'}(\Omega)$,

$$\Big|\int_\Omega (Tf)(x)F(x)\ d^n x\Big| = \lim_{j\to\infty}\Big|\int_\Omega (Tf)(x)\varphi_j(x)\ d^n x\Big| \le S$$

As $\Big|\int_\Omega (Tf)(x)F(x)\ d^n x\Big| = \|Tf\|_{L^{q_t}(\Omega)}$, we have $\|Tf\|_{L^{q_t}(\Omega)} = S$ when $q_t' < \infty$.

If $q_t' = \infty$, let $\Omega_R = \{\ x\in\Omega\ \big|\ |x| < R\ \}$. The set of simple functions is dense in $L^\infty(\Omega_R)$. So we can repeat the above argument to show that $\|\chi_{\Omega_R} Tf\|_{L^{q_t}(\Omega)} \le S$ for all $R$. By the Lebesgue dominated convergence theorem, $\|Tf\|_{L^{q_t}(\Omega)} \le S$ too.

(c) Since $\int_\Omega (Tf)(x)g(x)\ d^n x$ is linear in $f$ and in $g$, we may scale $f$ and $g$ so that $\|f\|_{L^{p_t}(\Omega)} = \|g\|_{L^{q_t}(\Omega)} = 1$. Let $f = \sum_{j=1}^m c_j\chi_{E_j}$ and $f = \sum_{j=1}^p d_j\chi_{F_j}$ with the $c_j$'s and $d_j$'s being nonzero complex numbers and the $E_j$'s and $F_j$'s being measurable subsets of $\Omega$ of finite measure with the different $E_j$'s disjoint and the different $F_j$'s disjoint. Set

$$\alpha(z) = \tfrac{1-z}{p_0} + \tfrac{z}{p_1} \qquad \beta(z) = 1 - \tfrac{1-z}{q_0} - \tfrac{z}{q_1}$$

Observe that $\alpha(t) = \frac{1}{p_t} \ne 0$, $\beta(t) = 1 - \frac{1}{q_t} = \frac{1}{q_t'}$. Write $c_j = |c_j|e^{i\theta_j}$ and $d_j = |d_j|e^{i\varphi_j}$ and define

$$f_z = \sum_{j=1}^m |c_j|^{\alpha(z)/\alpha(t)} e^{i\theta_j}\chi_{E_j}$$

$$g_z = \begin{cases} \sum_{j=1}^p |d_j|^{\beta(z)/\beta(t)} e^{i\varphi_j}\chi_{F_j} & \text{if } \beta(t) > 0 \\ g & \text{if } \beta(t) = 0 \end{cases}$$

and

$$F(z) = \int_\Omega (Tf_z)(x)g_z(x)\ d^n x$$

Since the $E_j$'s are disjoint and the $F_j$'s are disjoint

$$|f_z| = |f|^{\mathrm{Re}\,\alpha(z)/\alpha(t)} = |f|^{p_t/p_{\mathrm{Re}\,z}} \qquad |g_z| = |g|^{\mathrm{Re}\,\beta(z)/\beta(t)} = \begin{cases} |g|^{q_t'/q'_{\mathrm{Re}\,z}} & \text{if } q_t' < \infty \\ |g| & \text{if } q_t' = \infty \end{cases}$$

If $\mathrm{Re}\, z = 0$, then

$$\begin{aligned} |F(z)| &\le \|Tf_z\|_{L^{q_0}(\Omega)}\|g_z\|_{L^{q_0'}(\Omega)} \le M_0\|f_z\|_{L^{p_0}(\Omega)}\|g_z\|_{L^{q_0'}(\Omega)} \\ &= M_0\|f\|_{L^{p_t}(\Omega)}^{p_t/p_0} \begin{cases} \|g\|_{L^{q_t'}(\Omega)}^{q_t'/q_0'} & \text{if } q_t' < \infty \\ \|g\|_{L^{q_t'}(\Omega)} & \text{if } q_t' = \infty \end{cases} \\ &= M_0 \end{aligned}$$

since $q'_t = \infty$ only if $q'_0 = q'_1 = \infty$. If $\mathrm{Re}\, z = 1$, then

$$\begin{aligned} |F(z)| &\le \|Tf_z\|_{L^{q_1}(\Omega)} \|g_z\|_{L^{q'_1}(\Omega)} \le M_1 \|f_z\|_{L^{p_1}(\Omega)} \|g_z\|_{L^{q'_1}(\Omega)} \\ &= M_1 \|f\|_{L^{p_t}(\Omega)}^{p_t/p_1} \begin{cases} \|g\|_{L^{q'_t}(\Omega)}^{q'_t/q'_1} & \text{if } q'_t < \infty \\ \|g\|_{L^{q'_t}(\Omega)} & \text{if } q'_t = \infty \end{cases} \\ &= M_1 \end{aligned}$$

As $F(z)$ is an entire function of $z$, the three lines theorem, Lemma 2.3.1, gives

$$\Big| \int_\Omega (Tf)(x) g(x)\, d^n x \Big| = \big|F(t)\big| \le M_0^{1-t}\, M_1^t$$

(d) The case $p_0 = p_1$ has already been proven in part (a), so it suffices to consider $p_0 \ne p_1$. By parts (b) and (c),

$$\|Tf\|_{L^{q_t}(\Omega)} \le M_0^{1-t}\, M_1^t$$

for all $f \in \Sigma(\Omega)$ with $\|f\|_{L^{p_t}(\Omega)} \le 1$.

Let $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ with $\|f\|_{L^{p_t}(\Omega)} \le 1$. As, by hypothesis, $Tf \in L^{q_0}(\Omega) \cap L^{q_1}(\Omega)$, we have, by problem 2.3.3, that $Tf \in L^{q_t}(\Omega)$. Hence, it remains only to show that $\|Tf\|_{L^{q_t}(\Omega)} \le M_0^{1-t}\, M_1^t$. We claim that there is a sequence $f_j \in \Sigma(\Omega)$, with $|f_j(x)| \le |f(x)|$ for all $x$, that converges pointwise to $f$. If $f$ is pure real, it suffices to take, for $j \in \mathbb{N}$, $f_j = \sum_{\ell=2}^{j^2} \frac{\ell-1}{j} \big(\chi_{E_\ell} - \chi_{F_\ell}\big)$ where

$$\begin{aligned} E_\ell &= \big\{\ x \in \Omega \ \big|\ \tfrac{\ell-1}{j} < f(x) \le \tfrac{\ell}{j}\ \big\} \\ F_\ell &= \big\{\ x \in \Omega \ \big|\ \tfrac{-\ell}{j} \le f(x) < \tfrac{-\ell+1}{j}\ \big\} \end{aligned}$$

Observe that each $E_\ell$ and $F_\ell$ is of finite measure because $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ and at least one of $p_0, p_1$ is finite. If $f$ is complex valued, we just apply a construction like this separately to the real and imaginary components. The sequence $\{f_j\}_{j\in\mathbb{N}}$ converges to $f$ in both $L^{p_0}(\Omega)$ and $L^{p_1}(\Omega)$. For finite values of $p_0$ and/or $p_1$, this is a consequence of the Lebesgue dominated convergence theorem. For infinite values of $p_0$ or $p_1$, this is a consequence of the observation that when $f$ is bounded, $f_j$ converges to $f$ uniformly. As $T$ is a bounded map from $L^{p_i}(\Omega)$ to $L^{q_i}(\Omega)$, the sequence $\{Tf_j\}_{j\in\mathbb{N}}$ converges to $Tf$ in both $L^{q_0}(\Omega)$ and $L^{q_1}(\Omega)$. By problem 2.3.3, the sequence $\{Tf_j\}_{j\in\mathbb{N}}$ converges to $Tf$ in $L^{q_t}(\Omega)$. Hence

$$\begin{aligned} \|Tf\|_{L^{q_t}(\Omega)} &= \lim_{j\to\infty} \|Tf_j\|_{L^{q_t}(\Omega)} \le \limsup_{j\to\infty} M_0^{1-t}\, M_1^t \|f_j\|_{L^{p_t}(\Omega)} \\ &\le \limsup_{j\to\infty} M_0^{1-t}\, M_1^t \|f\|_{L^{p_t}(\Omega)} = M_0^{1-t}\, M_1^t \end{aligned}$$

■

**Problem 2.3.5** Let $\ell, L \in \mathbb{Z}$ and $s \in \mathbb{R}$ with $\ell \le s \le L$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and assume that $\varphi \in C^{\max\{|\ell|,|L|\}}(\partial\Omega)$. Prove that the map $f \in C^\infty(\partial\Omega) \mapsto \varphi f$ has a unique extension to a bounded, linear map on $H^s(\partial\Omega)$ and there is a constant $C$, depending only on $\Omega$, $\max\{|\ell|,|L|\}$ and $n$, such that

$$\|\varphi u\|_{s,\partial\Omega} \le C\|\varphi\|_{C^{|\ell|}(\partial\Omega)}^{\frac{L-s}{L-\ell}} \|\varphi\|_{C^{|L|}(\partial\Omega)}^{\frac{s-\ell}{L-\ell}} \|u\|_{s,\partial\Omega} \le C\|\varphi\|_{C^{\max\{|\ell|,|L|\}}(\partial\Omega)} \|u\|_{s,\partial\Omega}$$

for all $f \in H^s(\partial\Omega)$.

**Solution.** It suffices to prove the specified bound for $f \in C^\infty(\partial\Omega)$. Let $\big(\mathcal{U}(p), \psi_p\big)$ be a coordinate system as in Notation 2.1.16 and let $\chi_i \in C_0^\infty\big(\mathcal{U}(p_i)\big)$, $1 \le i \le N$, be a partition of unity as in Definition 2.1.18. For each $1 \le i \le N$, choose an open subset $\mathcal{V}_i$ of $\mathbb{R}^n$ that contains the support of $\chi_i$ and whose closure is contained in $\mathcal{U}(p_i)$. Then $\psi_{p_i}(\overline{\mathcal{V}_i})$ is compact and all the derivatives of $\psi_{p_i}^{-1}$ are bounded on $\psi_{p_i}(\overline{\mathcal{V}_i})$. Choose a $X_i \in C_0^\infty(\mathcal{V}_i)$ that is identically one on the support of $\chi_i$ and set $\Phi = (X_i\varphi) \circ \psi_{p_i}^{-1}$. By part (a) of Lemma 2.1.14,

$$\begin{aligned} \big|\Phi v\big|_{\ell',n-1} &\le C_{|\ell'|,n-1} \|\Phi\|_{C^{|\ell'|}(\mathbb{R}^{n-1})} \big|v\big|_{\ell',n-1} \\ &\le C' \|\varphi\|_{C^{|\ell'|}(\partial\Omega)} \big|v\big|_{\ell',n-1} \end{aligned}$$

for all $v \in H^{\ell'}(\mathbb{R}^{n-1})$ and both $\ell' = \ell, L$. Then, by Lemma 2.3.2, with $s_0 = \ell$, $s_1 = L$ and $t = \frac{s-\ell}{L-\ell}$

$$\big|\Phi v\big|_{s,n-1} \le C' \|\varphi\|_{C^{|\ell|}(\partial\Omega)}^{\frac{L-s}{L-\ell}} \|\varphi\|_{C^{|L|}(\partial\Omega)}^{\frac{s-\ell}{L-\ell}} \big|v\big|_{\ell',n-1}$$

for all $v \in H^s(\mathbb{R}^{n-1})$. In particular

$$\big|(\chi_i\varphi f) \circ \psi_{p_i}^{-1}\big|_{s,n-1} = \big|\Phi\ (\chi_i f) \circ \psi_{p_i}^{-1}\big|_{s,n-1} \le C' \|\varphi\|_{C^{|\ell|}(\partial\Omega)}^{\frac{L-s}{L-\ell}} \|\varphi\|_{C^{|L|}(\partial\Omega)}^{\frac{s-\ell}{L-\ell}} \big|(\chi_i f) \circ \psi_{p_i}^{-1}\big|_{s,n-1}$$

and the desired bounds follows. ■

**Problem 2.3.6(Hausdorff–Young Inequality)** Let $1 \le p \le 2$ and let $2 \le q \le \infty$ obey $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^q(\mathbb{R}^n)$ and

$$\|\hat{f}\|_{L^q(\mathbb{R}^n, \frac{d^nk}{(2\pi)^n})} \le \|f\|_{L^p(\mathbb{R}^n, d^nx)}$$

**Solution.** We apply the Riesz–Thorin interpolation theorem with $p_0 = 1$, $p_1 = 2$, $q_0 = \infty$, $q_1 = 2$, $\Omega = \mathbb{R}^n$ and $Tf = \hat{f}$. Since

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^n, \frac{d^nk}{(2\pi)^n})} \le \|f\|_{L^1(\mathbb{R}^n, d^nx)} \qquad \|\hat{f}\|_{L^2(\mathbb{R}^n, \frac{d^nk}{(2\pi)^n})} = \|f\|_{L^2(\mathbb{R}^n, d^nx)}$$

we have $M_0 = M_1 = 1$ (or if you insist on using the measure $d^n k$ in place of $\frac{d^n k}{(2\pi)^n}$, $M_0 = 1$ and $M_1 = (2\pi)^{n/2}$). Choosing $t = 2\big(1 - \frac{1}{p}\big)$ gives

$$\tfrac{1}{p_t} = \tfrac{1-t}{p_0} + \tfrac{t}{p_1} = 1 - t + \tfrac{t}{2} = \tfrac{1}{p} \qquad \tfrac{1}{q_t} = \tfrac{1-t}{q_0} + \tfrac{t}{q_1} = \tfrac{t}{2} = \tfrac{1}{q}$$

or $p_t = p$ and $q_t = q$. So

$$\|\hat f\|_{L^q(\mathbb{R}^n, \frac{d^n k}{(2\pi)^n})} \le \|f\|_{L^p(\mathbb{R}^n, d^n x)}$$

(or $\|\hat f\|_{L^q(\mathbb{R}^n, d^n k)} \le (2\pi)^{n/q} \|f\|_{L^p(\mathbb{R}^n, d^n x)}$) for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. For $1 < p < 2$, $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. So the bound extends by continuity to all of $L^p(\mathbb{R}^n)$.

■

**Problem 2.3.7** Let $2 < p \le \infty$ and let $1 \le q < 2$ obey $\frac{1}{p} + \frac{1}{q} = 1$. Prove that there does NOT exist a constant $C$ such that

$$\|\hat f\|_{L^q(\mathbb{R}^n, \frac{d^n k}{(2\pi)^n})} \le C\|f\|_{L^p(\mathbb{R}^n, d^n x)}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Hint: consider $f(x) = e^{-(a+ib)x^2}$ with $a > 0$.

**Solution.** Since

$$\int e^{-x^2}\ d^n x = \left[\int e^{-x^2}\ d^2 x\right]^{n/2} = \left[\int e^{-r^2}\ r dr d\theta\right]^{n/2} = \left[2\pi \int_0^\infty e^{-r^2}\ r dr\right]^{n/2} = \pi^{n/2}$$

we have

$$\text{(S2.4)} \qquad \int e^{-z(x+z')^2}\ d^n x = \left[\tfrac{\pi}{z}\right]^{\frac{n}{2p}}$$

for all $z > 0$, $z' \in \mathbb{R}$, by translating and scaling. Since the left hand side is analytic in $z$ and $z'$ for $\operatorname{Re} z > 0$ and $z' \in \mathbb{C}$, (S2.4) is valid for all $z, z' \in \mathbb{C}$ with $\operatorname{Re} z > 0$. On the right hand side, $\zeta^{1/2}$ is the branch of the square root that is defined on $\operatorname{Re}\zeta > 0$ and is positive for $\zeta > 0$. Let $f_{a,b}(x) = e^{-(a+ib)x^2}$. Then, writing $a + ib = z$,

$$\hat f_{a,b}(k) = \int e^{-ik\cdot x} e^{-zx^2} d^n x = e^{-k^2/(4z)} \int e^{-z[x+ik/(2z)]^2} d^n x = e^{-\frac{k^2}{4(a+ib)}} \left[\tfrac{\pi}{a+ib}\right]^{n/2}$$

so that

$$\|f_{a,b}\|_{L^p(\mathbb{R}^n, d^n x)} = \left[\int e^{-pax^2}\ d^n x\right]^{1/p} = \left[\tfrac{\pi}{pa}\right]^{\frac{n}{2p}}$$

$$\|\hat f_{a,b}\|_{L^q(\mathbb{R}^n, \frac{d^n k}{(2\pi)^n})} = \left[\tfrac{\pi}{|a+ib|}\right]^{n/2} \left[\int e^{-q\frac{ak^2}{4(a^2+b^2)}}\ \frac{d^n k}{(2\pi)^n}\right]^{1/q} = \left[\tfrac{\pi}{|a+ib|}\right]^{\frac{n}{2}} \left[\tfrac{4\pi}{qa}(a^2+b^2)\right]^{\frac{n}{2q}} \left[\tfrac{1}{2\pi}\right]^{\frac{n}{q}}$$

The ratio

$$\begin{aligned}\frac{\|\hat f_{a,b}\|_{L^q(\mathbb{R}^n,\frac{d^nk}{(2\pi)^n})}}{\|f_{a,b}\|_{L^p(\mathbb{R}^n,d^nx)}} &= \Big\{\tfrac{\pi}{|a+ib|}\big[\tfrac{4\pi}{qa}(a^2+b^2)\big]^{\frac1q}\big[\tfrac{pa}{\pi}\big]^{\frac1p}\big[\tfrac{1}{2\pi}\big]^{\frac2q}\Big\}^{n/2}\\ &= \Big\{\tfrac{\pi}{|a+ib|}\big[\tfrac{1}{qa\pi}(a^2+b^2)\big]^{\frac1q}\big[\tfrac{pa}{\pi}\big]^{\frac1p}\Big\}^{n/2}\\ &= \Big\{\tfrac{1}{|a+ib|}\big[\tfrac{1}{qa}(a^2+b^2)\big]^{\frac1q}\big[pa\big]^{\frac1p}\Big\}^{n/2}\\ &= \Big\{\tfrac{p^{1/p}}{q^{1/q}}(a^2+b^2)^{\frac1q-\frac12}a^{\frac1p-\frac1q}\Big\}^{n/2}\\ &= \Big\{\tfrac{p^{1/p}}{q^{1/q}}\big(\tfrac{a^2+b^2}{a^2}\big)^{\frac1q-\frac12}\Big\}^{n/2}\end{aligned}$$

For $q<2$ this grows to infinity as $b$ grows to infinity. ■

**Problem 2.3.8**

(a) Let $1\le p,q,r\le\infty$ obey $\frac1p+\frac1q+\frac1r=2$. Prove that if $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$ and $h\in L^r(\mathbb{R}^n)$, then

$$\left|\int f(x)g(x-y)h(y)\ d^nxd^ny\right|\le\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\|h\|_{L^r(\mathbb{R}^n)}$$

(b) Let $1\le p,q,r\le\infty$ obey $\frac1p+\frac1q=1+\frac1r$. Prove that if $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$, then $f*g\in L^r(\mathbb{R}^n)$ and

$$\|f*g\|_{L^r(\mathbb{R}^n)}\le\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

**Solution.** Let $p',q',r'$ be the dual indices for $p,q,r$. That is

$$\tfrac1p+\tfrac1{p'}=1\qquad\tfrac1q+\tfrac1{q'}=1\qquad\tfrac1r+\tfrac1{r'}=1$$

First consider the case in which all of $p,q,r$ are finite. Since

$$\tfrac1{r'}+\tfrac1{q'}=2-\tfrac1r-\tfrac1q=\tfrac1p\qquad\tfrac1{r'}+\tfrac1{p'}=2-\tfrac1r-\tfrac1p=\tfrac1q\qquad\tfrac1{p'}+\tfrac1{q'}=2-\tfrac1p-\tfrac1q=\tfrac1r$$

the absolute value of the integrand factors into the product of

$$\begin{aligned}\alpha(x,y)&=|f(x)|^{p/r'}|g(x-y)|^{q/r'}\\ \beta(x,y)&=|g(x-y)|^{q/p'}|h(y)|^{r/p'}\\ \gamma(x,y)&=|f(x)|^{p/q'}|h(y)|^{r/q'}\end{aligned}$$

Since

$$\tfrac{1}{p'} + \tfrac{1}{q'} + \tfrac{1}{r'} = 3 - \tfrac{1}{p} - \tfrac{1}{q} - \tfrac{1}{r} = 1$$

Hölder's inequality gives

$$\left| \int f(x)g(x-y)h(y)\ d^n x d^n y \right| \le \|\alpha\|_{L^{r'}(\mathbb{R}^{2n})} \|\beta\|_{L^{p'}(\mathbb{R}^{2n})} \|\gamma\|_{L^{q'}(\mathbb{R}^{2n})}$$

If $p', q', r'$ are finite, the desired inequality now follows from

(S2.5)
$$\begin{aligned} \|\alpha\|_{L^{r'}(\mathbb{R}^{2n})} &= \Big[ \int |f(x)|^p |g(x-y)|^q\ d^n x d^n y \Big]^{1/r'} \\ &= \Big[ \int |f(x)|^p |g(z)|^q\ d^n x d^n z \Big]^{1/r'} \\ &= \|f\|^{p/r'}_{L^p(\mathbb{R}^n)} \|g\|^{q/r'}_{L^q(\mathbb{R}^n)} \\ \|\beta\|_{L^{p'}(\mathbb{R}^{2n})} &= \|g\|^{q/p'}_{L^q(\mathbb{R}^n)} \|h\|^{r/p'}_{L^r(\mathbb{R}^n)} \\ \|\gamma\|_{L^{q'}(\mathbb{R}^{2n})} &= \|f\|^{p/q'}_{L^p(\mathbb{R}^n)} \|h\|^{r/q'}_{L^r(\mathbb{R}^n)} \end{aligned}$$

Even if some of $p', q', r'$ are infinite, the final bounds of (S2.5) still hold. For example, if $r' = \infty$, $\alpha \equiv 1$ so that both $\|\alpha\|_{L^{r'}(\mathbb{R}^{2n})} = 1$ and $\|f\|^{p/r'}_{L^p(\mathbb{R}^n)} \|g\|^{q/r'}_{L^q(\mathbb{R}^n)} = 1$.

Now suppose that $p, q, r$ are not all finite. Since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, one of them is $\infty$ and the other two are 1. We deal with the case $p = \infty$, $q = r = 1$. The other two cases are similar. The absolute value of the integrand factors into the product of

$$\begin{aligned} \alpha(x,y) &= |f(x)| \\ \beta(x,y) &= |g(x-y)||h(y)| \end{aligned}$$

Hölder's inequality gives

$$\left| \int f(x)g(x-y)h(y)\ d^n x d^n y \right| \le \|\alpha\|_{L^\infty(\mathbb{R}^{2n})} \|\beta\|_{L^1(\mathbb{R}^{2n})} = \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} \|h\|_{L^1(\mathbb{R}^n)}$$

as desired.

(b) If $p \neq 1$, apply the bound of part (a) with

$$f(x) = e^{-i\theta(x)} |g * h(x)|^{p'/p}$$

where $\theta(x)$ is chosen so that $g * h(x) = e^{i\theta(x)} |(g * h)(x)|$. This gives

$$\|g * h\|^{p'}_{L^{p'}(\mathbb{R}^n)} \le \|g * h\|^{p'/p}_{L^{p'}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}$$

or

$$\|g * h\|_{L^{p'}(\mathbb{R}^n)} \le \|g\|_{L^q(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}$$

If $p = 1$, that is $p' = \infty$, this conclusion is still valid by Hölder's inequality. Substituting

$$g = F \qquad h = G \qquad p' = R \qquad q = P \qquad r = Q$$

gives

$$\|F * G\|_{L^R(\mathbb{R}^n)} \le \|F\|_{L^P(\mathbb{R}^n)} \|G\|_{L^Q(\mathbb{R}^n)}$$

under the constraint

$$1 - \tfrac{1}{R} + \tfrac{1}{P} + \tfrac{1}{Q} = 2$$

Switching from upper to lower case gives the desired bound. ■

**Problem 2.3.9** Let $\Omega$ be an open subset of $\mathbb{R}^n$. Prove that if $u \in H_0^2(\Omega)$, then

$$\sum_{i=1}^n \big\| \tfrac{\partial u}{\partial x_i} \big\|_{L^2(\Omega)}^2 \le \sum_{i=1}^n \|u\|_{L^2(\Omega)} \; \big\| \tfrac{\partial^2 u}{\partial x_i^2} \big\|_{L^2(\Omega)}$$

**Solution.** First consider $u \in C_0^\infty(\Omega)$. Then, by integration by parts (a.k.a the divergence theorem),

$$\sum_{i=1}^n \int_\Omega \big| \tfrac{\partial u}{\partial x_i}(x) \big|^2 \; d^n x = \sum_{i=1}^n \int_\Omega \tfrac{\partial u}{\partial x_i}(x) \overline{\tfrac{\partial u}{\partial x_i}(x)} \; d^n x = -\sum_{i=1}^n \int_\Omega u(x) \overline{\tfrac{\partial^2 u}{\partial x_i^2}(x)} \; d^n x$$

The boundary terms disappeared because $u$ vanishes in a neighbourhood of $\partial\Omega$. By Cauchy–Schwarz

$$\sum_{i=1}^n \int_\Omega \big| \tfrac{\partial u}{\partial x_i}(x) \big|^2 \; d^n x \le \sum_{i=1}^n \|u\|_{L^2(\Omega)} \; \big\| \tfrac{\partial^2 u}{\partial x_i^2} \big\|_{L^2(\Omega)}$$

By taking limits this also applies to $u \in H_0^2(\Omega)$. ■

**Problem 2.3.10** Let $\Omega$ be a convex open subset of $\mathbb{R}^n$ with smooth boundary and $S$ a measurable subset of $\Omega$. Prove that if $u \in H^1(\Omega)$, then,

$$\big\| u - (u)_S \big\|_{L^1(\Omega)} \le \tfrac{d^n}{|S|} \omega_n^{1-\frac{1}{n}} |\Omega|^{\frac{1}{n}} \big\| \nabla u \big\|_{L^1(\Omega)} \le \tfrac{d^n}{|S|} \omega_n^{1-\frac{1}{n}} |\Omega|^{\frac{1}{n}+\frac{1}{2}} \big\| \nabla u \big\|_{L^2(\Omega)}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$, $d$ is the diameter of $\Omega$, $|S|$ is the measure of $S$ and $(u)_S = \frac{1}{|S|}\int_S u(x)\ d^n x$ is the average value of $u$ on $S$.

**Solution.** By Lemma 2.2.13, $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, so it suffices to consider $u \in C^1(\overline{\Omega})$. Then, by Lemma 2.3.8,

$$\big|u(x) - (u)_S\big| \le \tfrac{d^n}{n|S|} \int_\Omega \tfrac{1}{|x-y|^{n-1}} |\nabla u(y)|\ d^n y$$

Integrating over $\Omega$ with respect to $x$,

$$\begin{aligned}\big\|u - (u)_S\big\|_{L^1(\Omega)} &\le \tfrac{d^n}{n|S|}\big\|\nabla u\big\|_{L^1(\Omega)} \sup_{y\in\Omega}\int_\Omega \tfrac{1}{|x-y|^{n-1}}\ d^n y \\ &\le \tfrac{d^n}{n|S|}\tfrac{1-\delta}{\mu-\delta}\omega_n^{r(1-\mu)}|\Omega|^{r(\mu-\delta)}\big\|\nabla u\big\|_{L^1(\Omega)} \\ &= \tfrac{d^n}{|S|}\omega_n^{1-1/n}|\Omega|^{1/n}\big\|\nabla u\big\|_{L^1(\Omega)}\end{aligned}$$

by (2.3.1) with $r = 1$, $\delta = 0$ and $\mu = \frac{1}{n}$ chosen so that $n(1-\mu)r = n-1$. The other claim follows by Cauchy–Schwarz.

■

**Problem 2.3.11** Let $\langle X, \mu\rangle$ and $\langle Y, \nu\rangle$ be measure spaces and let $k(x, y)$ be a measurable function on $X \times Y$. Set, for $0 < \alpha, \beta \le \infty$,

$$L_\alpha = \sup_{x\in X}\begin{cases}\big\{\int_Y |k(x,y)|^\alpha\ d\nu(y)\big\}^{1/\alpha} & \text{if } 0<\alpha<\infty \\ \sup_{y\in Y}|k(x,y)| & \text{if } \alpha = \infty\end{cases}$$

$$R_\beta = \sup_{y\in Y}\begin{cases}\Big\{\int_X |k(x,y)|^\beta\ d\mu(x)\Big\}^{1/\beta} & \text{if } 0<\beta<\infty \\ \sup_{x\in X}|k(x,y)| & \text{if } \beta = \infty\end{cases}$$

Consider the map

$$(Kf)(x) = \int_Y k(x,y)f(y)\ d\nu(y)$$

(with domain to be specified).

(a) Let $1 \le \alpha \le \infty$ and $\alpha' = \big(1 - \frac{1}{\alpha}\big)^{-1}$ be the dual index to $\alpha$. If $L_\alpha < \infty$, then $K$ is a bounded linear operator from $L^{a'}(Y, \nu)$ to $L^\infty(X, \mu)$ with operator norm bounded by $L_\alpha$.

(b) Assume that $X$ is a $\sigma$–finite measure and $1 \le \beta \le \infty$. If $R_\beta < \infty$, then $K$ is a bounded linear operator from $L^1(Y, \nu)$ to $L^\beta(X, \mu)$ with operator norm bounded by $R_\beta$.

**Solution.** (a) By Hölder,

$$|(Kf)(x)| \le \int |k(x,y)||f(y)|\, d\nu(y) \le \Big\{ \int |k(x,y)|^{\alpha}\, d\nu(y) \Big\}^{\frac{1}{\alpha}} \Big\{ \int |f(y)|^{\alpha'}\, d\nu(y) \Big\}^{\frac{1}{\alpha'}}$$
$$\le L_\alpha \|f\|_{L^{\alpha'}}$$

as desired.

(b) For any $f \in L^1(Y,\nu)$ and $g \in L^{\beta'}(X,\mu)$

$$\int_{X\times Y} |g(x)|\,|k(x,y)|\,|f(y)|\, d\mu(x)d\nu(y) = \int_Y d\nu(y)\ |f(y)| \int_X d\mu(x)\ |g(x)|\,|k(x,y)|$$
$$\le \int_Y d\nu(y)\ |f(y)| \Big\{ \int |k(x,y)|^{\beta}\, d\mu(x) \Big\}^{\frac{1}{\beta}} \Big\{ \int |g(x)|^{\beta'}\, d\nu(y) \Big\}^{\frac{1}{\beta'}}$$
$$\le R_\beta \|g\|_{L^{\beta'}(X,\mu)} \|f\|_{L^1(Y,\nu)}$$

Since, for any measurable function $h(x)$,

$$\|h\|_{L^\beta(X,\mu)} = \sup \Big\{ \int_X |g(x)|\,|h(x)|\, d\mu(x) \ \Big|\ g \in L^{\beta'}(X,\mu),\ \|g\|_{L^{\beta'}(X,\mu)} \le 1 \Big\}$$

the claim follows. ■