§2. Sobolev Spaces

This chapter provides some background concerning Sobolev spaces. In the first section, we define the spaces $H^s(\Omega)$, $H^s(\partial\Omega)$ and $H^s_0(\Omega)$. We start with Ω being an arbitrary open subset of $\mathbb{R}^n$, but later restrict Ω to being a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary (see Definition 2.1.15), which we denote $\partial\Omega$. Roughly speaking, the spaces consist of all functions whose $s^{\rm th}$ order derivatives are L^2 and which, in the case of $H^s_0(\Omega)$, vanish appropriately on $\partial\Omega$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we shall use $\partial^\alpha u(x)$ to denote the partial derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x)$. The order of this partial derivative is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. There is more than one way to precisely implement this rough picture. We choose to define

- $H^\ell(\Omega)$, resp. $H^\ell_0(\Omega)$, with $\ell \in \mathbb{N}_0$, to be the completion of $\{\, u \in C^\ell(\Omega) \bigm| \|u\|_{\ell,\Omega} < \infty \,\}$, resp. $C^\infty_0(\Omega)$, under the norm $\|u\|^2_{\ell,\Omega} = \sum_{|\alpha|\le\ell} \int_\Omega \big|\partial^\alpha u(x)\big|^2 \, d^n x$ (Definition 2.1.1)
- and $H^\ell(\Omega)$, with ℓ a negative integer, to be the space $H_0^{-\ell}(\Omega)^*$ of bounded linear functionals on $H_0^{-\ell}(\Omega)$ (Definition 2.1.10).

We shall make some natural identifications, like identifying $H^0(\Omega) = L^2(\Omega)$ with $L^2(\Omega)^*$, and show that

- if $\ell, \ell' \in \mathbb{Z}$ with $\ell \le \ell'$, then $H^{\ell'}(\Omega) \subset H^\ell(\Omega)$ and $\|v\|_{\ell,\Omega} \le \|v\|_{\ell',\Omega}$ for all $v \in H^{\ell'}(\Omega)$ (Remark 2.1.13) and
- we can define ∂^α as a bounded linear map from $H^\ell(\Omega)$ to $H^{\ell-|\alpha|}(\Omega)$ for all $\ell \in \mathbb{Z}$ and all $\alpha \in \mathbb{N}_0^n$ (part (b) of Lemma 2.1.14) and then
- if $\ell \in \mathbb{N}_0$, $H^\ell(\Omega) = \{\ u \in L^2(\mathbb{R}^n) \bigm| \partial^\alpha u \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell\ \}$ (part (a) of Proposition 2.2.15)

We shall also equivalently characterize, in part (b) of Proposition 2.2.15, $H^\ell(\Omega)$, $\ell \in \mathbb{N}_0$, as the set of all $u \in L^2(\Omega)$ for which there is a constant C such that

$$\big|\left<\partial^\alpha f, u\right>_{L^2(\Omega)}\big| \le C\big\|f\big\|_{L^2(\Omega)} \tag{2.1}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and all $f \in C^\infty_0(\Omega)$. The map

$$\begin{aligned} D^\alpha_u : C^\infty_0(\Omega) &\to \mathbb{C} \\ f &\mapsto (-1)^{|\alpha|} \left<\partial^\alpha f, u\right>_{L^2(\Omega)} \end{aligned}$$

is called the weak (or distributional) α–derivative of u. It is defined for all $u \in L^2(\Omega)$ and all $\alpha \in \mathbb{N}_0^n$, but is not a function on Ω. One says that the weak derivative D^α_u is in $L^2(\Omega)$ if there exists a $v_\alpha \in L^2(\Omega)$ such that $\left<f, v_\alpha\right>_{L^2(\Omega)} = (-1)^{|\alpha|}\left<\partial^\alpha f, u\right>_{L^2(\Omega)}$ for all $f \in C^\infty_0(\Omega)$. In the special case that $u \in C^{|\alpha|}(\Omega)$, we can take $v_\alpha = \partial^\alpha u$, by repeated integration by parts. By the Riesz representation theorem, the bound (2.1) provides a necessary and sufficient condition for the weak derivative D_u^α to be in $L^2(\Omega)$. So part (b) of Proposition 2.2.15 characterizes $H^\ell(\Omega)$, as the set of all $u \in L^2(\Omega)$ all of whose weak derivatives of order up to ℓ are in $L^2(\Omega)$.

In the second section, we consider the problem of restricting functions on Ω, that a priori need only be defined almost everywhere, to $\partial\Omega$. We shall see that if the function u, on Ω, has $s > \frac{1}{2}$ derivatives in L^2, then one can define a natural restriction of u to $\partial\Omega$ and this restriction has $s - \frac{1}{2}$ derivatives in L^2. Of course, for this to make sense, we have define what it means for a function to have fractional derivatives.

Finally, in the third section, we prove a number of useful inequalities.

§2.1. Definitions of Sobolev Spaces

We wish to define a Hilbert space consisting of functions on Ω that have s derivatives, in some reasonable sense. When s is a positive integer, this is easy.

Definition 2.1.1 **($H^\ell(\Omega)$, $H_0^\ell(\Omega)$, $\ell \in \mathbb{N}_0$)** Let Ω be an open subset of $\mathbb{R}^n$ and $\ell \in \mathbb{N}_0$. Define

$$\|u\|_{\ell,\Omega} = \left\{ \sum_{|\alpha|\le\ell} \int_\Omega \left|\partial^\alpha u(x)\right|^2 \, d^n x \right\}^{1/2}$$

for each $u \in C^\ell(\Omega)$ for which the right hand side makes sense. Then $H^\ell(\Omega)$ is the completion[1] of the pre–Hilbert space $\left\{ \ u \in C^\ell(\Omega) \ \middle| \ \|u\|_{\ell,\Omega} < \infty \ \right\}$ equipped with the inner product

$$\langle u, v\rangle_{\ell,\Omega} = \sum_{|\alpha|\le\ell} \int_\Omega \partial^\alpha u(x)\overline{\partial^\alpha v(x)} \, d^n x$$

We sometimes drop the Ω from the notation. Similarly, $H_0^\ell(\Omega)$ is the completion of $C_0^\infty(\Omega)$.

Problem 2.1.1 Let $\Omega = (-1, 1)$ and $\ell = 1$. Think of $f(x) = |x|$ and $g(x) = \operatorname{sgn} x$ as functions in $L^2(\Omega)$. Find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C^1(\Omega)$ such that

- ◦ $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the norm $\| \ \|_{\ell,\Omega}$
- ◦ $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to f.
- ◦ $\left\{\frac{df_n}{dx}\right\}_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to g.

By definition, each element of $H^\ell(\Omega)$ is an equivalence class of Cauchy sequences. We can identify f with the Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ and so think of f as an element of $H^\ell(\Omega)$.

---

[1] Readers that are not familiar with completion should read the first Chapter of [RS]. It is also common to define $H^\ell(\Omega)$ to be the space of distributions $u \in \mathcal{D}'(\Omega)$, all of whose derivatives $\partial^\alpha u$ of order $|\alpha| \le \ell$ are L^2 functions, not just distributions. We shall not assume that readers are familiar with distributions.

Remark 2.1.2 By definition, each element of $H^\ell(\Omega)$ is an equivalence class of Cauchy sequences. A sequence $\{u_i\}_{i\in\mathbb{N}} \subset \big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\}$ is Cauchy with respect to the norm $\|\ \cdot\ \|_{\ell,\Omega}$ if and only if $\{\partial^\alpha u_i\}_{i\in\mathbb{N}}$ is Cauchy with respect to the norm $\|\ \cdot\ \|_{L^2(\Omega)}$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$. Since $L^2(\Omega)$ is complete, this in turn is the case if and only if, for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$, the sequence $\partial^\alpha u_i$ converges in $L^2(\Omega)$ to some $v_\alpha \in L^2(\Omega)$. Thus we may equivalently view $H^\ell(\Omega)$ as the set of all $v \in L^2(\Omega)$ for which there is a sequence $\{u_i\}_{i\in\mathbb{N}} \subset \big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\}$ obeying

- u_i converges to v in $L^2(\Omega)$ and
- for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$, the sequence $\partial^\alpha u_i$ converges in $L^2(\Omega)$.

In this way, we may view all of $H^\ell(\Omega)$, $H_0^\ell(\Omega)$, $\ell \in \mathbb{N}_0$ as subsets of $L^2(\Omega)$. Furthermore, if $\ell', \ell \in \mathbb{N}_0$ with $\ell' \ge \ell$, we have $H_0^{\ell'}(\Omega) \subset H^{\ell'}(\Omega) \subset H^\ell(\Omega) \subset L^2(\Omega)$ and $\|u\|_{\ell,\Omega} \le \|u\|_{\ell',\Omega}$ for all $u \in H^{\ell'}(\Omega)$.

Problem 2.1.2 Prove that $C_0^\ell(\Omega) \subset H_0^\ell(\Omega)$.

Problem 2.1.3 Let $\ell \in \mathbb{N}_0$.

(a) Let $\varphi : \Omega \to \mathbb{C}$ and all of its derivatives of order at most ℓ be bounded and continuous. Prove that the map $u \in \big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\} \mapsto \varphi u$, where, as you would expect $(\varphi u)(x) = \varphi(x)u(x)$, has a unique extension to a bounded, linear map on $H^\ell(\Omega)$. Prove, in particular, that there is a constant $C_{\ell,n}$, depending only on ℓ and n, such that

$$\|\varphi u\|_{\ell,\Omega} \le C_{\ell,n} \sum_{k=0}^{\ell} \|\varphi\|_{C^k(\Omega)} \|u\|_{\ell-k,\Omega}$$

for all $u \in H^\ell(\Omega)$. Here $\|\varphi\|_{C^k(\Omega)} = \sup\limits_{\substack{x\in\Omega \\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\le k}} \big|\partial^\alpha \varphi(x)\big|$.

(b) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$. Prove that the map $u \in \big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\} \mapsto \partial^\alpha u$ has a unique extension to a bounded, linear map from $H^\ell(\Omega)$ to $H^{\ell-|\alpha|}(\Omega)$.

Problem 2.1.4 Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a C^1 function with bounded derivative. Let $H^1_{\mathbb{R}}(\Omega)$ denote the set of real valued functions in $H^1(\Omega)$.

(a) Prove that if $u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, then $F \circ u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$.
(b) Prove that the map $u \mapsto F \circ u$, with domain $C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, is continuous with respect to the topology of $H^1_{\mathbb{R}}(\Omega)$. That is, prove that if $\varepsilon > 0$ and $v \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$, then there is a $\delta > 0$ such that $\|F \circ u - F \circ v\|_{1,\Omega} < \varepsilon$ for all $u \in C^1(\Omega) \cap H^1_{\mathbb{R}}(\Omega)$ obeying $\|u - v\|_{1,\Omega} < \delta$.
(c) Prove that the map $u \mapsto F \circ u$ has a continuous extension to $H^1_{\mathbb{R}}(\Omega)$.

Problem 2.1.5 Let Ω be a bounded open subset of $\mathbb{R}^n$. Prove that if $u \in H^1_{\mathbb{R}}(\Omega)$, then $|u|, \max\{0,u\} \in H^1_{\mathbb{R}}(\Omega)$.

Example 2.1.3 Here is an example which shows that $H^\ell_0(\Omega)$ can be a strictly proper subspace of $H^\ell(\Omega)$. We take $n=1$, $\Omega=(0,\infty)$. Start with $f \in C^\infty_0(\Omega)$. Since f is supported in Ω, we can think of it as being defined on all $\mathbb{R}$, taking value zero everywhere except in $(0,\infty)$. In particular $f^{(j)}(0)=0$. So, for $x>0$,

$$\begin{aligned}
\big|f^{(j)}(x)\big| &= \big|f^{(j)}(x) - f^{(j)}(0)\big| = \Big|\int_0^x f^{(j+1)}(t)\ dt\Big| \\
&= \Big|\int_0^\infty \chi_{[0,x]}(t) f^{(j+1)}(t)\ dt\Big| \le \Big[\int_0^\infty \chi_{[0,x]}(t)^2\ dt\Big]^{1/2}\Big[\int_0^\infty \big|f^{(j+1)}(t)\big|^2\ dt\Big]^{1/2} \\
&= \sqrt{x}\Big[\int_0^\infty \big|f^{(j+1)}(t)\big|^2\ dt\Big]^{1/2} \le \sqrt{x}\|f\|_{\ell,\Omega}
\end{aligned}$$

provided $\ell \ge j+1$. Now suppose that $f \in H^\ell_0(\Omega)$. It is a limit, in $H^\ell(\Omega)$, of functions $f_i \in C^\infty_0(\Omega)$. We can always choose the sequence so that $\|f_i\|_{\ell,\Omega} \le 2\|f\|_{\ell,\Omega}$ for every i. Thus, if $j \le \ell-1$, $\big|f_i^{(j)}(x)\big| \le \sqrt{x}\|f_i\|_{\ell,\Omega} \le 2\sqrt{x}\|f\|_{\ell,\Omega}$ for all i. We can think of this as saying that the first $\ell-1$ derivatives of every $f \in H^\ell_0(\Omega)$ vanish on the boundary of Ω. Certainly, given any constant C, if $0 < x < \min\big\{1, \frac{1}{100C^2}\big\}$ and $|f(x)| \le C\sqrt{x}$ then $\big|e^{-x} - f(x)\big| \ge e^{-1} - \frac{1}{10} > 0$. So the function $e^{-x} \in H^\ell(\Omega)$ cannot be approximated in $L^2(\Omega)$, let alone $H^\ell(\Omega)$, by functions $f_i(x)$ that obey $|f_i(x)| \le C\sqrt{x}$.

Problem 2.1.6 Let $\Omega = (0,\infty)$.

(a) Prove that there is a constant C such that if $u \in C^1(\Omega)$ and $\|u\|_{1,\Omega} < \infty$, then $\lim\limits_{x\to 0} u(x)$ exists and obeys

$$\big|\lim_{x\to 0} u(x)\big| \le C\|u\|_{1,\Omega}$$

(b) Prove that there is a unique bounded linear map $B : H^1(\Omega) \to \mathbb{C}$ such that $Bu = \lim\limits_{x\to 0} u(x)$ for all $u \in C^1(\Omega)$ with $\|u\|_{1,\Omega} < \infty$.

(c) Prove that $H^1_0(\Omega)$ is of codimension one in $H^1(\Omega)$. This means that there exists a $u_0 \in H^1(\Omega)$ such that each $u \in H^1(\Omega)$ has a unique representation of the form $u = \alpha u_0 + u_1$ with $\alpha \in \mathbb{C}$ and $u_1 \in H^1_0(\Omega)$.

(d) Let $I=(a,b)$ be a finite open interval in $\mathbb{R}$. What is the codimension of $H^1_0(I)$ in $H^1(I)$?

An important special case of $H^\ell(\Omega)$ is $H^\ell(\mathbb{R}^n)$. Many properties of functions, $f(x)$, in this space become easy to understand when expressed in terms of their Fourier transforms

$$\hat{f}(k) = \int e^{-ik\cdot x} f(x)\ d^n x \tag{2.1.1a}$$

Recall the following basic properties of Fourier transforms. See, for example, [RS2, §IX.1].

(a) Let

$$\mathcal{S}(\mathbb{R}^n) = \left\{\ u \in C^\infty(\mathbb{R}^n)\ \middle|\ \sup_{x\in\mathbb{R}^n} \big|(1+|x|^m)\partial^\alpha u(x)\big| < \infty \text{ for all } m \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n\ \right\}$$

be the space of all C^∞ functions on $\mathbb{R}^n$ all of whose derivatives (including the function itself) decay faster than any polynomial at infinity. It is called Schwartz space. If $f(x) \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f}(k) \in \mathcal{S}(\mathbb{R}^n)$. In fact, one usually first defines the map $f \mapsto \hat{f}$ just for $f \in \mathcal{S}(\mathbb{R}^n)$ and then extends it to all $f \in L^2(\mathbb{R}^n)$ by continuity, using

(b) The L^2 norm of a function is the same as the L^2 norm of its Fourier transform, up to some factors of 2π that depend on your Fourier transform conventions. For the definition (2.1.1a),

$$\int \big|f(x)\big|^2\ d^n x = \int \big|\hat{f}(k)\big|^2\ \tfrac{d^n k}{(2\pi)^n} \tag{2.1.1b}$$

(c) The inverse Fourier transform of $g(k)$ is

$$\check{g}(x) = \int e^{ik\cdot x} g(k)\ \tfrac{d^n k}{(2\pi)^n} \tag{2.1.1c}$$

(d) The Fourier transform of $\frac{\partial}{\partial x_m} f(x)$ is $ik_m \hat{f}(k)$ so that the Fourier transform of $\partial^\alpha f(x)$ is $i^{|\alpha|} k^\alpha \hat{f}(k)$ where $k^\alpha = k_1^{\alpha_1} \cdots k_n^{\alpha_n}$.

(e) The Fourier transform (with respect to x) of the translate $f(x+z)$ is $e^{ik\cdot z}\hat{f}(k)$.

Problem 2.1.7 The goal of this problem is to prove the Paley-Wiener theorem, which says that a function f is C^∞ and supported in the closed ball $\bar{B}_R = \left\{\ x \in \mathbb{R}^n\ \middle|\ |x| \le R\ \right\}$ if and only if $\hat{f}(k)$ extends to a holomorphic function on $\mathbb{C}^n$ which obeys

$$\big|\hat{f}(k)\big| \le \tfrac{C_N}{1+|k|^{2N}} e^{R|\operatorname{Im} k|} \tag{2.1.2}$$

for all $N \in \mathbb{N}$.

(a) Let $f \in C_0^\infty(\mathbb{R}^n)$ be supported in $\bar{B}_R$. Prove that $\hat{f}(k)$ extends to a holomorphic function on $\mathbb{C}^n$ and that, for each $N \in \mathbb{N}$, there is a constant C_N such that (2.1.2) holds.

(b) Assume that the Fourier transform $\hat{f}(k)$ of a function $f(x)$ extends to a holomorphic function on $\mathbb{C}^n$ and that, for each $N \in \mathbb{N}$, there is a constant C_N such that (2.1.2) holds. Let $p \in \mathbb{R}^n$. Prove that

$$f(x) = e^{-p\cdot x} \int e^{ik\cdot x} \hat{f}(k+ip)\ \tfrac{d^n k}{(2\pi)^n}$$

(c) Prove that, under the hypotheses of part (b), $f(x)$ is supported in $\bar{B}_R$.

Using properties (b) and (d) of the Fourier transform, we have that, when $\Omega = \mathbb{R}^n$, the (square of the) norm of Definition 2.1.1 is

$$\|u\|_{\ell,\mathbb{R}^n}^2 = \int_{\mathbb{R}^n} \Big(\sum_{|\alpha|\le\ell} k^{2\alpha} \Big) \big|\hat{u}(k)\big|^2 \ \tfrac{d^n k}{(2\pi)^n}$$

It is traditional to replace the norm $\|u\|_{\ell,\mathbb{R}^n}$ with

$$|u|_\ell = \Big[\int_{\mathbb{R}^n} \big(1+|k|^2\big)^\ell \big|\hat{u}(k)\big|^2 \ \tfrac{d^n k}{(2\pi)^n} \Big]^{1/2}$$

There are constants c and C, depending only on n and ℓ, such that

$$\sum_{|\alpha|\le\ell} k^{2\alpha} \le c^2 \big(1+|k|^2\big)^\ell \qquad \big(1+|k|^2\big)^\ell \le C^2 \Big(\sum_{|\alpha|\le\ell} k^{2\alpha} \Big)$$

for all $k \in \mathbb{R}^n$. Consequently the norms $\|\ \cdot\ \|_{\ell,\mathbb{R}^n}$ and $|\ \cdot\ |_\ell$ are equivalent in the sense that

$$\|u\|_{\ell,\mathbb{R}^n} \le c|u|_\ell \qquad |u|_\ell \le C\|u\|_{\ell,\mathbb{R}^n}$$

for all $u \in H^\ell(\mathbb{R}^n)$. So a sequence of functions converges with respect to one of the norms if and only if it converges with respect to the other.

Since $\big(1+|k|^2\big)^s$ makes sense and is positive for all real s, the right hand side of

$$|u|_s = \Big[\int_{\mathbb{R}^n} \big(1+|k|^2\big)^s \big|\hat{u}(k)\big|^2 \ \tfrac{d^n k}{(2\pi)^n} \Big]^{1/2}$$

(which we also write $|u|_{s,n}$ when we wish to specify the dimension explicitly) makes sense (though may be $+\infty$) for all functions $u \in L^2(\mathbb{R}^n)$. So we may define

Definition 2.1.4 **($H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$)** The Sobolev space $H^s(\mathbb{R}^n)$ is the completion[2] of

$$\{\ u \in L^2(\mathbb{R}^n)\ \big|\ |u|_s < \infty\ \}$$

equipped with the inner product

$$\langle u, v\rangle_s = \int_{\mathbb{R}^n} \big(1+|k|^2\big)^s \hat{u}(k)\overline{\hat{v}(k)} \ \tfrac{d^n k}{(2\pi)^n}$$

---

[2] Readers already familiar with distributions can avoid taking a completion by defining $H^s(\mathbb{R}^n)$ to be the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transforms, $\hat{u}(k)$, obey $\int (1+|k|^2)^s |\hat{u}(k)|^2 \ d^n k < \infty$.

Lemma 2.1.5 *Let* $s \in \mathbb{R}$*.*

(a) The map $u \in \mathcal{S}(\mathbb{R}^n) \mapsto \hat{u} \in \mathcal{S}(\mathbb{R}^n)$ *has a unique extension to a bounded, linear map*

$$\mathcal{F}_s : H^s(\mathbb{R}^n) \to \Big\{\ g : \mathbb{R}^n \to \mathbb{C},\ \textit{measureable}\ \Big|\ \textstyle\int_{\mathbb{R}^n} \big(1+|k|^2\big)^s |g(k)|^2\ \frac{d^nk}{(2\pi)^n} < \infty\ \Big\}$$

This extension is unitary (that is, one–to–one, onto and inner product preserving) when the target space is equipped with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^n} \big(1+|k|^2\big)^s f(k)\overline{g(k)}\ \tfrac{d^nk}{(2\pi)^n}$$

We also denote $\mathcal{F}_s u = \hat{u}$*.*

(b) Let $\alpha \in \mathbb{N}_0^n$*. The map* $u \in \mathcal{S}(\mathbb{R}^n) \mapsto \partial^\alpha u \in \mathcal{S}(\mathbb{R}^n)$ *has a unique extension (that we persist in denoting* ∂^α*) to a bounded, linear map*

$$\partial^\alpha : H^s(\mathbb{R}^n) \to H^{s-|\alpha|}(\mathbb{R}^n)$$

Proof: The proofs of both parts are trivial applications of the B.L.T. theorem [RS, Theorem I.7]. ■

Remark 2.1.6 If $s > s'$, then there is a natural identification of $H^s(\mathbb{R}^n)$ with a subset of $H^{s'}(\mathbb{R}^n)$. So any element u of $H^s(\mathbb{R}^n)$ can also be viewed as an element of $H^{s'}(\mathbb{R}^n)$ for any $s' < s$. As $\mathcal{F}_{s'}u$ is independent of s', it is safe to use $\hat{u}$ to denote $\mathcal{F}_{s'}u$ for all $s' \le s$. similarly, the absence of "s" in the notation $\partial^\alpha u$ is harmless.

Problem 2.1.8 Let $s \in \mathbb{R}$. Prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Problem 2.1.9 Let $r < s < t$. Prove that, for each $\varepsilon > 0$, there is a $C > 0$, depending only on r, s, t and ε such that $\|f\|_s \le \varepsilon \|f\|_t + C\|f\|_r$ for all $f \in H^t(\mathbb{R}^n)$.

Problem 2.1.10 Let $s \in \mathbb{R}$.

(a) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. By Lemma 2.3.5, the map $u \in \mathcal{S}(\mathbb{R}^n) \mapsto fu \in \mathcal{S}(\mathbb{R}^n)$ has a unique extension to a bounded, linear map on $H^s(\mathbb{R}^n)$. Prove that

$$\partial^\alpha(fu) = \sum_{\substack{\beta \in \mathbb{N}_0^n \\ \beta \le \alpha}} \tbinom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta}u)$$

for all $u \in H^s(\mathbb{R}^n)$. Here $\beta \le \alpha$ if and only if $\beta_i \le \alpha_i$ for all $1 \le i \le n$, $(\alpha-\beta)_i = \alpha_i - \beta_i$ for all $1 \le i \le n$ and $\binom{\alpha}{\beta} = \prod_{i=1}^n \frac{\alpha_i!}{\beta_i!(\alpha_i-\beta_i)!}$.

(b) Let $u, v \in H^s(\mathbb{R}^n)$ and let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Make up a definition for "$u = v$ on $\mathcal{O}$".

(c) Prove that differentiation is local in the sense that if $u, v \in H^s(\mathbb{R}^n)$ and $\mathcal{O}$ is any open subset of $\mathbb{R}^n$ with $u = v$ on $\mathcal{O}$, then $\partial^\alpha u = \partial^\alpha v$ on $\mathcal{O}$ for all $\alpha \in \mathbb{N}_0^n$.

Problem 2.1.11 (Sobolev Imbedding) Let $\alpha \in \mathbb{N}_0^n$ and $s > |\alpha| + \frac{n}{2}$. Prove that if $u \in H^s(\mathbb{R}^n)$, then $\partial^\alpha u$ is continuous and there is a constant C, depending only on s, n and $|\alpha|$, such that

$$\sup_{x \in \mathbb{R}^n} \big|\partial^\alpha u(x)\big| \le C \| u \|_s$$

Problem 2.1.12 (Logarithmic Convexity) Let s and s' be real numbers and $0 \le \mu \le 1$. Prove that

$$\| u \|_{\mu s + (1-\mu)s'} \le \| u \|_s^\mu \ \| u \|_{s'}^{1-\mu}$$

for all $u \in H^{\max\{s,s'\}}(\mathbb{R}^n)$.

Problem 2.1.13 Let $\ell \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^n$ be open and $u \in H^\ell(\Omega)$.

(a) Let $K \subset \Omega$ be compact. Suppose that $u(x) = 0$ for all $x \in \Omega \setminus K$. Prove that $u \in H_0^\ell(\Omega)$.

(b) Let $K \subset \Omega$. Suppose that $u(x) = 0$ for all $x \in \Omega \setminus K$ and that, for each $R > 0$, $K \cap \{|x| \le R\}$ is compact. Prove that $u \in H_0^\ell(\Omega)$.

While it is possible [A] to define and work with $H^s(\Omega)$ and $H_0^s(\Omega)$ for positive noninteger s, we only need the latter. The motivation for our definition is the observation that Fourier transforming converts differentiation with respect to x into multiplication by ik. We can think of multiplication by $|k|^\sigma$ as corresponding to taking a fractional derivative in x–space.

Definition 2.1.7 ($H_0^s(\Omega)$, $s \ge 0$) If s happens to be an integer, we use Definition 2.1.1. Otherwise, we define $H_0^s(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ under the norm $\| \cdot \|_s$.

Problem 2.1.14 Let $s < t$ and Ω be a bounded open subset of $\mathbb{R}^n$. Since $\| u \|_s \le \| u \|_t$, $H_0^t(\Omega) \subset H_0^s(\Omega)$. The goal of this problem is to prove Rellich's theorem, which states that any bounded subset of $H_0^t(\Omega)$ is precompact when viewed as a subset of $H_0^s(\Omega)$.

(a) Let ψ be a real valued function in $C_0^\infty(\mathbb{R}^n)$ that is identically one on Ω. Prove that if $u \in C_0^\infty(\Omega)$ then, for all $\alpha \in \mathbb{N}_0^n$,

$$\big|\partial_k^\alpha \hat{u}(k)\big| \le \| u \|_t \| \psi_k^\alpha \|_{-t}$$

where $\psi_k^\alpha(x) = x^\alpha e^{ik\cdot x}\psi(x)$.

(b) Prove that if $u \in H_0^t(\Omega)$ then, for all $\alpha \in \mathbb{N}_0^n$, $\partial_k^\alpha \hat{u}(k)$ exists, is continuous and obeys the bound of part (a).

(c) Let $r > 0$. Prove that if $\{u_i\}_{i\in\mathbb{N}} \subset \left\{\ u \in H_0^t(\Omega)\ \middle|\ |u|_t \le r\ \right\}$, then there is a subsequence u_{i_j} which converges in $H_0^s(\Omega)$. *Hints:* (1) Let $R > 0$. Use the Arzelà–Ascoli theorem to prove the existence of a subsequence that converges uniformly on $|k| \le R$. (2) Bound the two terms in

$$\begin{aligned}|u_{i_j} - u_{i_\ell}|_s^2 = \int_{|k|\le R} &(1+|k|^2)^s \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 \tfrac{d^nk}{(2\pi)^n} \\ &+ \int_{|k|>R} (1+|k|^2)^{-(t-s)}(1+|k|^2)^t \big|\hat{u}_{i_j}(k) - \hat{u}_{i_\ell}(k)\big|^2 \tfrac{d^nk}{(2\pi)^n}\end{aligned}$$

separately.

Our next job is to define $H^s(\Omega)$ for negative s. For motivation, we again find a characterization of $H^s(\mathbb{R}^n)$ that does not use the Fourier transform. Recall that a bounded linear functional on a Banach space $\mathcal{B}$ is a linear map $\mathcal{L} : \mathcal{B} \to \mathbb{C}$ obeying

$$\|\mathcal{L}\| = \sup_{\substack{v\in\mathcal{B}\\ \|v\|\ne 0}} \tfrac{|\mathcal{L}v|}{\|v\|} < \infty$$

and that the dual space, $\mathcal{B}^*$, of $\mathcal{B}$ is the Banach space of all bounded linear functionals on $\mathcal{B}$. The following Proposition proves that $H^s(\mathbb{R}^n)^* \cong H^{-s}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Proposition 2.1.8 *Let $s \in \mathbb{R}$.*

(a) Let $u \in H^s(\mathbb{R}^n)$ and $v \in H^{-s}(\mathbb{R}^n)$. Then the map

$$\begin{aligned} H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) &\to \mathbb{C} \\ (u,v) &\mapsto \int \hat{u}(k)\overline{\hat{v}(k)}\ \tfrac{d^nk}{(2\pi)^n}\end{aligned}$$

is sesquilinear (i.e. linear in the first argument and conjugate linear in the second) and obeys

$$\Big|\int \hat{u}(k)\overline{\hat{v}(k)}\ \tfrac{d^nk}{(2\pi)^n}\Big| \le |u|_s\, |v|_{-s}$$

(b) If $\mathcal{L} \in H^s(\mathbb{R}^n)^*$, then there exists $v \in H^{-s}(\mathbb{R}^n)$ such that

$$\mathcal{L}u = \int \hat{u}(k)\overline{\hat{v}(k)}\ \tfrac{d^nk}{(2\pi)^n}$$

Furthermore $\|\mathcal{L}\| = |v|_{-s}$.

Proof: (a) By Cauchy–Schwarz

$$\begin{aligned}\left|\int \hat{u}(k)\overline{\hat{v}(k)}\ \tfrac{d^n k}{(2\pi)^n}\right| &\le \int \left|\left(1+|k|^2\right)^{s/2}\hat{u}(k)\right|\ \left|\left(1+|k|^2\right)^{-s/2}\hat{v}(k)\right|\ \tfrac{d^n k}{(2\pi)^n}\\ &\le |u|_s\,|v|_{-s}\end{aligned}$$

The linearity in u and conjugate linearity in v are obvious.

(b). Let $\mathcal{L} \in H^s(\mathbb{R}^n)^*$. By the Riesz representation theorem [RS, Theorem II.4] there is a $g \in H^s(\mathbb{R}^n)$ such that

$$\mathcal{L}u = \langle u, g\rangle_s = \int \left(1+|k|^2\right)^s \hat{u}(k)\,\overline{\hat{g}(k)}\ \tfrac{d^n k}{(2\pi)^n}$$

Furthermore $\|\mathcal{L}\| = |g|_s$. Then the $v \in H^{-s}(\mathbb{R}^n)$ with $\hat{v}(k) = \left(1+|k|^2\right)^s \hat{g}(k)$, using the Fourier transform of Lemma 2.1.5, fulfills the requirements of the Lemma. ■

Problem 2.1.15

(a) Let $s \in \mathbb{R}$ and $v \in H^{-s}(\mathbb{R}^n)$. Prove that

$$|v|_{-s,n} = \sup_{\substack{u\in H^s(\mathbb{R}^n)\\ |u|_{s,n}\le 1}} \left|\int \hat{u}(k)\overline{\hat{v}(k)}\ \tfrac{d^n k}{(2\pi)^n}\right|$$

(b) Let $v \in L^2(\mathbb{R}^n)$ and $s \ge 0$. Prove that

$$|v|_{-s,n} = \sup_{\substack{u\in H^s(\mathbb{R}^n)\\ |u|_{s,n}\le 1}} \left|\langle u, v\rangle_{L^2(\mathbb{R}^n)}\right|$$

(c) Let $t < s$ and $v \in H^t(\mathbb{R}^n)$. Prove that if

$$M = \sup\left\{ \left|\int \hat{u}(k)\overline{\hat{v}(k)}\ \tfrac{d^n k}{(2\pi)^n}\right| \ \middle|\ \hat{u} \in C_0^\infty(\mathbb{R}^n),\ |u|_{-s,n} \le 1 \right\} < \infty$$

then $v \in H^s(\mathbb{R}^n)$ and $|v|_{s,n} = M$.

Corollary 2.1.9 *Let $\ell \in \mathbb{N}_0$. There are constants c, C, depending only on n and ℓ such that for each $u \in H^{-\ell}(\mathbb{R}^n)$ there are $f_\alpha \in L^2(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le \ell$ such that*

$$u = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|\le\ell}} \partial^\alpha f_\alpha \qquad \textit{and} \qquad c|u|_{-\ell}^2 \le \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|\le\ell}} \|f_\alpha\|_{L^2(\mathbb{R}^n)}^2 \le C|u|_{-\ell}^2$$

Proof: By Lemma 2.1.5.a, $(1+|k|^2)^{-\ell/2}\hat{u}(k) \in L^2(\mathbb{R}^n)$. Let L be any integer with $L \ge \frac{\ell}{2}$. Set $F(k) = (1+|k|^2)^{-L}\hat{u}(k)$ and write

$$(1+|k|^2)^L = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} k^\alpha P_\alpha(k)$$

where $P_\alpha(k)$ is a polynomial in $k_1, \cdots, k_n$ of degree at most $2L-\ell$. There are many such representations. Pick any one you like. For each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ determine f_α by $\hat{f}_\alpha = (-i)^{|\alpha|}P_\alpha(k)F(k)$. Since $P_\alpha(k)^2$ is of degree at most $4L-2\ell$, there is a constant $\tilde{C}$, which depends on n, L and the P_α's we chose, but not on u, such that $P_\alpha(k)^2 \le \tilde{C}(1+|k|^2)^{2L-\ell}$. Hence

$$\begin{aligned}\|f_\alpha\|^2_{L^2(\mathbb{R}^n)} &= \int P^2_\alpha(k)|F(k)|^2 \tfrac{d^nk}{(2\pi)^n} = \int \tfrac{P^2_\alpha(k)}{(1+|k|^2)^{2L}}|\hat{u}(k)|^2 \tfrac{d^nk}{(2\pi)^n} \le \tilde{C}\int \tfrac{|\hat{u}(k)|^2}{(1+|k|^2)^\ell} \tfrac{d^nk}{(2\pi)^n} \\ &= \tilde{C}|\!|\!|u|\!|\!|^2_{-\ell}\end{aligned}$$

and

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} (ik)^\alpha \hat{f}_\alpha(k) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} k^\alpha P_\alpha(k)F(k) = (1+|k|^2)^L F(k) = \hat{u}(k)$$

The final remaining bound is

$$\begin{aligned}|\!|\!|u|\!|\!|^2_{-\ell} &= \int \tfrac{|\hat{u}(k)|^2}{(1+|k|^2)^\ell} \tfrac{d^nk}{(2\pi)^n} = \int \tfrac{|\Sigma_\alpha (ik)^\alpha \hat{f}_\alpha(k)|^2}{(1+|k|^2)^\ell} \tfrac{d^nk}{(2\pi)^n} \\ &\le (\ell+1)^n \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} \int \tfrac{|k^{2\alpha}|}{(1+|k|^2)^\ell}|\hat{f}_\alpha(k)|^2 \tfrac{d^nk}{(2\pi)^n} \\ &\le (\ell+1)^n \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} \int |\hat{f}_\alpha(k)|^2 \tfrac{d^nk}{(2\pi)^n} = (\ell+1)^n \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} \|f_\alpha\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

We used Cauchy–Schwarz to get from the first line to the second line. ■

Definition 2.1.10 **($H^s(\Omega)$, $s<0$)** If $s<0$, we define $H^s(\Omega) = H_0^{-s}(\Omega)^*$. The norm on $H_0^{-s}(\Omega)^*$ is denoted $\|\cdot\|_{s,\Omega}$.

Let $s<0$. By Problem 2.1.8, $H^{-s}(\mathbb{R}^n) = H_0^{-s}(\mathbb{R}^n)$. So Proposition 2.1.8 states that $H^s(\mathbb{R}^n) \cong H^{-s}(\mathbb{R}^n)^* = H_0^{-s}(\mathbb{R}^n)^*$. Thus, Definition 2.1.10, in the special case $\Omega = \mathbb{R}^n$, is consistent with our Definition 2.1.4 of $H^s(\mathbb{R}^n)$. But in general, $H_0^{-s}(\Omega)$ can be a closed strict subspace of $H^{-s}(\Omega)$. By the Hahn–Banach theorem [RS, Theorem III.6], every bounded linear functional on $H_0^{-s}(\Omega)$ extends to a bounded linear functional on $H^{-s}(\Omega)$. But, when $H_0^{-s}(\Omega)$ is a closed strict subspace of $H^{-s}(\Omega)$, every bounded linear functional on $H_0^{-s}(\Omega)$ can be extended to many different bounded linear functionals on $H^{-s}(\Omega)$ and $H^{-s}(\Omega)^*$ is much larger than $H_0^{-s}(\Omega)^*$. We shall prove, in Theorem 2.1.11, below, that every bounded linear functional $\mathcal{L} \in H_0^\ell(\Omega)^*$, $\ell \in \mathbb{N}_0$, is of the form

$$\mathcal{L}u = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} \langle \partial^\alpha u, f_\alpha\rangle_{L^2(\Omega)}$$

for some $f_\alpha \in L^2(\Omega)$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le \ell$. The same representation, and even the same proof, also applies to $\mathcal{L} \in H^\ell(\Omega)^*$. The difference is that, when the f_α's are sufficiently regular and we integrate by parts to move the ∂_α's from the u to the f_α's, there are no boundary terms when $u \in H_0^\ell(\Omega)$ while there can be boundary terms when $u \in H^\ell(\Omega)$. This makes it more useful to define $H^s(\Omega) = H_0^{-s}(\Omega)^*$ rather than $H^s(\Omega) = H^{-s}(\Omega)^*$.

Theorem 2.1.11 *Let $\ell \in \mathbb{N}_0$.*

(a) Let $\mathcal{L} \in H_0^\ell(\Omega)^*$. There exist $f_\alpha \in L^2(\Omega)$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le \ell$ such that

$$\mathcal{L}u = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} \langle \partial^\alpha u, f_\alpha\rangle_{L^2(\Omega)} \tag{2.1.3}$$

for all $u \in H_0^\ell(\Omega)$. Furthermore

$$\|\mathcal{L}\| = \min\Big\{ \Big[\sum_{|\alpha|\le\ell} \|f_\alpha\|^2_{L^2(\Omega)} \Big]^{1/2} \ \Big|\ (f_\alpha)_{|\alpha|\le\ell} \text{ obeys } (2.1.3) \Big\}$$

(b) If $v \in L^2(\Omega)$ then

$$\mathcal{L}_v u = \langle u, \bar{v}\rangle_{L^2(\Omega)}$$

defines a bounded linear functional on $H_0^\ell(\Omega)$ with $\|\mathcal{L}_v\|_{-\ell,\Omega} \le \|v\|_{L^2(\Omega)}$. Furthermore $\{\ \mathcal{L}_v \ \big|\ v \in C_0^\infty(\Omega)\ \}$ is dense in $H_0^\ell(\Omega)^*$.

Proof: (a) Let $N = \#\{\ \alpha \in \mathbb{N}_0^n \ \big|\ |\alpha| \le \ell\ \}$ and L^2_N be the Hilbert space $\oplus_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} L^2(\Omega)$. The elements of L^2_N are of the form $\big(f_\alpha\big)_{|\alpha|\le\ell}$ with each $f_\alpha \in L^2(\Omega)$. The inner product on L^2_N is

$$\Big\langle \big(f_\alpha\big)_{|\alpha|\le\ell}, \big(g_\alpha\big)_{|\alpha|\le\ell} \Big\rangle = \sum_{|\alpha|\le\ell} \langle f_\alpha, g_\alpha\rangle_{L^2(\Omega)}$$

If $|\alpha| \le \ell$, the map $v \in \big\{\, v \in C^\ell(\Omega) \bigm| \|v\|_{\ell,\Omega} < \infty \,\big\} \mapsto \partial^\alpha v$ has a unique extension to a bounded, linear map from $H^\ell(\Omega)$ to $L^2(\Omega)$. Define the map

$$\begin{aligned} P : H_0^\ell(\Omega) &\to L_N^2(\Omega) \\ u &\mapsto \big(\partial^\alpha u\big)_{|\alpha|\le\ell} \end{aligned}$$

This is a unitary map from $H_0^\ell(\Omega)$ to its range W, which is a closed linear subspace of $L_N^2(\Omega)$. Hence

$$\tilde{\mathcal{L}}(Pu) = \mathcal{L}(u)$$

defines a bounded linear functional on W with norm $\|\mathcal{L}\|$. As W is itself a Hilbert space, the Riesz representation theorem [RS, Theorem II.4] implies that there is a vector $(f_\alpha)_{|\alpha|\le\ell}$ in W such that

$$\tilde{\mathcal{L}}\big((v_\alpha)_{|\alpha|\le\ell}\big) = \Big\langle \big(v_\alpha\big)_{|\alpha|\le\ell}, (f_\alpha)_{|\alpha|\le\ell} \Big\rangle = \sum_{|\alpha|\le\ell} \langle v_\alpha, f_\alpha\rangle_{L^2(\Omega)}$$

and that $\|\tilde{\mathcal{L}}\| = \big\|(f_\alpha)_{|\alpha|\le\ell}\big\|_{L_N^2(\Omega)}$. Hence

$$\mathcal{L}(u) = \tilde{\mathcal{L}}(Pu) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le\ell}} \langle \partial^\alpha u, f_\alpha\rangle_{L^2(\Omega)}$$

If $(\tilde f_\alpha)_{|\alpha|\le\ell}$ is any other vector in $L_N^2(\Omega)$ that obeys

$$\tilde{\mathcal{L}}\big((v_\alpha)_{|\alpha|\le\ell}\big) = \Big\langle \big(v_\alpha\big)_{|\alpha|\le\ell}, (\tilde f_\alpha)_{|\alpha|\le\ell} \Big\rangle = \sum_{|\alpha|\le\ell} \Big\langle v_\alpha, \tilde f_\alpha\Big\rangle_{L^2(\Omega)}$$

for all $(v_\alpha)_{|\alpha|\le\ell} \in W$, then

$$\Big\langle \big(v_\alpha\big)_{|\alpha|\le\ell}, (\tilde f_\alpha - f_\alpha)_{|\alpha|\le\ell} \Big\rangle = 0$$

for all $(v_\alpha)_{|\alpha|\le\ell} \in W$. This says that $(\tilde f_\alpha - f_\alpha)_{|\alpha|\le\ell} \in W^\perp$, the orthogonal complement of W in $L_N^2(\Omega)$. Consequently

$$\big\|(\tilde f_\alpha)_{|\alpha|\le\ell}\big\|_{L_N^2(\Omega)} \ge \big\|(f_\alpha)_{|\alpha|\le\ell}\big\|_{L_N^2(\Omega)} = \|\tilde{\mathcal{L}}\| = \|\mathcal{L}\|$$

(b) Let $v \in L^2(\Omega)$. Since $H_0^\ell(\Omega) \subset L^2(\Omega)$, $\mathcal{L}_v$ is a well–defined linear functional on $H_0^\ell(\Omega)$. The boundedness and the claimed bound follow from

$$|\mathcal{L}_v u| = \big|\, \langle u, \bar v\rangle_{L^2(\Omega)} \,\big| \le \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)} \|u\|_{\ell,\Omega}$$

Now we prove the denseness. Since $H_0^\ell(\Omega)$ is a Hilbert space, the Riesz representation theorem says that, for each $\mathcal{L} \in H_0^\ell(\Omega)^*$, there is a vector $u_{\mathcal{L}} \in H_0^\ell(\Omega)$ such that $\mathcal{L}w = \langle w, u_{\mathcal{L}} \rangle_{H_0^\ell(\Omega)}$ for all $w \in H_0^\ell(\Omega)$. The map $\mathcal{L} \mapsto u_{\mathcal{L}}$ is an isometry from $H_0^\ell(\Omega)^*$ to $H_0^\ell(\Omega)$. So if $\{\ \mathcal{L}_v \ \big|\ v \in C_0^\infty(\Omega)\ \}$ is not dense in $H_0^\ell(\Omega)^*$, then $\{\ u_{\mathcal{L}_v} \ \big|\ v \in C_0^\infty(\Omega)\ \}$ is not dense in $H_0^\ell(\Omega)$ and there is a nonzero vector $w \in H_0^\ell(\Omega)$ that is orthogonal to $\{\ u_{\mathcal{L}_v} \ \big|\ u \in C_0^\infty(\Omega)\ \}$. But

$$\begin{aligned} &w \perp \{\ u_{\mathcal{L}_v} \ \big|\ v \in C_0^\infty(\Omega)\ \} \\ &\qquad\qquad \Longrightarrow\ 0 = \langle w, u_{\mathcal{L}_v} \rangle_{H_0^\ell(\Omega)} = \mathcal{L}_v(w) = \langle w, \bar{v} \rangle_{L^2(\Omega)} \ \text{ for all } v \in C_0^\infty(\Omega) \end{aligned}$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, w must be zero as an element of $L^2(\Omega)$ and hence also as an element of $H_0^\ell(\Omega)$, which contradicts the assumption that $\{\ \mathcal{L}_v \ \big|\ v \in C_0^\infty(\Omega)\ \}$ is not dense in $H_0^\ell(\Omega)^*$. ■

Remark 2.1.12 For each $\ell \in \mathbb{N}$, we identify each $v \in L^2(\Omega)$ with the corresponding $\mathcal{L}_v \in H^{-\ell}(\Omega) = H_0^\ell(\Omega)^*$ of Theorem 2.1.11. Using this identification, $H^{-\ell}(\Omega)$ is the completion of $L^2(\Omega)$ under the norm

$$\|v\|_{-\ell,\Omega} = \sup_{0 \ne u \in H_0^\ell(\Omega)} \big|\, \langle u, \bar{v} \rangle_{L^2(\Omega)} \,\big| / \|u\|_{\ell,\Omega}$$

We also identify each $v \in H^0(\Omega) = L^2(\Omega)$ with the corresponding $\mathcal{L}_v \in H_0^0(\Omega)^*$ of Theorem 2.1.11, so that $L^2(\Omega) = H^0(\Omega) = H_0^0(\Omega)^*$.

Remark 2.1.13 Let $\ell', \ell \in \mathbb{N}_0$ with $\ell' \le \ell$. We saw in Remark 2.1.2 that $H_0^\ell(\Omega) \subset H_0^{\ell'}(\Omega)$ and $\|u\|_{\ell',\Omega} \le \|u\|_{\ell,\Omega}$ for all $u \in H_0^\ell(\Omega)$. Now consider any $v \in H^{-\ell'}(\Omega) = H_0^{\ell'}(\Omega)^*$. It obeys

$$\big|v(u)\big| \le \|v\|_{-\ell',\Omega}\|u\|_{\ell',\Omega} \le \|v\|_{-\ell',\Omega}\|u\|_{\ell,\Omega}$$

for all $u \in H_0^\ell(\Omega)$. Thus the restriction of v to $H_0^\ell(\Omega)$ is an element of $H^{-\ell}(\Omega) = H_0^\ell(\Omega)^*$, which we again call v, and $\|v\|_{-\ell,\Omega} \le \|v\|_{-\ell',\Omega}$.

Combining this with the corresponding part of Remark 2.1.2, we have that if $\ell, \ell' \in \mathbb{Z}$ with $\ell \le \ell'$, then $H^{\ell'}(\Omega) \subset H^\ell(\Omega)$ and $\|v\|_{\ell,\Omega} \le \|v\|_{\ell',\Omega}$ for all $v \in H^{\ell'}(\Omega)$.

Lemma 2.1.14 *Let* $\ell \in \mathbb{Z}$.

(a) Let $\varphi : \Omega \to \mathbb{C}$ *and all of its derivatives of order at most* $|\ell|$ *be bounded and continuous. The map* $v \in \{\ v \in C^{|\ell|}(\Omega) \ \big|\ \|v\|_{\max\{0,\ell\},\Omega} < \infty\ \} \mapsto \varphi v$ *has a unique extension to a bounded, linear map on* $H^\ell(\Omega)$ *and there is a constant* $C_{|\ell|,n}$*, depending only on* $|\ell|$ *and* n*, such that*

$$\|\varphi u\|_{\ell,\Omega} \le C_{|\ell|,n} \|\varphi\|_{C^{|\ell|}(\Omega)} \|u\|_{\ell,\Omega}$$

for all $u \in H^\ell(\Omega)$*. Here* $\|\varphi\|_{C^\ell(\Omega)} = \sup\limits_{\substack{x\in\Omega \\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\le\ell}} \big|\partial^\alpha\varphi(x)\big|$.

(b) Let $\alpha \in \mathbb{N}_0^n$*. In the case* $0 < \ell < |\alpha|$*, assume that* $\big\{\ v \in C^{|\alpha|}(\Omega)\ \big|\ \|v\|_{|\alpha|,\Omega} < \infty\ \big\}$ *is dense in* $H^\ell(\Omega)$*. The map* $v \in \big\{\ v \in C^{\max\{\ell,|\alpha|\}}(\Omega)\ \big|\ \|v\|_{\max\{\ell,|\alpha|\},\Omega} < \infty\ \big\} \mapsto \partial^\alpha v$ *has a unique extension to a bounded, linear map from* $H^\ell(\Omega)$ *to* $H^{\ell-|\alpha|}(\Omega)$*.*

Problem 2.1.16 Prove Lemma 2.1.14.

Problem 2.1.17 This problem illustrates the need for the multiplier φ of part (a) of Lemma 2.1.14 to be relatively smooth if the product φv is to be well–defined when v is quite unsmooth. In this problem, φ will be a characteristic function with a discontinuity at 0 and there will be two different v's. One will be a Dirac delta function and the other the derivative of a Dirac delta function. Both will be supported on the point 0. Let $\Omega = (-1,1)$. Let $w \in C_0^\infty(\mathbb{R})$ be supported in $\big[-\frac{1}{2},\frac{1}{2}\big]$. Assume that w is even and obeys $\int_{\mathbb{R}} w(x)\,dx = 1$ and $w'(x) \ge 0$ for $x < 0$. Define, for $\varepsilon > 0$ and $u \in C_0^\infty(\Omega)$,

$$\begin{array}{lll} \delta(u) = u(0) & \delta_\varepsilon(u) = \displaystyle\int_{-1}^{1} \tfrac{1}{\varepsilon} w\big(\tfrac{x}{\varepsilon}\big) u(x)\,dx & \delta_{\varepsilon,+}(u) = \displaystyle\int_{-1}^{1} \tfrac{1}{\varepsilon} w\big(\tfrac{x-\varepsilon}{\varepsilon}\big) u(x)\,dx \\ \delta'(u) = -u'(0) & \delta'_\varepsilon(u) = \displaystyle\int_{-1}^{1} \tfrac{1}{\varepsilon^2} w'\big(\tfrac{x}{\varepsilon}\big) u(x)\,dx & \end{array}$$

(a) Prove that δ, δ_ε and $\delta_{\varepsilon,+}$ all have unique continuous extensions to elements of $H^{-1}(\Omega)$. Prove that δ' and δ'_ε have unique continuous extensions to elements of $H^{-2}(\Omega)$.

(b) Prove that

$$\lim_{\varepsilon\to 0+} \delta_\varepsilon = \delta \qquad \text{and} \qquad \lim_{\varepsilon\to 0+} \delta_{\varepsilon,+} = \delta$$

in $H^{-2}(\Omega)$ and

$$\lim_{\varepsilon\to 0+} \delta'_\varepsilon = \delta'$$

in $H^{-3}(\Omega)$.

(c) Let φ be the characteristic function of $(0,1)$. Prove that

$$\lim_{\varepsilon\to 0+} \varphi\delta_\varepsilon = \tfrac{1}{2}\delta \qquad \text{and} \qquad \lim_{\varepsilon\to 0+} \varphi\delta_{\varepsilon,+} = \delta$$

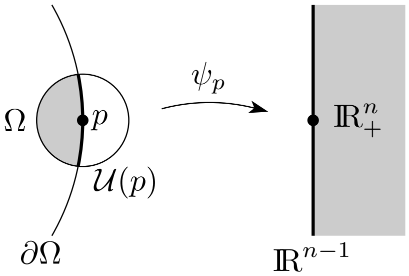
in $H^{-2}(\Omega)$ and that $\lim\limits_{\varepsilon\to 0+} \varphi\delta'_\varepsilon$ diverges in $H^{-\ell}(\Omega)$ for all $\ell \in \mathbb{N}$.

We also want to define the space $H^\ell(\partial\Omega)$ in the same way as the space $H^\ell(\Omega)$, but for functions that are only defined on $\partial\Omega$. To do so, we need a measure on $\partial\Omega$. The easy way to create such a measure is to use local coordinates.

Definition 2.1.15 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

(a) We say that Ω has C^k boundary (or that $\partial\Omega$ is C^k or that $\partial\Omega \in C^k$) if, for each $p \in \partial\Omega$, there is an open neighbourhood $\mathcal{U}(p)$ of p and a C^k diffeomorphism $\psi_p : \mathcal{U}(p) \to \mathbb{R}^n$ such that

$$\begin{aligned} \psi_p\big(\mathcal{U}(p) \cap \Omega\big) &= \left\{\ x \in \mathbb{R}^n \ \middle|\ x_n > 0\ \right\} = \mathbb{R}^n_+ \\ \psi_p\big(\mathcal{U}(p) \cap \partial\Omega\big) &= \left\{\ x \in \mathbb{R}^n \ \middle|\ x_n = 0\ \right\} = \mathbb{R}^{n-1} \end{aligned} \tag{2.1.4}$$



(b) We say that Ω has smooth boundary (or that $\partial\Omega$ is smooth or that $\partial\Omega \in C^\infty$) if each $\psi_p,\ p \in \partial\Omega$, of (a) is a C^∞ diffeomorphism.

For the rest of this chapter, we assume that Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary.

Problem 2.1.18 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $k \in \mathbb{N}$. Prove that Ω has C^k boundary if and only if, for each $p \in \partial\Omega$, there is an open neighbourhood $\mathcal{U}(p)$ of p and a C^k function $\phi_p : \mathcal{U}(p) \to \mathbb{R}$ such that $\nabla\phi_p(p) \neq 0$, $\mathcal{U}(p) \cap \partial\Omega = \phi_p^{-1}(0)$ and $\mathcal{U}(p) \cap \Omega = \phi_p^{-1}\big((0,\infty)\big)$.

Notation 2.1.16 We call a system $\big(\mathcal{U}(p), \psi_p\big)$, $p \in \partial\Omega$ as in Definition 2.1.15, a coordinate system for $\partial\Omega$.

Definition 2.1.17 We say that $f : \partial\Omega \to \mathbb{C}$ is C^∞ and write $f \in C^\infty(\partial\Omega)$ if there is a coordinate system $\big(\mathcal{U}(p), \psi_p\big)$ such that $f \circ \psi_p^{-1} : \mathbb{R}^{n-1} \to \mathbb{C}$ is C^∞.

Definition 2.1.18 **($H^s(\partial\Omega)$, $s \in \mathbb{R}$)** Since $\partial\Omega$ is compact, there exist $p_i,\ i = 1, \cdots, N$, such that $\partial\Omega \subset \cup_{i=1}^N \mathcal{U}(p_i)$. Choose functions $\chi_i \in C_0^\infty\big(\mathcal{U}(p_i)\big)$, $1 \le i \le N$, taking values in $[0,1]$ so that $\sum_{i=1}^N \chi_i = 1$ on some neighbourhood of $\partial\Omega$.

(a) For $f \in C^\infty(\partial\Omega)$ we define

$$\|f\|^2_{s,\partial\Omega} = \sum_{i=1}^N \big|(\chi_i f) \circ \psi_{p_i}^{-1}\big|^2_{s,n-1}$$

(b) We define $H^s(\partial\Omega)$ to be the completion of $C^\infty(\partial\Omega)$ under the norm $\| \ \cdot \ \|_{s,\partial\Omega}$.

Problem 2.1.19 Prove that $\| \ \cdot \ \|_{s,\partial\Omega}$, given in Definition 2.1.18, is a norm. Prove further that

$$\langle f, g\rangle_{s,\partial\Omega} = \sum_{i=1}^{N} \big\langle (\chi_i f)\circ\psi_{p_i}^{-1}, (\chi_i g)\circ\psi_{p_i}^{-1}\big\rangle_{s,n-1}$$

is an inner product.

We prove in Proposition 2.1.20, below, that all of the norms, defined through different choices of partition of unity $\{\chi_i\}$ and coordinate systems $\big(\mathcal{U}(p), \psi_p\big)$ in Definition 2.1.18.a, are equivalent. That is, if $\| \ \cdot \ \|$ and $\| \ \cdot \ \|'$ are two such norms, then there are constants c and c' such that

$$\|f\| \le c\|f\|' \qquad \|f\|' \le c'\|f\|$$

So a sequence of functions converges with respect to one of the norms if and only if it converges with respect to the other.

Lemma 2.1.19 *Let $\varphi \in C_0^\infty(\mathbb{R}^m)$ and let $\psi : \mathbb{R}^m \to \mathbb{R}^m$ be a C^∞ diffeomorphism. Then the map*

$$\begin{aligned} C_0^\infty(\mathbb{R}^m) &\to C_0^\infty(\mathbb{R}^m) \\ f &\mapsto (\varphi f)\circ\psi^{-1} \end{aligned}$$

extends to a bounded linear map on $H^s(\mathbb{R}^m)$.

Proof: By the B.L.T. theorem and Problem 2.1.8, it suffices to prove that there is a constant C such that $\left|(\varphi f)\circ\psi^{-1}\right|_{s,m} \le C|f|_{s,m}$ for all $f \in C_0^\infty(\mathbb{R}^n)$. The case $s \in \mathbb{N}_0$ is Problem 2.1.20, below. The case $s \ge 0$ then follows by the interpolation Lemma 2.3.2.

So let $s < 0$. Recalling, from Problem 2.1.15, that

$$|v|_s = \sup_{|u|_{-s}\le 1} \big| \langle u, v\rangle_{L^2(\mathbb{R}^m)} \big|$$

for all $v \in L^2(\mathbb{R}^n)$, we have

$$\left|(\varphi f)\circ\psi^{-1}\right|_s = \sup_{|u|_{-s}\le 1} \big| \langle u, (\varphi f)\circ\psi^{-1}\rangle_{L^2(\mathbb{R}^m)} \big| = \sup_{|u|_{-s}\le 1} \Big| \int u(x)\overline{(\varphi f)\big(\psi^{-1}(x)\big)}\ d^m x\Big|$$

Making the change of variables $x = \psi(y)$ and using $\left|\frac{\partial x}{\partial y}\right|$ to denote the associated Jacobian determinant

$$\left|(\varphi f)\circ\psi^{-1}\right|_s = \sup_{|u|_{-s}\le 1} \Big| \int \left|\tfrac{\partial x}{\partial y}(y)\right| u\big(\psi(y)\big)\overline{(\varphi f)(y)}\ d^m y\Big|$$

Set $\tilde\varphi(x) = \left|\frac{\partial x}{\partial y}\big(\psi^{-1}(x)\big)\right|\overline{\varphi\big(\psi^{-1}(x)\big)}$ and observe that $\tilde\varphi \in C_0^\infty(\mathbb{R}^m)$, since $\varphi \in C_0^\infty(\mathbb{R}^m)$ and ψ is a C^∞ diffeomorphism of $\mathbb{R}^m$. By the $s > 0$ case of this Lemma, with ψ replaced by ψ^{-1},

$$\begin{aligned}\big|(\varphi f)\circ\psi^{-1}\big|_s &= \sup_{|u|_{-s}\le 1}\left|\int (\tilde\varphi u)\big(\psi(y)\big)\overline{f(y)}\ d^m y\right| \le \sup_{|u|_{-s}\le 1}\big|(\tilde\varphi u)\circ\psi\big|_{-s}\ |f|_s \\ &\le \sup_{|u|_{-s}\le 1} C|u|_{-s}\ |f|_s \qquad\qquad \le C\,|f|_s\end{aligned}$$

■

Problem 2.1.20 Prove Lemma 2.1.19 in the special case that $s \in \mathbb{N}_0$.

Proposition 2.1.20 *All of the norms, defined through different choices of partition of unity $\{\chi_i\}$ and coordinate systems $\big(\mathcal{U}(p), \psi_p\big)$ in Definition 2.1.18.a, are equivalent.*

Proof: Let $\big(\mathcal{U}(p), \psi_p\big)$ and $\big(\tilde{\mathcal{U}}(p), \tilde\psi_p\big)$ be two coordinate systems as in Notation 2.1.16 and let $\chi_i \in C_0^\infty\big(\mathcal{U}(p_i)\big)$, $1 \le i \le N$ and $\tilde\chi_i \in C_0^\infty\big(\tilde{\mathcal{U}}(\tilde p_i)\big)$, $1 \le i \le \tilde N$ be two partitions of unity as in Definition 2.1.18. If $f \in C^\infty(\partial\Omega)$, then

$$(\chi_i f)\circ\psi_{p_i}^{-1} = \sum_{j=1}^{\tilde N}(\tilde\chi_j\chi_i f)\circ\psi_{p_i}^{-1} = \sum_{j=1}^{\tilde N}(\tilde\chi_j\chi_i f)\circ\tilde\psi_{\tilde p_j}^{-1}\circ\tilde\psi_{\tilde p_j}\circ\psi_{p_i}^{-1}$$

Note that $(\tilde\chi_j\chi_i)\circ\psi_{p_i}^{-1}(x)$ vanishes except when $x \in \psi_{p_i}\big(\tilde{\mathcal{U}}(\tilde p_j)\cap\mathcal{U}(p_i)\big)$ and then $\psi_{p_i}^{-1}(x)$ is in the domain of $\tilde\psi_{\tilde p_j}$. Applying Lemma 2.1.19, with $\varphi = \chi_i\circ\tilde\psi_{\tilde p_j}^{-1}$, $\psi^{-1} = \tilde\psi_{\tilde p_j}\circ\psi_{p_i}^{-1}$ and f replaced by $(\tilde\chi_j f)\circ\tilde\psi_{\tilde p_j}^{-1}$,

$$\begin{aligned}\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{s,n-1} &\le \sum_{j=1}^{\tilde N}\big|(\tilde\chi_j\chi_i f)\circ\tilde\psi_{\tilde p_j}^{-1}\circ\tilde\psi_{\tilde p_j}\circ\psi_{p_i}^{-1}\big|_{s,n-1} \le \sum_{j=1}^{\tilde N} C\big|(\tilde\chi_j f)\circ\tilde\psi_{\tilde p_j}^{-1}\big|_{s,n-1} \\ &\le C\tilde N \max_{1\le j\le\tilde N}\big|(\tilde\chi_j f)\circ\tilde\psi_{\tilde p_j}^{-1}\big|_{s,n-1}\end{aligned}$$

and hence

$$\begin{aligned}\sum_{i=1}^{N}\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{s,n-1}^2 &\le C^2\tilde N^2 N \max_{1\le j\le\tilde N}\big|(\tilde\chi_j f)\circ\tilde\psi_{\tilde p_j}^{-1}\big|_{s,n-1}^2 \\ &\le C^2\tilde N^2 N\sum_{j=1}^{\tilde N}\big|(\tilde\chi_j f)\circ\tilde\psi_{\tilde p_j}^{-1}\big|_{s,n-1}^2\end{aligned}$$

completing the proof. ■

Problem 2.1.21 Prove that $\| \cdot \|_{0,\partial\Omega}$ is equivalent to

$$\|f\|_{L^2(\partial\Omega)} = \sqrt{\int_{\partial\Omega} |f(x)|^2 \, d\sigma(x)}$$

where $d\sigma(x)$ is the surface measure on $\partial\Omega$.

Problem 2.1.22 Let $s \in \mathbb{R}$. Prove that there are constants C_s and c_s such that

$$\big| \langle f, \bar{g} \rangle_{L^2(\partial\Omega)} \big| \le C_s \|f\|_{-s,\partial\Omega} \|g\|_{s,\partial\Omega}$$

for all $f, g \in C^\infty(\partial\Omega)$ and

$$\|f\|_{-s,\partial\Omega} \le c_s \sup \big\{ \, \big| \langle f, \bar{g} \rangle_{L^2(\partial\Omega)} \big| \; \big| \; g \in C^\infty(\partial\Omega), \; \|g\|_{s,\partial\Omega} \le 1 \, \big\}$$

for all $f \in C^\infty(\partial\Omega)$. Here, as you would expect,

$$\langle f, g \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} f(x) \overline{g(x)} \, d\sigma(x)$$

where $d\sigma(x)$ is the surface measure on $\partial\Omega$.

Problem 2.1.23 Let $0 \le s \in \mathbb{R}$. By Problem 2.1.22, if $f \in C^\infty(\partial\Omega)$ then

$$g \in C^\infty(\partial\Omega) \mapsto \mathcal{L}_f g = \langle f, \bar{g} \rangle_{L^2(\partial\Omega)}$$

extends to a bounded linear functional on $H^s(\partial\Omega)$ with norm bounded by $C_s \|f\|_{-s,\partial\Omega}$. Prove that the map $f \mapsto \mathcal{L}_f$ has a unique continuous extension to an isomorphism

$$\mathcal{L} : H^{-s}(\partial\Omega) \to H^s(\partial\Omega)^*$$

and that $\big\{ \, \mathcal{L}_f \; \big| \; f \in C^\infty(\partial\Omega) \, \big\}$ is dense in $H^s(\partial\Omega)^*$.

Problem 2.1.24 (Sobolev Imbedding) Let $\ell \in \mathbb{N}_0$ and $s > \ell + \frac{n}{2}$. Prove that if $u \in H^s(\partial\Omega)$, then $u \in C^\ell(\partial\Omega)$ and there is a constant C, depending only on Ω and $|\ell|$, such that

$$\|u\|_{C^\ell(\partial\Omega)} \le C \|u\|_{s,\partial\Omega}$$

Problem 2.1.25 (Logarithmic Convexity) Let s and s' be real numbers and $0 \le \mu \le 1$. Let N be the number of neighbourhoods in the cover of $\partial\Omega$ used in Definition 2.1.18. Prove that

$$\|f\|_{\mu s + (1-\mu)s', \partial\Omega} \le N \|f\|_{s,\partial\Omega}^{\mu} \; \|u\|_{s',\partial\Omega}^{1-\mu}$$

for all $f \in H^{\max\{s,s'\}}(\partial\Omega)$.

Lemma 2.1.21 *Let $\ell \in \mathbb{N}_0$.*

(a) Let $\mathcal{U}$ and $\mathcal{V}$ be bounded open sets in $\mathbb{R}^n$ and let $\psi : \mathcal{U} \to \mathcal{V}$ be a $C^{\max\{1,\ell\}}$ diffeomorphism with $\psi \in C^1(\overline{\mathcal{U}})$ and $\psi^{-1} \in C^{\max\{1,\ell\}}(\overline{\mathcal{V}})$. Then the map

$$\begin{aligned} \left\{\ u \in C^\ell(\mathcal{U}) \ \middle|\ \|u\|_{\ell,\mathcal{U}} < \infty \ \right\} &\to C^\ell(\mathcal{V}) \\ u &\mapsto u \circ \psi^{-1} \end{aligned}$$

extends to a bounded linear map from $H^\ell(\mathcal{U})$ to $H^\ell(\mathcal{V})$.

(b) Let $\mathcal{U}$, $\mathcal{V}$ and ψ be as in part (a), except that we do not require $\mathcal{U}$ or $\mathcal{V}$ to be bounded and we do not require $\psi \in C^1(\overline{\mathcal{U}})$ or $\psi^{-1} \in C^{\max\{1,\ell\}}(\overline{\mathcal{V}})$. Then if $\varphi \in C_0^\infty(\mathcal{U})$, the map

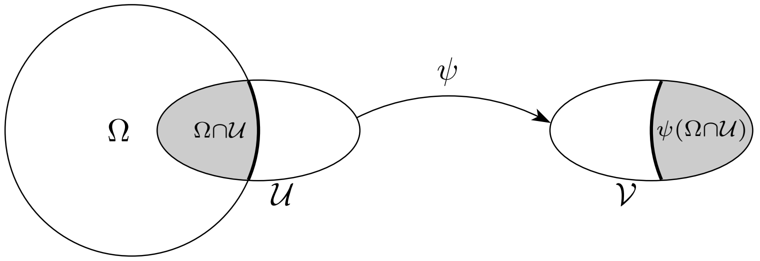
$$\begin{aligned} C_0^\infty(\mathcal{U}) &\to C_0^\ell(\mathcal{V}) \\ u &\mapsto (\varphi u) \circ \psi^{-1} \end{aligned}$$

extends to a bounded linear map from $H_0^\ell(\mathcal{U})$ to $H_0^\ell(\mathcal{V})$.

(c) Let $\mathcal{U}$, $\mathcal{V}$ and Ω be open subsets of $\mathbb{R}^n$ and let $\psi : \mathcal{U} \to \mathcal{V}$ be a $C^{\max\{1,\ell\}}$ diffeomorphism. Then if $\chi \in C_0^\infty(\mathcal{U})$, the map

$$\begin{aligned} \left\{\ u \in C^\ell(\Omega) \ \middle|\ \|u\|_{\ell,\Omega} < \infty \ \right\} &\to C^\ell\big(\psi(\Omega \cap \mathcal{U})\big) \\ u &\mapsto (\chi u) \circ \psi^{-1} \end{aligned}$$

extends to a bounded linear map from $H^\ell(\Omega)$ to $H^\ell\big(\psi(\Omega \cap \mathcal{U})\big)$.



Proof: (a) We start with $\ell = 0$. Making the change of variables $x = \psi(y)$ and using $\left|\frac{\partial x}{\partial y}\right|$ to denote the associated Jacobian determinant

$$\|u \circ \psi^{-1}\|_{0,\mathcal{V}}^2 = \int_{\mathcal{V}} \left|u\big(\psi^{-1}(x)\big)\right|^2 \, d^n x = \int_{\mathcal{U}} \left|u(y)\right|^2 \, \left|\tfrac{\partial x}{\partial y}(y)\right| \, d^n y$$

Since $\psi \in C^1(\overline{\mathcal{U}})$, $\sup_{y \in \overline{\mathcal{U}}} \left|\frac{\partial x}{\partial y}(y)\right|$ is finite and $\|u \circ \psi^{-1}\|_{0,\mathcal{V}}^2 \le \sup_{y \in \overline{\mathcal{U}}} \left|\frac{\partial x}{\partial y}(y)\right| \, \|u\|_{0,\mathcal{U}}^2$. By definition, the set $\left\{\ u \in C^0(\mathcal{U}) \ \middle|\ \|u\|_{0,\mathcal{U}} < \infty \ \right\}$ is dense in $H^0(\mathcal{U})$, so the B.L.T. theorem yields the desired result.

Next we consider $\ell = 1$. By the chain rule

$$\tfrac{\partial}{\partial x_i} u \circ \psi^{-1}(x) = \sum_{j=1}^{n} \tfrac{\partial u}{\partial y_j}\big(\psi^{-1}(x)\big)\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)$$

Since $\psi^{-1} \in C^1(\overline{\mathcal{V}})$, $\big|\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)\big|$ is bounded on $\overline{\mathcal{V}}$ for all $1 \le j \le n$ so, by Cauchy–Schwarz,

$$\begin{aligned}
\int_{\mathcal{V}} \big|\tfrac{\partial}{\partial x_i} u \circ \psi^{-1}(x)\big|^2 \, d^n x &= \int_{\mathcal{V}} \Big|\sum_{j=1}^{n} \tfrac{\partial u}{\partial y_j}\big(\psi^{-1}(x)\big)\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)\Big|^2 \, d^n x \\
&\le \int_{\mathcal{V}} \Big[\sum_{j=1}^{n} \big|\tfrac{\partial u}{\partial y_j}\big(\psi^{-1}(x)\big)\big|^2\Big]\Big[\sum_{j=1}^{n} \big|\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)\big|^2\Big] \, d^n x \\
&\le \sup_{x \in \overline{\mathcal{V}}} \Big[\sum_{j=1}^{n} \big|\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)\big|^2\Big] \int_{\mathcal{V}} \Big[\sum_{j=1}^{n} \big|\tfrac{\partial u}{\partial y_j}\big(\psi^{-1}(x)\big)\big|^2\Big] \, d^n x \\
&= \sup_{x \in \overline{\mathcal{V}}} \Big[\sum_{j=1}^{n} \big|\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)\big|^2\Big] \int_{\mathcal{U}} \Big[\sum_{j=1}^{n} \big|\tfrac{\partial u}{\partial y_j}(y)\big|^2\Big] \, \big|\tfrac{\partial x}{\partial y}(y)\big| \, d^n y \\
&\le \sup_{x \in \overline{\mathcal{V}}} \Big[\sum_{j=1}^{n} \big|\tfrac{\partial \psi_j^{-1}}{\partial x_i}(x)\big|^2\Big] \sup_{y \in \overline{\mathcal{U}}} \big|\tfrac{\partial x}{\partial y}(y)\big| \, \|u\|_{1,\mathcal{U}}^2
\end{aligned}$$

The proof for $\ell > 1$ is similar, since $\partial^\alpha\big[u \circ \psi^{-1}\big](x)$ is a finite sum of terms, each having one factor $(\partial^\beta u) \circ \psi^{-1}$, with $|\beta| \le |\alpha|$, and one or more factors $\partial^\gamma \psi_j^{-1}$ with $1 \le |\gamma| \le |\alpha|$.

(b) We already know, from part (a) of Problem 2.1.3, that multiplication by φ is a bounded operator on $H^\ell(\mathcal{U})$. Denote by Φ the support of φ. It is a compact subset of $\mathcal{U}$. Because ψ is continuous, $\psi(\Phi)$ is a compact subset of $\mathcal{V}$. For all $|\alpha| \le \ell$, $\partial^\alpha(\varphi u)\big(\psi^{-1}(x)\big)$ vanishes unless $x \in \psi(\Phi)$. Thus when we replace u with φu in the arguments of part (a), all of the integrals $\int_{\mathcal{V}} \cdot \, d^n x$ are restricted to $\int_{\psi(\Phi)} \cdot \, d^n x$ and all of the integrals $\int_{\mathcal{U}} \cdot \, d^n y$ are restricted to $\int_{\Phi} \cdot \, d^n y$. Since $\psi(\Phi)$ is compact, $\psi^{-1}(x)$ and all of its derivatives to order ℓ are bounded on $\psi(\Phi)$, even if $\psi^{-1} \notin C^{\max\{1,\ell\}}(\overline{\mathcal{V}})$. Since Φ is compact, $\tfrac{\partial x}{\partial y}(y)$ is bounded on Φ even if $\psi \notin C^1(\overline{\mathcal{U}})$. So the arguments of part (a) still imply that

$$\|(\varphi u) \circ \psi^{-1}\|_{\ell,\mathcal{V}} \le \text{const}\, \|u\|_{\ell,\mathcal{U}}$$

and the B.L.T. theorem implies that the map extends to the closure of $C_0^\infty(\mathcal{U})$, which is $H_0^\ell(\mathcal{U})$. By Problem 2.1.2, $C_0^\ell(\mathcal{V}) \subset H_0^\ell(\mathcal{V})$, so the image of $C_0^\infty(\mathcal{U})$ is contained in $H_0^\ell(\mathcal{V})$. Since $H_0^\ell(\mathcal{V})$ is a closed subset of $H^\ell(\mathcal{V})$, the image of the extended map is still contained in $H_0^\ell(\mathcal{V})$.

(c) The proof is similar to that of part (b). Let $\mathcal{W}$ denote $\psi(\Omega \cap \mathcal{U})$ and Φ denote the support of χ. As Φ is a compact subset of $\mathcal{U}$, $\overline{\Phi \cap \Omega}$ is also a compact subset of $\mathcal{U}$. Because ψ is continuous, $\psi(\overline{\Phi \cap \Omega})$ is a compact subset of $\mathcal{V}$. For all $x \in \mathcal{W}$ and $|\alpha| \le \ell$,

$\partial^\alpha(\chi u)\big(\psi^{-1}(x)\big)$ vanishes unless $x \in \psi(\Phi) \cap \psi(\Omega \cap \mathcal{U}) = \psi(\Phi \cap \Omega)$. Thus when we replace u with χu in the arguments of part (a)

- all of the integrals $\int_{\mathcal{V}} \cdot\, d^n x$ of part (a) become integrals $\int_{\mathcal{W}} \cdot\, d^n x$ and are restricted to $\int_{\psi(\Phi\cap\Omega)} \cdot\, d^n x$ and
- all of the integrals $\int_{\mathcal{U}} \cdot\, d^n y$ of part (a) become integrals $\int_{\Omega\cap\mathcal{U}} \cdot\, d^n y$ and are restricted to $\int_{\Phi\cap\Omega} \cdot\, d^n y$.

Since $\psi(\overline{\Phi \cap \Omega})$ is compact, $\psi^{-1}(x)$ and all of its derivatives to order ℓ are bounded on $\psi(\Phi\cap\Omega)$, even if $\psi^{-1} \notin C^{\max\{1,\ell\}}(\overline{\mathcal{V}})$. Since Φ is compact, $\frac{\partial x}{\partial y}(y)$ is bounded on $\Phi \cap \Omega$ even if $\psi \notin C^1(\overline{\mathcal{U}})$. So the arguments of part (a) still imply that

$$\|(\chi u)\circ\psi^{-1}\|_{\ell,\mathcal{W}} \le \text{const}\,\|u\|_{\ell,\mathcal{U}\cap\Omega} \le \text{const}\,\|u\|_{\ell,\Omega}$$

and the B.L.T. theorem implies that the map extends to the closure of the domain of definition $\big\{\ u \in C^\ell(\Omega)\ \big|\ \|u\|_{\ell,\Omega} < \infty\ \big\}$, which is $H^\ell(\Omega)$. ■

§2.2. Traces – Restriction to the Boundary

Definition 2.2.1 $(C^\ell(\,\overline{\Omega}\,))$ Let Ω be an open subset of $\mathbb{R}^n$. We define $C^\ell(\,\overline{\Omega}\,)$ to be the set of all functions $u \in C^\ell(\mathbb{R}^n)$ for which $\partial^\alpha u$ is bounded and uniformly continuous on Ω for all $|\alpha| \le \ell$. We also write $C^\infty(\,\overline{\Omega}\,) = \cap_{\ell=0}^\infty C^\ell(\,\overline{\Omega}\,)$. If $u \in C^\ell(\,\overline{\Omega}\,)$ and $|\alpha| \le \ell$, then $\partial^\alpha u$ has a unique continuous extension to $\overline{\Omega}$, which we also denote $\partial^\alpha u$.

The goal of this section is to prove

Theorem 2.2.2 *Let $\ell \in \mathbb{N}$ and let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Define the restriction map*

$$\begin{aligned} r : C^\infty(\,\overline{\Omega}\,) &\to C^\infty(\partial\Omega) \\ u &\mapsto u \restriction \partial\Omega \end{aligned}$$

There exists a unique map

$$R : H^\ell(\Omega) \to H^{\ell-1/2}(\partial\Omega)$$

and constants C, C' such that

(i) R extends r. That is, $Ru = ru$ for all u in the domain, $C^\infty(\,\overline{\Omega}\,)$, of r.

(ii) R is bounded. That is, $\|Ru\|_{\ell-1/2,\partial\Omega} \le C\|u\|_{\ell,\Omega}$.

(iii) R is surjective (onto).

(iv) For each $f \in H^{\ell-1/2}(\partial\Omega)$*, there is a* $u \in H^\ell(\Omega)$ *such that*

$$Ru = f \quad \textit{and} \qquad \|u\|_{\ell,\Omega} \le C'\|f\|_{\ell-1/2,\partial\Omega}$$

That is, R *has a bounded right inverse.*

(v) R *has kernel* $H^1_0(\Omega) \cap H^\ell(\Omega)$*.*

Remark 2.2.3 Theorem 2.2.2 is true for all real $\ell > \frac{1}{2}$, except that, when $\frac{1}{2} < \ell < 1$, R has kernel $H^\ell_0(\Omega)$. See [A, Theorem 7.53].

Remark 2.2.4 There are three main parts to the proof of Theorem 2.2.2, which is given following Lemma 2.2.13.

- The first part is to show that r is bounded. Once this is done, and once we prove in Lemma 2.2.13, below, that the domain of r is dense in $H^\ell(\Omega)$, we can extend r by continuity to all of $H^\ell(\Omega)$ and call the result R. Parts (i) and (ii) of the Theorem and also the uniqueness of R are then automatic. For pedagogical purposes, we prove the boundedness of r first for $\Omega = \mathbb{R}^n$ with $\partial\Omega$ replaced by $\{x_n = 0\}$ (Lemma 2.2.5), second for $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, $\partial\Omega = \{x_n = 0\}$ (Proposition 2.2.10) and finally for Ω as in the theorem.
- The second part is to prove that each $f \in H^{\ell-1/2}(\partial\Omega)$ can be extended to a $u \in H^\ell(\Omega)$ in a bounded way. This will prove part (iv), which implies part (iii). For pedagogical purposes, we prove extendibility first for $\Omega = \mathbb{R}^n$ with $\partial\Omega$ replaced by $\{x_n = 0\}$ (Lemma 2.2.6), second for $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, $\partial\Omega = \{x_n = 0\}$ (Proposition 2.2.10) and finally for Ω as in the theorem.
- Since r vanishes on $C^\infty_0(\Omega)$, which is dense in $H^\ell_0(\Omega)$, the boundedness of r will also imply that the kernel of R contains $H^\ell_0(\Omega)$. Note that, when ℓ is large, many derivatives of any $f \in H^\ell_0(\Omega)$ must vanish on $\partial\Omega$. On the other hand, to be in the kernel of R only f itself — not its derivatives — must vanish on $\partial\Omega$. In fact, if $u \in H^1_0(\Omega) \cap H^\ell(\Omega)$, then there is a sequence of functions $u_j \in C^\infty_0(\Omega)$ that converge to u with respect to the norm $\|\ \cdot\ \|_{1,\Omega}$. Since, by part (ii) of the theorem with $\ell = 1$,

  $$\|Ru\|_{1/2,\partial\Omega} = \|Ru - Ru_j\|_{1/2,\partial\Omega} \le C\|u - u_j\|_{1,\Omega} \text{ for all } j \implies Ru = 0$$

  the kernel of R (acting on $H^\ell(\Omega)$) contains all of $H^1_0(\Omega) \cap H^\ell(\Omega)$. To prove part (v), we have to show that any function in the kernel of R can be approximated, in the $H^1(\Omega)$ norm, by functions in $C^\infty_0(\Omega)$. For pedagogical purposes, we prove this first for $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, $\partial\Omega = \{x_n = 0\}$ (Proposition 2.2.11) and then for Ω as in the theorem.

We start with a particularly simple geometry, to illustrate what's going on. In the illustration, we replace Ω by $\mathbb{R}^n$ and $\partial\Omega$ by $\{x_n = 0\}$, which we identify with $\mathbb{R}^{n-1}$. To save writing, we denote $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$.

Lemma 2.2.5 (Restriction from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$) *Let $s > \frac{1}{2}$. The linear transformation $r : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-1})$ defined by*

$$(ru)(x') = u(x', 0)$$

has a unique extension to a bounded linear map

$$R : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$$

Proof: We shall prove that there is a constant C (depending only on s) such that

$$\|ru\|_{s-\frac{1}{2},n-1} \le C\|u\|_{s,n}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. The Lemma will then follow by the B.L.T. theorem.

If $k \in \mathbb{R}^n$, write $k = (k', k_n)$ with $k' = (k_1, \cdots, k_{n-1})$. The definition of the Fourier transform gives

$$\int e^{ik'\cdot x'} (\widehat{ru})(k') \ \tfrac{d^{n-1}k'}{(2\pi)^{n-1}} = (ru)(x') = u(x', 0) = \int e^{ik'\cdot x'} \hat{u}(k', k_n) \ \tfrac{d^{n-1}k'}{(2\pi)^{n-1}} \tfrac{dk_n}{2\pi}$$

so that

$$(\widehat{ru})(k') = \int \hat{u}(k', k_n) \ \tfrac{dk_n}{2\pi}$$

Thus, by Cauchy–Schwarz,

$$\begin{aligned}
\big|(\widehat{ru})(k')\big|^2 &= \Big| \int \hat{u}(k', k_n) \ \big(1 + k'^2 + k_n^2\big)^{s/2} \big(1 + k'^2 + k_n^2\big)^{-s/2} \ \tfrac{dk_n}{2\pi} \Big|^2 \\
&\le \Big[\int \tfrac{1}{(1+k'^2+k_n^2)^s} \ \tfrac{dk_n}{2\pi} \Big] \Big[\int \big|\hat{u}(k', k_n)\big|^2 \ \big(1 + k'^2 + k_n^2\big)^s \ \tfrac{dk_n}{2\pi} \Big] \\
&= \Big[\tfrac{1}{(1+k'^2)^{s-1/2}} \int \tfrac{1}{(1+p^2)^s} \ \tfrac{dp}{2\pi} \Big] \Big[\int \big|\hat{u}(k', k_n)\big|^2 \big(1 + k'^2 + k_n^2\big)^s \ \tfrac{dk_n}{2\pi} \Big] \\
&\qquad\qquad\qquad\qquad \text{where } k_n = p\sqrt{1 + k'^2} \\
&= \tfrac{C^2}{(1+k'^2)^{s-1/2}} \int \big|\hat{u}(k', k_n)\big|^2 \ \big(1 + k'^2 + k_n^2\big)^s \ \tfrac{dk_n}{2\pi}
\end{aligned}$$

where the constant $C^2 = \int \frac{1}{(1+p^2)^s} \ \frac{dp}{2\pi}$ depends only on s and is finite because $s > \frac{1}{2}$. Hence

$$\begin{aligned}
\|ru\|^2_{s-\frac{1}{2},n-1} &= \int (1 + k'^2)^{s-\frac{1}{2}} |\widehat{ru}(k')|^2 \ \tfrac{d^{n-1}k'}{(2\pi)^{n-1}} \\
&\le C^2 \int \big|\hat{u}(k', k_n)\big|^2 \ \big(1 + k'^2 + k_n^2\big)^s \ \tfrac{d^n k}{(2\pi)^n} \\
&= C\|u\|^2_{s,n}
\end{aligned}$$

■

Problem 2.2.1 Let $k \in \mathbb{N}$ and $s \in \mathbb{R}$ obey $k < n$ and $s > \frac{k}{2}$. Identify $\mathbb{R}^{n-k}$ with $\left\{ x \in \mathbb{R}^n \;\middle|\; x_{n-k+1} = \cdots = x_n = 0 \right\}$ and write $x \in \mathbb{R}^n$ as $x = (x', y)$ with $x' \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. Prove that the linear transformation $r : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-k})$ defined by

$$(ru)(x') = u(x', 0)$$

has a unique extension to a bounded linear map

$$R : H^s(\mathbb{R}^n) \to H^{s-\frac{k}{2}}(\mathbb{R}^{n-k})$$

Problem 2.2.2 Let $\frac{1}{2} < s \le \ell \in \mathbb{N}$ and let $u \in C^\ell(\mathbb{R}^n)$. Suppose that each derivative of u of order at most ℓ is bounded by a constant times $(1 + |x|)^{-\alpha}$ for some $\alpha > \frac{n}{2}$. Then $u \in H^\ell(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ and $u(x', 0) \in H^\ell(\mathbb{R}^{n-1}) \subset H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Prove that $(Ru)(x') = u(x', 0)$, where $R : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is the map of Lemma 2.2.5.

Problem 2.2.3 Let $s > \frac{1}{2}$. Define, for each $t \in \mathbb{R}$, $R_t : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ to be the unique bounded linear extension of the linear transformation $r_t : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-1})$ defined by

$$(ru)(x') = u(x', t)$$

Prove that R_t is strongly continuous in t. That is, prove that

$$\lim_{t \to t_0} \left| R_t f - R_{t_0} f \right|^2_{s-\frac{1}{2}, n-1} = 0$$

for each $t_0 \in \mathbb{R}$ and each $f \in H^s(\mathbb{R}^n)$.

Lemma 2.2.6 (Extension from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$) *Let $s > \frac{1}{2}$. Then there is a constant C (depending only on n and s) such that, for each $f \in H^{s-1/2}(\mathbb{R}^{n-1})$ there is a $u \in H^s(\mathbb{R}^n)$ obeying*

$$Ru = f \quad \text{and} \qquad |u|_{s,n} \le C |f|_{s-1/2, n-1}$$

where R is the map of Lemma 2.2.5.

Proof: As in the last lemma, write $k = (k', k_n) \in \mathbb{R}^n$ with $k' = (k_1, \cdots, k_{n-1})$. We claim that if $f \in \mathcal{S}(\mathbb{R}^{n-1})$, then the u determined by

$$\hat{u}(k', k_n) = \hat{f}(k') \; 2\sqrt{\pi} e^{-k_n^2/(1+|k'|^2)} / \sqrt{1 + |k'|^2}$$

does the job for that f. Once the claim is proven, we will be able to extend the map $f \mapsto u$ by continuity (i.e. the B.L.T. theorem) to $H^{s-1/2}(\mathbb{R}^{n-1})$ and the Lemma will follow, since R is itself continuous.

We first check that $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$. Every derivative of $e^{-k_n^2/(1+|k'|^2)}/\sqrt{1+|k'|^2}$ is a polynomial in k times $e^{-k_n^2/(1+|k'|^2)}\big(1+|k'|^2\big)^{-\ell/2}$ for some $\ell \in \mathbb{N}$. Since, for each $m \in \mathbb{N}_0$,

$$\begin{aligned}|k_n|^m e^{-k_n^2/(1+|k'|^2)} &= (1+|k'|^2)^{m/2}\Big[k_n^2/(1+|k'|^2)\Big]^{m/2} e^{-k_n^2/(1+|k'|^2)}\\ &\le \text{const}_m\,(1+|k'|^2)^{m/2}\end{aligned}$$

each polynomial in k times $e^{-k_n^2/(1+|k'|^2)}\big(1+|k'|^2\big)^{-\ell/2}$ is bounded by a polynomial in k'. As $\hat{f} \in \mathcal{S}(\mathbb{R}^{n-1})$, this ensures that $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$.

To verify that $Ru = f$, we need to verify that $u(x', 0) = f(x')$, or equivalently $\int \hat{u}(k', k_n)\,\frac{dk_n}{2\pi} = \hat{f}(k')$. But this is an immediate consequence of

$$2\sqrt{\pi}\int e^{-k_n^2/(1+|k'|^2)}\ \frac{dk_n}{2\pi\sqrt{1+|k'|^2}} = \frac{1}{\sqrt{\pi}}\int e^{-p^2}\ dp = 1$$

We made the change of variables $k_n = p\sqrt{1+|k'|^2}$. The verification of the boundedness requirement is

$$\begin{aligned}\|u\|_{s,n}^2 &= \int \big(1+|k|^2\big)^s |\hat{u}(k)|^2\ \frac{d^nk}{(2\pi)^n}\\ &= 4\pi\int \big(1+|k|^2\big)^s\big(1+|k'|^2\big)^{-1} e^{-2k_n^2/(1+|k'|^2)}|\hat{f}(k')|^2\ \frac{d^{n-1}k'}{(2\pi)^{n-1}}\frac{dk_n}{2\pi}\\ &= 2\Big[\int \big(1+p^2\big)^s e^{-2p^2}\ dp\Big]\int (1+|k'|^2)^{s-1+1/2}|\hat{f}(k')|^2\ \frac{d^{n-1}k'}{(2\pi)^{n-1}}\\ &= \Big[\int 2\big(1+p^2\big)^s e^{-2p^2}\ dp\Big]\ \|f\|_{s-1/2,n-1}^2\end{aligned}$$

■

We next consider another, slightly more realistic, but still simple, geometry. Namely $\Omega = \mathbb{R}^n_+ = \big\{\ x \in \mathbb{R}^n\ \big|\ x_n > 0\ \big\}$, $\partial\Omega = \mathbb{R}^{n-1} = \big\{\ x \in \mathbb{R}^n\ \big|\ x_n = 0\ \big\}$.

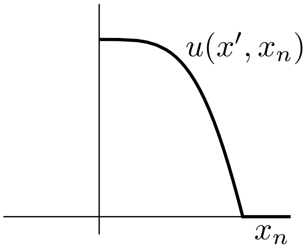
Lemma 2.2.7 *Let $\ell \in \mathbb{N}_0$. The set of restrictions of functions in $C_0^\infty(\mathbb{R}^n)$ to $\mathbb{R}^n_+$ is dense in $H^\ell(\mathbb{R}^n_+)$. In particular, $C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$ is dense in $H^\ell(\mathbb{R}^n_+)$.*

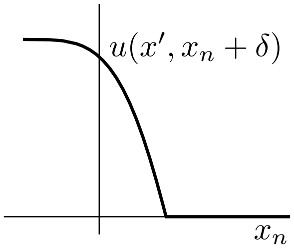
Proof: Let $u \in H^\ell(\mathbb{R}^n_+)$ and $\varepsilon > 0$. Let, for any measurable function w on $\mathbb{R}^n$, P_+w be the restriction of w to $\mathbb{R}^n_+$. Observe that

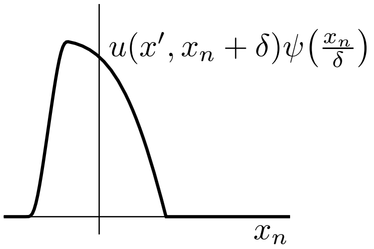
$$\|P_+w\|_{\ell,\mathbb{R}^n_+} = \Big\{\sum_{|\alpha|\le\ell}\int_{\mathbb{R}^n_+}\big|\partial^\alpha w(x)\big|^2\ d^nx\Big\}^{1/2} \le \Big\{\sum_{|\alpha|\le\ell}\int_{\mathbb{R}^n}\big|\partial^\alpha w(x)\big|^2\ d^nx\Big\}^{1/2} = \|w\|_{\ell,\mathbb{R}^n}$$

Step 1. We first find a $v \in H^\ell(\mathbb{R}^n)$ such that $\|u - P_+ v\|_{\ell,\mathbb{R}^n_+} < \frac{\varepsilon}{2}$. Let $\psi \in C^\infty(\mathbb{R})$ obey

$$\psi(t) = \begin{cases} 1 & \text{for } t \ge -\frac{1}{2} \\ 0 & \text{for } t \le -\frac{3}{4} \end{cases} \text{ and define, for } \delta > 0,\ u_\delta(x) = \begin{cases} u(x', x_n + \delta)\psi\big(\frac{x_n}{\delta}\big) & \text{if } x_n > -\delta \\ 0 & \text{if } x_n \le -\frac{3}{4}\delta \end{cases}$$







Using the same procedure as in part (a) of Problem 2.1.3, we see that $u_\delta \in H^\ell(\mathbb{R}^n)$ (though $\boldsymbol{|} u_\delta \boldsymbol{|}_\ell$ may diverge as $\delta \to 0$). For $x_n \ge 0$, $u_\delta(x) = u(x', x_n + \delta)$ and hence

$$\|u - P_+ u_\delta\|^2_{\ell,\mathbb{R}^n_+} = \sum_{|\alpha| \le \ell} \int_{\mathbb{R}^n_+} \big|\partial^\alpha u(x) - \partial^\alpha u(x', x_n + \delta)\big|^2 \, d^n x$$

We shall show in the next paragraph that translation is continuous in $L^2(\mathbb{R}^n_+)$. Hence we can pick $\delta_0 > 0$ sufficiently small that $\|u - P_+ u_{\delta_0}\|_{\ell,\mathbb{R}^n_+} < \frac{\varepsilon}{2}$ and set $v = u_{\delta_0}$.

To complete this step, we show that translation is continuous in $L^2(\Omega)$ for any open $\Omega \subset \mathbb{R}^n$. Let $f \in L^2(\Omega)$. Define f to be zero on $\mathbb{R}^n \setminus \Omega$. Then for all $t \in \mathbb{R}^n$

$$\int_\Omega \big|f(x) - f(x+t)\big|^2 \, d^n x \le \int_{\mathbb{R}^n} \big|f(x) - f(x+t)\big|^2 \, d^n x = \int_{\mathbb{R}^n} \big|1 - e^{ik\cdot t}\big|^2 \, |\hat{f}(k)|^2 \, \tfrac{d^n k}{(2\pi)^n}$$

by (2.1.1b) and the translation property of the Fourier transform. The integrand converges pointwise to zero as $t \to 0$ and is bounded uniformly by the integrable function $4|\hat{f}(k)|^2$. So, by the Lebesgue dominated convergence theorem,

$$\lim_{t \to 0} \int_\Omega \big|f(x) - f(x+t)\big|^2 \, d^n x = 0$$

Step 2. By Problem 2.1.8 there exists a $w \in C_0^\infty(\mathbb{R}^n)$ such that $\|v - w\|_{\ell,\mathbb{R}^n} < \frac{\varepsilon}{2}$. Hence

$$\|u - P_+ w\|_{\ell,\mathbb{R}^n_+} \le \|u - P_+ v\|_{\ell,\mathbb{R}^n_+} + \|P_+ v - P_+ w\|_{\ell,\mathbb{R}^n_+} \le \|u - P_+ v\|_{\ell,\mathbb{R}^n_+} + \|v - w\|_{\ell,\mathbb{R}^n} < \varepsilon$$

and we are done. ■

Problem 2.2.4 Let $\ell \in \mathbb{N}_0$, $\varepsilon, \varepsilon', R > 0$ and $u \in H^\ell(\mathbb{R}^n_+)$. Suppose that $u(x) = 0$ for all $|x| \geq R$. Show that there is a function $v \in C_0^\infty(\mathbb{R}^n)$ that obeys $v(x) = 0$ for all $|x| \geq R+\varepsilon'$ and $\|u - P_+ v\|_{\ell,\mathbb{R}^n_+} < \varepsilon$, where P_+ is the restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_+$.

Lemma 2.2.8 (Extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$) *Let $\ell \in \mathbb{N}_0$. There exists a bounded linear operator*

$$E : H^\ell(\mathbb{R}^n_+) \to H^\ell(\mathbb{R}^n)$$

such that $Eu(x) = u(x)$ for all $u \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$, $x \in \overline{\mathbb{R}^n_+}$.

Proof: Let $u \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$. Write $x = (x', x_n) \in \mathbb{R}^n$ with $x' = (x_1, \cdots, x_{n-1})$. Define

$$Eu(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ \sum_{j=1}^{\ell+1} \beta_j u\big(x', -\frac{x_n}{j}\big) & \text{if } x_n < 0 \end{cases} \tag{2.2.1}$$

where the β_j's are to chosen, independent of u, so as to ensure that Eu is C^ℓ across $x_n = 0$. This will ensure that $Eu \in C^\ell(\mathbb{R}^n)$. Since

$$\int_{x_n<0} \big|\tfrac{\partial^\alpha}{\partial x^\alpha} u\big(x', -\tfrac{x_n}{j}\big)\big|^2 \, d^n x = j\big(\tfrac{1}{j}\big)^{2\alpha_n} \int_{x_n>0} \big|\partial^\alpha u(x)\big|^2 \, d^n x \tag{2.2.2}$$

it will also ensure that E is bounded. It will then suffice to extend E by continuity to all of $H^\ell(\mathbb{R}^n_+)$.

To ensure that $\partial^\alpha Eu$ is continuous across $x_n = 0$, when $\alpha_n = 0$, it suffices to choose the β_j's so that $\sum_{j=1}^{\ell+1} \beta_j = 1$. To ensure that $\partial^\alpha Eu$ is continuous across $x_n = 0$, when $\alpha_n = 1$, it suffices to choose the β_j's so that $\sum_{j=1}^{\ell+1} -\frac{\beta_j}{j} = 1$. In general, to ensure that $\partial^\alpha Eu$ is continuous across $x_n = 0$, when $\alpha_n = m$, it suffices to choose the β_j's so that

$$\sum_{j=1}^{\ell+1} \big(-\tfrac{1}{j}\big)^m \beta_j = 1 \tag{2.2.3$_\mathrm{m}$}$$

The system of linear equations (2.2.3$_m$), $0 \leq m \leq \ell$ has a solution provided

$$D = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1 & -\frac{1}{2} & \cdots & -\frac{1}{\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^\ell & \big(-\frac{1}{2}\big)^\ell & \cdots & \big(-\frac{1}{\ell+1}\big)^\ell \end{vmatrix} \neq 0$$

But D is a Vandermonde determinant. By Problem 2.2.5, below, it has the value $\prod_{1 \leq i < j \leq \ell+1} \big(-\frac{1}{j} + \frac{1}{i}\big)$, which is certainly not zero. ■

Problem 2.2.5 Let $\{x_i\}_{i\in\mathbb{N}}$ be any sequence of complex numbers. Define the Vandermonde determinant

$$V_n = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

(a) Use row and column operations to prove that $V_n = V_{n-1}\prod_{1\le i<n}\big(x_n - x_i\big)$.
(b) Prove that $V_n = \prod_{1\le i<j\le n}\big(x_j - x_i\big)$.

Problem 2.2.6 Let $\ell \in \mathbb{N}_0$ and $\ell' \in \mathbb{Z}$ with $\ell' \le \ell$. Denote by E_ℓ the operator of Lemma 2.2.8. Prove that there is a constant C such that

$$\big|E_\ell u\big|_{\ell',n} \le C\|u\|_{\ell',\mathbb{R}^n_+}$$

for all $u \in H^\ell(\mathbb{R}^n_+)$.

Problem 2.2.7 Let $\ell',\ell \in \mathbb{N}_0$ with $\ell' \le \ell$ and $\alpha \in \mathbb{N}_0^n$ with $\alpha_n = 0$. Denote by E_ℓ the operator of Lemma 2.2.8. By Problem 2.2.6, E_ℓ has a unique bounded extension to $H^{\ell'}(\mathbb{R}^n_+)$ and it in turn has a unique bounded extension to $H^{\ell'-|\alpha|}(\mathbb{R}^n_+)$. We persist in denoting both of them E_ℓ. Recall, from Lemma 2.1.14, that ∂^α is a bounded operator from $H^{\ell'}(\mathbb{R}^n_+)$ to $H^{\ell'-|\alpha|}(\mathbb{R}^n_+)$ and from $H^{\ell'}(\mathbb{R}^n)$ to $H^{\ell'-|\alpha|}(\mathbb{R}^n)$. Let $u \in H^{\ell'}(\mathbb{R}^n_+)$. Prove $\partial^\alpha E_\ell u = E_\ell \partial^\alpha u$ and that $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$ if and only if $\partial^\alpha E_\ell u \in H^\ell(\mathbb{R}^n)$.

Problem 2.2.8

(a) Let $\big\{a_\ell\big\}_{\ell\in\mathbb{N}_0}$ be any sequence of real numbers. Prove that there is a function $f \in C^\infty(\mathbb{R})$ such that $f^{(m)}(0) = a_m$ for all $m \in \mathbb{N}_0$.

(b) Let $\big\{a_\ell(x')\big\}_{\ell\in\mathbb{N}_0}$ be any sequence of C^∞ functions on $\mathbb{R}^{n-1}$. Prove that there is a function $f \in C^\infty(\mathbb{R}^n)$ such that $\frac{\partial^m f}{\partial x_n^m}(x',0) = a_m(x')$ for all $m \in \mathbb{N}_0$ and $x' \in \mathbb{R}^{n-1}$.

(c) Prove that if $f \in C^\infty(\overline{\mathbb{R}^n_+})$, then there exists $F \in C^\infty(\mathbb{R}^n)$ such that $f(x) = F(x)$ for all $\mathbb{R}^n_+$.

We defined $H^\ell(\mathbb{R}^n_+)$, rather abstractly, as the completion of $C^\ell(\mathbb{R}^n_+)$ under a certain norm. Part (a) of the next proposition (with $\ell' = 0$) shows that $H^\ell(\mathbb{R}^n_+)$ is precisely the set of all $u \in L^2(\mathbb{R}^n_+)$ all of whose derivatives $\partial^\alpha u$ with order $|\alpha| \le \ell$ are again in $L^2(\mathbb{R}^n_+)$. We also defined (in part (b) of Lemma 2.1.14) $\partial^\alpha : H^\ell(\mathbb{R}^n_+) \to H^{\ell-|\alpha|}(\mathbb{R}^n_+)$, rather abstractly, as the continuous extension of $\partial^\alpha : C^{\max\{\ell,|\alpha|\}}(\mathbb{R}^n_+) \to C^{\max\{\ell-|\alpha|,0\}}(\mathbb{R}^n_+)$. Part (b) of the next proposition replaces these derivatives with, more explicit, weak derivatives and shows that $H^\ell(\mathbb{R}^n_+)$ is precisely the set of all $u \in L^2(\mathbb{R}^n_+)$ all of whose weak derivatives of order at most ℓ are again in $L^2(\mathbb{R}^n_+)$. Recall that the weak (or distributional) α–derivative of u is the map

$$\begin{aligned} D^\alpha_u : C^\infty_0(\Omega) &\to \mathbb{C} \\ f &\mapsto (-1)^{|\alpha|} \langle \partial^\alpha f, u\rangle_{L^2(\Omega)} \end{aligned}$$

and is defined for all $u \in L^2(\Omega)$ and all $\alpha \in \mathbb{N}^n_0$. Recall also, from the discussion following (2.1), that one says that the weak derivative D^α_u is in $L^2(\Omega)$ if there exists a $v_\alpha \in L^2(\Omega)$ such that $\langle f, v_\alpha\rangle_{L^2(\Omega)} = (-1)^{|\alpha|} \langle \partial^\alpha f, u\rangle_{L^2(\Omega)}$ for all $f \in C^\infty_0(\Omega)$. These characterizations of $H^\ell(\mathbb{R}^n_+)$, as the set of all $u \in L^2(\mathbb{R}^n_+)$ all of whose derivatives of order at most ℓ are again in $L^2(\mathbb{R}^n_+)$, provide a good mental image for $H^\ell(\mathbb{R}^n_+)$. But we will not explicitly use them later in the book. So Proposition 2.2.9 and Proposition 2.2.15 (the analogous result for $H^\ell(\Omega)$) can be safely skipped on first reading.

Proposition 2.2.9 *Let $\ell \in \mathbb{N}_0$.*

(a) Let $u \in H^{\ell'}(\mathbb{R}^n_+)$ with $\ell' \in \mathbb{N}_0$ obeying $\ell' \le \ell$. Then $u \in H^\ell(\mathbb{R}^n_+)$ if and only if $\partial^\alpha u \in H^{\ell'}(\mathbb{R}^n_+)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell - \ell'$.

(b) Let $u \in L^2(\mathbb{R}^n_+)$. Then $u \in H^\ell(\mathbb{R}^n_+)$ if and only if there is a constant C such that

$$\big| \langle \partial^\alpha f, u\rangle_{L^2(\mathbb{R}^n_+)} \big| \le C \big\| f \big\|_{L^2(\mathbb{R}^n_+)}$$

for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell$ and all $f \in C^\infty_0(\mathbb{R}^n_+)$. Also, $u \in H^\ell(\mathbb{R}^n_+)$ if and only if, for each $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell$, there is a $v_\alpha \in L^2(\mathbb{R}^n_+)$ such that

$$(-1)^{|\alpha|} \langle \partial^\alpha f, u\rangle_{L^2(\mathbb{R}^n_+)} = \langle f, v_\alpha\rangle_{L^2(\mathbb{R}^n_+)}$$

for all $f \in C^\infty_0(\mathbb{R}^n_+)$.

Proof: (a) The implication that if $u \in H^\ell(\mathbb{R}^n_+)$, then $\partial^\alpha u \in H^{\ell'}(\mathbb{R}^n_+)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell - \ell'$ is an immediate consequence of part (b) of Lemma 2.1.14. This Lemma also implies that if $\partial^\alpha u \in H^{\ell'}(\mathbb{R}^n_+)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell - \ell'$ then $\partial^\alpha u \in L^2(\mathbb{R}^n_+)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell$, so it suffices to consider $\ell' = 0$.

So let $u \in L^2(\mathbb{R}^n_+)$ and assume that $\partial^\alpha u \in L^2(\mathbb{R}^n_+)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell$. Denote by E_ℓ the operator of Lemma 2.2.8 and its extensions to $H^{\ell'}(\mathbb{R}^n_+)$ for $\ell' < \ell$, as in Problem 2.2.7. It suffices to prove that $E_\ell u \in H^\ell(\mathbb{R}^n)$, since u is the restriction of $E_\ell u$ to $\mathbb{R}^n_+$. To prove that $E_\ell u \in H^\ell(\mathbb{R}^n)$, it suffices to prove that $\partial^\alpha E_\ell u \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell$, since then $k^\alpha \widehat{E_\ell u}(k) \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le \ell$ and this implies that $\big(1+|k|^2\big)^{\ell/2}\widehat{E_\ell u}(k) \in L^2(\mathbb{R}^n)$. We have already proven, in Problem 2.2.7, that $\partial^\alpha E_\ell u \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and $\alpha_n = 0$.

So assume that $\alpha_n > 0$ and set $\alpha' = \alpha - (0,\cdots,0,\alpha_n)$. We need to prove that $\partial^\alpha E_\ell u = \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} E_\ell \partial^{\alpha'} u \in L^2(\mathbb{R}^n)$. Recall, from (2.2.1), that, for $v \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$,

$$E_\ell v(x) = \begin{cases} v(x) & \text{if } x_n \ge 0 \\ \sum_{j=1}^{\ell+1} \beta_j v\big(x', -\frac{x_n}{j}\big) & \text{if } x_n < 0 \end{cases}$$

where $x = (x', x_n) \in \mathbb{R}^n$ with $x' = (x_1,\cdots,x_{n-1})$. Hence

$$\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} E_\ell v(x) = \begin{cases} \frac{\partial^{\alpha_n} v}{\partial x_n^{\alpha_n}}(x) & \text{if } x_n \ge 0 \\ \sum_{j=1}^{\ell+1} \beta_j \big(-\frac{1}{j}\big)^{\alpha_n} \frac{\partial^{\alpha_n} v}{\partial x_n^{\alpha_n}}\big(x', -\frac{x_n}{j}\big) & \text{if } x_n < 0 \end{cases} \tag{2.2.4}$$

Define, for $w \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^{\ell-\alpha_n}(\mathbb{R}^n_+)$,

$$Fw(x) = \begin{cases} w(x) & \text{if } x_n \ge 0 \\ \sum_{j=1}^{\ell+1} \beta_j \big(-\frac{1}{j}\big)^{\alpha_n} w\big(x', -\frac{x_n}{j}\big) & \text{if } x_n < 0 \end{cases} \tag{2.2.5}$$

For the β_j's chosen in Lemma 2.2.8, Fw is $C^{\ell-\alpha_n}$ across $x_n = 0$ so that $Fw \in C^{\ell-\alpha_n}(\mathbb{R}^n)$ and F is bounded as a map from $H^{\ell-\alpha_n}(\mathbb{R}^n_+)$ to $H^{\ell-\alpha_n}(\mathbb{R}^n_+)$. So we may extend F by continuity to all of $H^{\ell-\alpha_n}(\mathbb{R}^n_+)$. As in Problem 2.2.6, F has a unique bounded extension to an operator from $H^{\ell'}(\mathbb{R}^n_+)$ to $H^{\ell'}(\mathbb{R}^n)$ for each $\ell' \in \mathbb{Z}$ with $\ell' \le \ell - \alpha_n$. Comparing (2.2.4) and (2.2.5), we see that $\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} E_\ell v = F \frac{\partial^{\alpha_n} v}{\partial x_n^{\alpha_n}}$ for all $v \in C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$. By the continuity of all of the operators involved and the denseness of $C^\infty(\overline{\mathbb{R}^n_+}) \cap H^\ell(\mathbb{R}^n_+)$, $\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} E_\ell v = F \frac{\partial^{\alpha_n} v}{\partial x_n^{\alpha_n}}$ for all $v \in H^{\ell'}(\mathbb{R}^n_+)$ with $\ell' \le \ell$. Applying this with $v = \partial^{\alpha'} u$, we have that

$$\partial^\alpha E_\ell u = \tfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} E_\ell \partial^{\alpha'} u = F \partial^\alpha u$$

By assumption, $\partial^\alpha u \in L^2(\mathbb{R}^n_+)$. So $F\partial^\alpha u \in L^2(\mathbb{R}^n_+)$ and hence $\partial^\alpha E_\ell u \in L^2(\mathbb{R}^n_+)$.

(b) First, suppose that $u \in H^\ell(\mathbb{R}^n_+)$. By Lemma 2.2.7, there is a sequence of functions $u_i \in C^\infty(\overline{\mathbb{R}^n_+})$ that converges in $H^\ell(\mathbb{R}^n_+)$ to u. In particular, $\lim\limits_{i\to\infty} \|u_i\|_{\ell,\mathbb{R}^n_+} = \|u\|_{\ell,\mathbb{R}^n_+}$ and, if $f \in C_0^\infty(\mathbb{R}^n_+)$,

$$\begin{aligned} \big|\langle \partial^\alpha f, u\rangle_{L^2(\mathbb{R}^n_+)}\big| &= \lim_{i\to\infty} \big|\langle \partial^\alpha f, u_i\rangle_{L^2(\mathbb{R}^n_+)}\big| = \lim_{i\to\infty} \left| \int_{\mathbb{R}^n_+} \partial^\alpha f(x)\, \overline{u_i(x)}\, d^n x \right| \\ &= \lim_{i\to\infty} \left| \int_{\mathbb{R}^n_+} f(x)\, \overline{\partial^\alpha u_i(x)}\, d^n x \right| \le \limsup_{i\to\infty} \|f\|_{L^2(\mathbb{R}^n_+)} \|\partial^\alpha u_i\|_{L^2(\mathbb{R}^n_+)} \\ &\le \|f\|_{L^2(\mathbb{R}^n_+)} \limsup_{i\to\infty} \|u_i\|_{\ell,\mathbb{R}^n_+} = \|f\|_{L^2(\mathbb{R}^n_+)} \|u\|_{\ell,\mathbb{R}^n_+} \end{aligned}$$

provided $|\alpha| \le \ell$. The Riesz representation theorem provides the existence of the v_α required for the second statement.

For the converse, suppose that $u \in L^2(\mathbb{R}^n_+) = H^0(\mathbb{R}^n_+)$ obeys

$$\big|\,\langle \partial^\alpha f, u\rangle_{L^2(\mathbb{R}^n_+)}\,\big| \le C\big\|f\big\|_{L^2(\mathbb{R}^n_+)}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and all $f \in C_0^\infty(\mathbb{R}^n_+)$. (For the second statement, we can use $C = \max_{|\alpha|\le\ell}\big\|v_\alpha\big\|_{L^2(\mathbb{R}^n_+)}$.) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$. By part (b) of Lemma 2.1.14, $\partial^\alpha u$ exists as an element of $H^{-|\alpha|}(\mathbb{R}^n_+) = H_0^{|\alpha|}(\mathbb{R}^n_+)^*$. For $v \in \big\{\ v \in C^\ell(\mathbb{R}^n_+)\ \big|\ \|v\|_{\ell,\mathbb{R}^n_+} < \infty\ \big\}$, $\partial^\alpha v$ is defined, as an element of $H^{-|\alpha|}(\mathbb{R}^n_+)$ by

$$\big(\partial^\alpha v\big)(f) = \langle f, \partial^\alpha \bar v\rangle_{L^2(\mathbb{R}^n_+)}$$

for all $f \in H_0^{|\alpha|}(\mathbb{R}^n_+)$. In particular, for $f \in C_0^\infty(\mathbb{R}^n_+)$,

$$\begin{aligned}\big(\partial^\alpha v\big)(f) &= \langle f, \partial^\alpha \bar v\rangle_{L^2(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+} f(x)\,\partial^\alpha v(x)\ d^nx = (-1)^{|\alpha|}\int_{\mathbb{R}^n_+} v(x)\,\partial^\alpha f(x)\ d^nx\\ &= (-1)^{|\alpha|}\,\langle \partial^\alpha f, \bar v\rangle_{L^2(\mathbb{R}^n_+)}\end{aligned}$$

By continuity,

$$\big(\partial^\alpha u\big)(f) = (-1)^{|\alpha|}\,\langle \partial^\alpha f, \bar u\rangle_{L^2(\mathbb{R}^n_+)}$$

for all $u \in L^2(\mathbb{R}^n_+)$ and $f \in H_0^{|\alpha|}(\mathbb{R}^n_+)$. Thus for our u and α, $\big|\big(\partial^\alpha u\big)(f)\big| \le C\big\|f\big\|_{L^2(\mathbb{R}^n_+)}$. So, by the Riesz representation theorem, there is a $u_\alpha \in L^2(\mathbb{R}^n_+)$ such that $\big(\partial^\alpha u\big)(f) = \langle f, \overline{u_\alpha}\rangle_{L^2(\mathbb{R}^n_+)}$ for all $f \in H_0^{|\alpha|}(\mathbb{R}^n_+)$. In other words $\partial^\alpha u \in L^2(\mathbb{R}^n_+)$. Since this is the case for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$, $u \in H^\ell(\mathbb{R}^n_+)$ by part (a). ■

Proposition 2.2.10 *Let $\ell \in \mathbb{N}$. Define the restriction map*

$$\begin{aligned} r : \big\{\ u \in C^\infty(\overline{\mathbb{R}^n_+})\ \big|\ u \text{ has bounded support}\ \big\} &\to C_0^\infty(\mathbb{R}^{n-1})\\ u &\mapsto u\big|_{x_n=0}\end{aligned}$$

There exists a unique map

$$R : H^\ell(\mathbb{R}^n_+) \to H^{\ell-1/2}(\mathbb{R}^{n-1})$$

and constants C, C' such that

(i) R extends r.

(ii) R is bounded. That is, $|Ru|_{\ell-1/2,n-1} \le C\|u\|_{\ell,\mathbb{R}^n_+}$.

(iii) R is surjective (onto).

(iv) For each $f \in H^{\ell-1/2}(\mathbb{R}^{n-1})$, there is a $u \in H^\ell(\mathbb{R}^n_+)$ such that

$$Ru = f \quad \text{and} \qquad \|u\|_{\ell,\mathbb{R}^n_+} \le C'|f|_{\ell-1/2,n-1}$$

That is, R has a bounded right inverse.

Proof: By Lemma 2.2.7, $\{\ u \in C^\infty(\overline{\mathbb{R}^n_+})\ \big|\ u \text{ has bounded support}\ \}$ is dense in $H^\ell(\mathbb{R}^n_+)$, so there is at most one bounded extension.

Let E_+ be the extension map from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ of Lemma 2.2.8 and let $\tilde{R}$ be the restriction map from $\mathbb{R}^n$ to $x_n = 0$ of Lemma 2.2.5. Then

$$Ru = \tilde{R}E_+u$$

defines a bounded linear map from $H^\ell(\mathbb{R}^n_+)$ to $H^{\ell-1/2}(\mathbb{R}^{n-1})$. If $u \in C^\infty(\overline{\mathbb{R}^n_+})$ has bounded support, then $E_+u \in C^\ell_0(\mathbb{R}^n)$ (see (2.2.1)). By Problem 2.2.2, $(\tilde{R}E_+u)(x') = (E_+u)(x',0) = u(x',0)$, so R extends r. We have proven parts (i) and (ii).

Let E_0 be the extension map of Lemma 2.2.6, with $s = \ell$. Let R_+u be the restriction of $u \in H^\ell(\mathbb{R}^n)$ to $\mathbb{R}^n_+$. It is a bounded map from $H^\ell(\mathbb{R}^n)$ to $H^\ell(\mathbb{R}^n_+)$. Then $\mathcal{E} = R_+E_0$ is a bounded map from $H^{\ell-1/2}(\mathbb{R}^{n-1})$ through $H^\ell(\mathbb{R}^n)$ to $H^\ell(\mathbb{R}^n_+)$. Furthermore, if $f \in \mathcal{S}(\mathbb{R}^{n-1})$, then $E_0f \in \mathcal{S}(\mathbb{R}^n)$ (this was shown in the proof of Lemma 2.2.6) and $E_+\mathcal{E}f = E_+R_+E_0f$ is in $C^\ell(\mathbb{R}^n)$ with derivatives that decay faster than the inverse of any polynomial (see (2.2.1)). Hence, by Problem 2.2.2,

$$(R\mathcal{E}f)(x') = (\tilde{R}E_+\mathcal{E}f)(x') = (E_+R_+E_0f)(x',0) = (E_0f)(x',0) = f(x')$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{\ell-1/2}(\mathbb{R}^{n-1})$ and R and $\mathcal{E}$ are bounded, $R\mathcal{E}$ is the identity on $H^{\ell-1/2}(\mathbb{R}^{n-1})$. This proves parts (iii) and (iv). ■

Proposition 2.2.11 *Let $\ell \in \mathbb{N}$ and let R be the restriction operator of Proposition 2.2.10. The kernel of R is $H^\ell(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$.*

Proof: If $u \in H^1_0(\mathbb{R}^n_+) \cap H^\ell(\mathbb{R}^n_+)$, then there is a sequence of functions $u_j \in C^\infty_0(\mathbb{R}^n_+)$ that converge to u with respect to the norm $\|\ \cdot\ \|_{1,\mathbb{R}^n_+}$. Since, by part (ii) of Proposition 2.2.10 with $\ell = 1$,

$$|Ru|_{1/2,n-1} = |Ru - Ru_j|_{1/2,n-1} \le C\|u - u_j\|_{1,\mathbb{R}^n_+} \text{ for all } j \implies Ru = 0$$

the kernel of R (acting on $H^\ell(\mathbb{R}^n_+)$) contains all of $H^1_0(\mathbb{R}^n_+) \cap H^\ell(\mathbb{R}^n_+)$.

To prove the converse, we have to show that any function in the kernel of R can be approximated, in the $H^1(\mathbb{R}^n_+)$ norm, by functions in $C^\infty_0(\mathbb{R}^n_+)$. Let $\psi \in C^\infty_0(\mathbb{R})$ take values in $[0,1]$ and obey $\psi(t) = 0$ for all $|t| \le \frac{1}{2}$ and $\psi(t) = 1$ for all $|t| \ge 1$.

Step 1. We first prove that if $u \in H^1(\mathbb{R}^n_+)$, $\varepsilon > 0$ and $u_\varepsilon(x) = u(x',x_n)\psi\big(\frac{x_n}{\varepsilon}\big)$, then

$$\begin{aligned}
&\|u - u_\varepsilon\|^2_{1,\mathbb{R}^n_+} \le 2\tfrac{\Psi}{\varepsilon}\|Ru\|^2_{L^2(\mathbb{R}^{n-1})} + 2(1+\Psi)\|u\|^2_{1,\Omega_\varepsilon} \text{ where}\\
&\Omega_\varepsilon = \{\ x \in \mathbb{R}^n_+\ \big|\ x_n < \varepsilon\ \}\\
&\Psi = \max\{\ \psi'(t)^2\ \big|\ \tfrac{1}{2} \le t \le 1\ \}
\end{aligned} \tag{2.2.6}$$

Multiplication by $\psi\big(\frac{x_n}{\varepsilon}\big)$ is a bounded operator on $H^1(\mathbb{R}^n_+)$ (possibly with a norm that diverges as $\varepsilon\to 0$). So it suffices to prove (2.2.6) for $u\in C^\infty(\overline{\mathbb{R}^n_+})$.

It is clear that

$$\|u-u_\varepsilon\|^2_{L^2(\mathbb{R}^n_+)}=\big\|\big[1-\psi\big(\tfrac{x_n}{\varepsilon}\big)\big]u\big\|^2_{L^2(\mathbb{R}^n_+)}\le\int_{0<x_n<\varepsilon}|u(x)|^2\,d^nx \tag{2.2.7a}$$

so we only need to bound the first derivative terms. If $1\le j\le n-1$,

$$\tfrac{\partial}{\partial x_j}\big[\psi\big(\tfrac{x_n}{\varepsilon}\big)u(x)\big]=\psi\big(\tfrac{x_n}{\varepsilon}\big)\tfrac{\partial u}{\partial x_j}(x)$$

so that

$$\big\|\tfrac{\partial}{\partial x_j}[u-u_\varepsilon]\big\|^2_{L^2(\mathbb{R}^n_+)}=\big\|\big[1-\psi\big(\tfrac{x_n}{\varepsilon}\big)\big]\tfrac{\partial u}{\partial x_j}(x)\big\|^2_{L^2(\mathbb{R}^n_+)}\le\int_{0<x_n<\varepsilon}\big|\tfrac{\partial u}{\partial x_j}(x)\big|^2\,d^nx \tag{2.2.7b}$$

Finally,

$$\tfrac{\partial}{\partial x_n}\big[u(x)-\psi\big(\tfrac{x_n}{\varepsilon}\big)u(x)\big]=\big[1-\psi\big(\tfrac{x_n}{\varepsilon}\big)\big]\tfrac{\partial u}{\partial x_n}(x)-\tfrac{1}{\varepsilon}\psi'\big(\tfrac{x_n}{\varepsilon}\big)u(x)$$

Since $(a+b)^2\le 2a^2+2b^2$,

$$\big\|\tfrac{\partial}{\partial x_n}[u-u_\varepsilon]\big\|^2_{L^2(\mathbb{R}^n_+)}\le 2\big\|\big[1-\psi\big(\tfrac{x_n}{\varepsilon}\big)\big]\tfrac{\partial u}{\partial x_n}(x)\big\|^2_{L^2(\mathbb{R}^n_+)}+2\big\|\tfrac{1}{\varepsilon}\psi'\big(\tfrac{x_n}{\varepsilon}\big)u(x)\big\|^2_{L^2(\mathbb{R}^n_+)}$$

The first term is bounded by

$$2\big\|\big[1-\psi\big(\tfrac{x_n}{\varepsilon}\big)\big]\tfrac{\partial u}{\partial x_n}(x)\big\|^2_{L^2(\mathbb{R}^n_+)}\le 2\int_{0<x_n<\varepsilon}\big|\tfrac{\partial u}{\partial x_n}(x)\big|^2\,d^nx \tag{2.2.7c}$$

Since $\psi'\big(\frac{x_n}{\varepsilon}\big)$ vanishes for $0\le x_n\le\frac{\varepsilon}{2}$ and for $x_n\ge\varepsilon$,

$$\begin{aligned}2\big\|\tfrac{1}{\varepsilon}\psi'\big(\tfrac{x_n}{\varepsilon}\big)u(x)\big\|^2_{L^2(\mathbb{R}^n_+)}&=\tfrac{2}{\varepsilon^2}\int_{\frac{\varepsilon}{2}<x_n<\varepsilon}\psi'\big(\tfrac{x_n}{\varepsilon}\big)^2\big|u(x)\big|^2\,d^nx\\&=\tfrac{2}{\varepsilon}\int_{\frac12<y<1}\psi'(y)^2\big|u(x',\varepsilon y)\big|^2\,d^{n-1}x'dy\\&\le 2\tfrac{\Psi}{\varepsilon}\int_{\frac12<y<1}\big|u(x',\varepsilon y)\big|^2\,d^{n-1}x'dy\end{aligned}$$

where $x_n=\varepsilon y$. By the fundamental theorem of calculus

$$\begin{aligned}u(x',\varepsilon y)&=u(x',0)+\int_0^{\varepsilon y}\tfrac{\partial u}{\partial x_n}(x',x_n)\,dx_n\\&=(Ru)(x')+\int_0^{\varepsilon y}\tfrac{\partial u}{\partial x_n}(x',x_n)\,dx_n\end{aligned}$$

So, by Cauchy–Schwarz and the fact that $(a+b)^2\le 2a^2+2b^2$,

$$\big|u(x',\varepsilon y)\big|^2\le 2\big|(Ru)(x')\big|^2+2\varepsilon\int_0^\varepsilon\big|\tfrac{\partial u}{\partial x_n}(x',x_n)\big|^2\,dx_n$$

for all $0<y<1$. Hence

$$2\big\|\tfrac{1}{\varepsilon}\psi'\big(\tfrac{x_n}{\varepsilon}\big)u(x)\big\|^2_{L^2(\mathbb{R}^n_+)}\le 2\tfrac{\Psi}{\varepsilon}\int\big|(Ru)(x')\big|^2\,d^{n-1}x'+2\Psi\int_{0<x_n<\varepsilon}\big|\tfrac{\partial u}{\partial x_n}(x)\big|^2\,d^nx \tag{2.2.7d}$$

Adding (2.2.7a)–(2.2.7d) gives (2.2.6).

Step 2. We now apply (2.2.6) to complete the proof. Let $u \in H^\ell(\mathbb{R}^n_+)$ be in the kernel of R. By (2.2.6),

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|^2_{1,\mathbb{R}^n_+} \le 2(1+\Psi) \lim_{\varepsilon\to 0} \|u\|^2_{1,\Omega_\varepsilon} = 0$$

Since $u_\varepsilon(x)$ vanishes for $x_n < \frac{\varepsilon}{2}$, we have that $u_\varepsilon \in H^1_0(\mathbb{R}^n_+)$ by Problem 2.1.13b. Since $H^1_0(\mathbb{R}^n_+)$ is complete, $u \in H^1_0(\mathbb{R}^n_+)$. ■

We are now ready to deal with general bounded open sets Ω. We develop some generally useful results about $H^\ell(\Omega)$ and then give the proof of Theorem 2.2.2.

Lemma 2.2.12 *Let $\ell \in \mathbb{N}_0$ and let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ with $\overline{\Omega} \cap \overline{\mathcal{O}} = \emptyset$. Then there exists a bounded linear operator $E : H^\ell(\Omega) \to H^\ell(\mathbb{R}^n)$ such that*

$$\begin{aligned} Eu(x) &= u(x) \ \ \textit{a.e. for } x \in \Omega \\ Eu(x) &= 0 \qquad \textit{a.e. for } x \in \mathcal{O} \end{aligned}$$

Proof: Let N and, for $1 \le i \le N$, $\mathcal{U}(p_i)$, χ_i, ψ_{p_i} be as in Definition 2.1.18. Let $\mathcal{V}$ be a neighbourhood of $\partial\Omega$ on which $\sum_{i=1}^N \chi_i = 1$. Select a function $\varphi \in C_0^\infty(\Omega)$ that is identically one on $\Omega \setminus \mathcal{V}$. Since $\overline{\Omega}$ is compact, the distance from $\overline{\Omega}$ to $\overline{\mathcal{O}}$ is strictly positive and we may also choose a function $\chi \in C_0^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ that is identically one on Ω.

Let $u \in H^\ell(\Omega)$. By construction, $u = \varphi u + \sum_{i=1}^N \chi_i(1-\varphi)u$. By Lemma 2.1.21c, for each $1 \le i \le N$, $v_i = \chi_i(1-\varphi)u \circ \psi_{p_i}^{-1} \in H^\ell(\mathbb{R}^n_+)$ and

$$\|v_i\|_{\ell,\mathbb{R}^n_+} \le c_1 \|u\|_{\ell,\Omega}$$

Let E_+ be the extension operator of Lemma 2.2.8 and set

$$Eu = \chi\Big(\varphi u + \sum_{i=1}^N (E_+ v_i) \circ \psi_{p_i}\Big)$$

Since χ_i is supported in a compact subset of the domain, $\mathcal{U}(p_i)$, of ψ_{p_i} and ψ_{p_i} is continuous, there is an $r > 0$, independent of u, such that $v_i = \chi_i(1-\varphi)u \circ \psi_i^{-1}$ vanishes outside of $\{\ x \in \mathbb{R}^n_+ \ \big|\ |x| < r\ \}$. By Problem 2.2.4 and the construction (2.2.1) of Lemma 2.2.8, there is an $R > 0$, such that $E_+ v_i$ vanishes outside of $\mathcal{U}'_R = \{\ x \in \mathbb{R}^n \ \big|\ |x| < R\ \}$ for all u. Set $\mathcal{V}' = \psi_{p_i}^{-1}(\mathcal{U}'_{2R}) \subset \mathcal{U}(p_i)$. As both $\overline{\mathcal{U}'_{2R}}$ and $\overline{\mathcal{V}'}$ are compact, the restriction of $\psi_{p_i}^{-1}$ to $\mathcal{U}'_{2R}$ is in $C^\infty(\overline{\mathcal{U}'_{2R}})$ and the restriction of ψ_{p_i} to $\mathcal{V}'$ is in $C^\infty(\overline{\mathcal{V}'})$. By Lemma 2.1.21a, with $\psi = \psi_{p_i}^{-1}$, $\mathcal{U}$ replaced by $\mathcal{U}'_{2R}$ and $\mathcal{V}$ replaced by $\mathcal{V}'$, followed by Lemma 2.2.8,

$$\begin{aligned} \big|(E_+ v_i) \circ \psi_{p_i}\big|_{\ell,n} &= \big\|(E_+ v_i) \circ \psi_{p_i}\big\|_{\ell,\mathcal{V}'} \le c_2 \|E_+ v_i\|_{\ell,\mathcal{U}'_{2R}} \\ &= c_2 |E_+ v_i|_{\ell,n} \qquad\quad \le c_3 \|v_i\|_{\ell,\mathbb{R}^n_+} \\ &\le c_1 c_3 \|u\|_{\ell,\Omega} \end{aligned}$$

It now suffices to recall that, by Problem 2.1.3, multiplication by φ is bounded on $H^\ell(\Omega)$ and multiplication by χ is bounded on $H^\ell(\mathbb{R}^n)$. ■

Lemma 2.2.13 *Let $\ell \in \mathbb{N}_0$ and let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Then $C^\infty(\overline{\Omega})$ is dense in $H^\ell(\Omega)$.*

Proof: Let $u \in H^\ell(\Omega)$ and $Eu \in H^\ell(\mathbb{R}^n)$ its extension to $\mathbb{R}^n$ given by Lemma 2.2.12 with $\mathcal{O} = \big\{\ x \in \mathbb{R}^n \ \big|\ |x-y| > 1 \text{ for all } y \in \overline{\Omega}\ \big\}$. By Problem 2.1.8, there is a sequence of functions $g_m \in C_0^\infty(\mathbb{R}^n)$ that converges to Eu in $H^\ell(\mathbb{R}^n)$. If $P : H^\ell(\mathbb{R}^n) \to H^\ell(\Omega)$ is the operator that restricts functions on $\mathbb{R}^n$ to Ω, then

$$\begin{aligned}\lim_{m\to\infty} \|u - Pg_m\|_{\ell,\Omega} &= \lim_{m\to\infty} \|P(Eu - g_m)\|_{\ell,\Omega} \\ &\le \lim_{m\to\infty} |Eu - g_m|_{\ell,n} \\ &= 0\end{aligned}$$

■

Problem 2.2.9 Let $\ell \in \mathbb{N}_0$ and let $\Omega = \big\{\ (x,y) \in \mathbb{R}^2 \ \big|\ 0 < |x| < 1,\ 0 < y < 1\ \big\}$. Prove that $C^\infty(\overline{\Omega})$ is NOT dense in $H^\ell(\Omega)$.

Problem 2.2.10 **(Sobolev Imbedding)** Let $\alpha \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$ obey $\ell > |\alpha| + \frac{n}{2}$. Prove that if $u \in H^\ell(\Omega)$, then $\partial^\alpha u$ is continuous on $\overline{\Omega}$ and there is a constant C, depending only on Ω, ℓ and $|\alpha|$, such that

$$\sup_{x\in\overline{\Omega}} \big|\partial^\alpha u(x)\big| \le C\|u\|_{\ell,\Omega}$$

Lemma 2.2.14 **(Classical Regularity)** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. If $u \in H^\ell(\Omega)$ for all $\ell \in \mathbb{N}$, then $u \in C^\infty(\overline{\Omega})$.*

Proof: Just apply Problem 2.2.10 for all $\ell \in \mathbb{N}_0$. ■

Proof of Theorem 2.2.2: Let N and, for $1 \le i \le N$, $\mathcal{U}(p_i)$, χ_i, ψ_{p_i} be as in Definition 2.1.18.

(i), (ii) Let $u \in C^\infty(\,\overline{\Omega}\,)$. Then, using R_+ to denote the restriction map from $H^\ell(\mathbb{R}^n_+)$ to

$H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$,

$$\begin{aligned}
\|ru\|^2_{H^{\ell-\frac{1}{2}}(\partial\Omega)} &= \sum_{i=1}^N \big|(\chi_i ru)\circ\psi_{p_i}^{-1}\big|^2_{\ell-\frac{1}{2},n-1} && \text{by Definition 2.1.18}\\
&= \sum_{i=1}^N \big|R_+(\chi_i u\circ\psi_{p_i}^{-1})\big|^2_{\ell-\frac{1}{2},n-1}\\
&\le \sum_{i=1}^N C^2\|(\chi_i u)\circ\psi_{p_i}^{-1}\|^2_{\ell,\mathbb{R}^n_+} && \text{by Proposition 2.2.10}\\
&\le \sum_{i=1}^N C^2\tilde{C}^2\|u\|^2_{\ell,\Omega} && \text{by Lemma 2.1.21c}
\end{aligned}$$

The existence of a unique $R : H^\ell(\Omega) \to H^{\ell-1/2}(\partial\Omega)$ fulfilling the requirements of parts (i) and (ii) now follows by the B.L.T. theorem.

(iii), (iv) Let $f \in H^{\ell-1/2}(\partial\Omega)$. By definition, for each $1 \le i \le n$, $(\chi_i f)\circ\psi_{p_i}^{-1} \in H^{\ell-1/2}(\mathbb{R}^{n-1})$. By parts (iii) and (iv) of Proposition 2.2.10, there exists $v_i \in H^\ell(\mathbb{R}^n_+)$ such that $R_+v_i = (\chi_i f)\circ\psi_{p_i}^{-1}$ and $\|v_i\|_{\ell,\mathbb{R}^n_+} \le C'\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{\ell-\frac{1}{2},n-1}$. For each $1 \le i \le N$, select a function $\varphi_i \in C^\infty(\overline{\mathbb{R}^n_+})$ that has bounded support and is identically one on the support of $(\chi_i f)\circ\psi_{p_i}^{-1}$ and set

$$u = \sum_{i=1}^N (\varphi_i v_i)\circ\psi_{p_i}$$

By Lemma 2.1.21a (with the $\mathcal{U}$ of Lemma 2.1.21a being a bounded open subset of $\mathbb{R}^n_+$ that contains the support of φ_i and the ψ of Lemma 2.1.21a being $\psi_{p_i}^{-1}$) and the boundedness of multiplication by C^∞ functions all of whose derivatives are bounded, the map

$$v \in H^\ell(\mathbb{R}^n_+) \mapsto (\varphi_i v)\circ\psi_{p_i} \in H^\ell(\Omega)$$

is bounded. Similarly, the map $g \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1}) \mapsto \big((R_+\varphi_i)g\big)\circ\psi_{p_i} \in H^{\ell-\frac{1}{2}}(\partial\Omega)$ is bounded. Hence

$$R\big((\varphi_i v)\circ\psi_{p_i}\big) = (R_+\varphi_i)(R_+v)\circ\psi_{p_i}$$

is valid for all $v \in H^\ell(\mathbb{R}^n_+)$, not just $v \in C^\infty(\overline{\mathbb{R}^n_+})$. In particular,

$$R\big((\varphi_i v_i)\circ\psi_{p_i}\big) = (R_+\varphi_i)(R_+v_i)\circ\psi_{p_i} = (R_+\varphi_i)\ \big(\chi_i f\circ\psi_{p_i}^{-1}\big)\circ\psi_{p_i} = \chi_i f$$

and $Ru = \sum_{i=1}^N \chi_i f = f$.

For the bound of part (iv),

$$\|u\|_{\ell,\Omega} \le \sum_{i=1}^{N} \big\|(\varphi_i v_i)\circ\psi_{p_i}\big\|_{\ell,\Omega} \le \sum_{i=1}^{N} \|v_i\|_{\ell,\mathbb{R}^n_+} \le \sum_{i=1}^{N} C' \big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|_{\ell-\frac12,n-1}$$

$$\le C'\sqrt{N}\Big[\sum_{i=1}^{N}\big|(\chi_i f)\circ\psi_{p_i}^{-1}\big|^2_{\ell-\frac12,n-1}\Big]^{1/2} = C'\sqrt{N}\,\|f\|_{\ell-\frac12,\partial\Omega}$$

by Cauchy–Schwarz and Definition 2.1.18.

(v) Let $u \in H^\ell(\Omega)$ be in the kernel of R. Observe that

$$R_+\big((\chi_i u)\circ\psi_{p_i}^{-1}\big) = \big((R\chi_i)(Ru)\big)\circ\psi_{p_i}^{-1}\Big|_{\mathbb{R}^{n-1}} = 0$$

Therefore, by Proposition 2.2.11, $(\chi_i u)\circ\psi_{p_i}^{-1} \in H^1_0(\mathbb{R}^n_+)\cap H^\ell(\mathbb{R}^n_+)$ and there exist sequences $u^{(i)}_m \in C^\infty_0(\mathbb{R}^n_+)$ obeying

$$\lim_{m\to\infty} u^{(i)}_m = (\chi_i u)\circ\psi_{p_i}^{-1} \text{ in } H^1(\mathbb{R}^n_+)$$

Consequently $\lim_{m\to\infty} u^{(i)}_m \circ\psi_{p_1} = \chi_i u$ in $H^1(\Omega)$ and

$$\lim_{m\to\infty}\sum_{i=1}^{N} u^{(i)}_m\circ\psi_{p_1} = \sum_{i=1}^{N}\chi_i u \text{ in } H^1(\Omega)$$

As $u^{(i)}_m\circ\psi_{p_i}\in C^\infty_0(\Omega)$, $\sum\limits_{i=1}^{N}\chi_i u \in H^1_0(\Omega)$. By Problem 2.1.13,

$$\Big(1-\sum_{i=1}^{N}\chi_i\Big)u \in H^\ell_0(\Omega)\subset H^1_0(\Omega)$$

Thus the kernel is contained in $H^1_0(\Omega)\cap H^\ell(\Omega)$. We have already observed, in Remark 2.2.4, that the kernel contains all of $H^1_0(\Omega)\cap H^\ell(\Omega)$. So the proof is concluded. ■

Problem 2.2.11 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Define the restriction map

$$r : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$$
$$u \mapsto u \restriction \partial\Omega$$

Prove that there does NOT exist a bounded map

$$R : L^2(\Omega)\to L^2(\partial\Omega)$$

such that $Ru = ru$ for all $u\in C^\infty(\overline{\Omega})$.

Here is the analog of Proposition 2.2.9 for $H^\ell(\Omega)$. As we mentioned in the discussion preceeding Proposition 2.2.9, it can be safely skipped on first reading.

Proposition 2.2.15 *Let $\ell \in \mathbb{N}_0$ and let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary.*

(a) Let $u \in H^{\ell'}(\Omega)$ with $\ell' \in \mathbb{N}_0$ obeying $\ell' \le \ell$. Then $u \in H^\ell(\Omega)$ if and only if $\partial^\alpha u \in H^{\ell'}(\Omega)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell - \ell'$.

(b) Let $u \in L^2(\Omega)$. Then $u \in H^\ell(\Omega)$ if and only if there is a constant C such that

$$\big| \langle \partial^\alpha f, u \rangle_{L^2(\Omega)} \big| \le C \big\| f \big\|_{L^2(\Omega)}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and all $f \in C_0^\infty(\Omega)$. Also, $u \in H^\ell(\Omega)$ if and only if, for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$, there is a $v_\alpha \in L^2(\Omega)$ such that

$$(-1)^{|\alpha|} \langle \partial^\alpha f, u \rangle_{L^2(\Omega)} = \langle f, v_\alpha \rangle_{L^2(\Omega)}$$

for all $f \in C_0^\infty(\Omega)$.

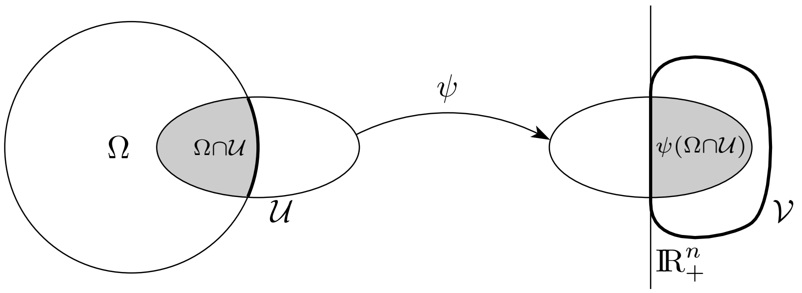
Proof: (a) The implication that if $u \in H^\ell(\Omega)$, then $\partial^\alpha u \in H^{\ell'}(\Omega)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell - \ell'$ is an immediate consequence of part (b) of Lemma 2.1.14, so we just prove the converse.

Throughout the next two paragraphs, assume that α runs over the elements of $\mathbb{N}_0^n$ that obey $|\alpha| \le \ell - \ell'$. Let N and, for $1 \le i \le N$, $\mathcal{U}(p_i)$, χ_i, ψ_{p_i} be as in Definition 2.1.18. Let $\mathcal{V}$ be a neighbourhood of $\partial\Omega$ on which $\sum_{i=1}^N \chi_i = 1$. Select a function $\varphi \in C_0^\infty(\Omega)$ that is identically one on $\Omega \setminus V$. We prove that each of the $N+1$ terms on the right hand side of

$$u = \varphi u + \sum_{i=1}^N \chi_i (1-\varphi) u$$

is in $H^\ell(\Omega)$. By the product rule and part (a) of Lemma 2.1.14, $\partial^\alpha(\varphi u) \in H^{\ell'}(\Omega)$. Since $\partial^\alpha(\varphi u)$ vanishes outside of a compact subset of Ω, $\partial^\alpha(\varphi u) \in H^{\ell'}(\mathbb{R}^n)$. Thus $k^\alpha \big(1+|k|^2\big)^{\ell'/2} \widehat{\varphi u}(k) \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell - \ell'$. This implies that $\big(1+|k|^2\big)^{\ell/2} \widehat{\varphi u}(k) \in L^2(\mathbb{R}^n)$ and hence that $\varphi u \in H^\ell(\mathbb{R}^n)$ and that $\varphi u \in H^\ell(\Omega)$.

Fix any $1 \le i \le N$. Denote $\psi = \psi_{p_i}$, $v = \chi_i(1-\varphi)u$ and $w = \chi_i(1-\varphi)u \circ \psi^{-1}$. By the chain rule, product rule and part (c) of Lemma 2.1.21, $\partial^\alpha w \in H^{\ell'}(\mathbb{R}^n_+)$. Hence, by part (a) of Lemma 2.2.9, $w \in H^\ell(\mathbb{R}^n_+)$. Let $\mathcal{U}$ be an open neighbourhood of the support of χ_i whose closure is contained in $\mathcal{U}(p_i)$. Then $\psi(\mathcal{U}) \cap \mathbb{R}^n_+ = \psi(\mathcal{U} \cap \Omega)$ is a bounded open subset of $\mathbb{R}^n$ and w vanishes outside of $\psi(\mathcal{U} \cap \Omega)$. Enlarge $\psi(\mathcal{U}) \cap \mathbb{R}^n_+$ to $\mathcal{V}$, a bounded open subset of $\mathbb{R}^n_+$ with smooth boundary.



By part (a) of Lemma 2.1.21, with $\mathcal{U}$ replaced by $\mathcal{V}$, $\mathcal{V}$ replaced by $\psi^{-1}(\mathcal{V})$, and ψ replaced by ψ^{-1}, we have $v \in H^\ell(\Omega)$.

(b) The proof is virtually identical to that of part (b) of Proposition 2.2.9. It suffices to replace $\mathbb{R}^n_+$ by Ω and replace the reference to Lemma 2.2.7 by a reference to Lemma 2.2.13.

■

§2.3. Inequalities

In this section we prove a number of useful inequalities. Here is a table giving the locations of the main inequalities.

Inequality	Name	Reference
$\|A\|_{(1-t)s_0+ts_1} \le \|A\|_{s_0}^{1-t}\|A\|_{s_1}^{t}$	interpolation	Lemma 2.3.2
$\|f * g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$	Young's inequality	Lemma 2.3.3
$\lvert\varphi u\rvert_s \le 2^{\frac{\lvert s\rvert}{2}} \Big(\int (1+\lvert p\rvert^2)^{\frac{\lvert s\rvert}{2}} \lvert\hat{\varphi}(p)\rvert \ \frac{d^n p}{(2\pi)^n}\Big) \lvert u\rvert_s$	boundedness of $u \mapsto \varphi u$	Lemma 2.3.5
$\|u\|^2_{L^2(\Omega)} \le C\Big[\sum_{i=1}^n \|\partial_i u\|^2_{L^2(\Omega)} + \|Ru\|^2_{L^2(\partial\Omega)}\Big]$	Poincaré's inequality	Proposition 2.3.7
$\big\|u - (u)_S\big\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}$	Poincaré's inequality	Proposition 2.3.10
$L^1 - L^\infty$ bound on operator norm of integral operators		Proposition 2.3.12

We start with the three lines theorem of complex analysis and apply it prove an interpolation lemma that says that the operator norm on Sobolev spaces is log convex.

Lemma 2.3.1 (Three Lines Theorem) *Let $F(z)$ be a function that is holomorphic in* $0 < \mathrm{Re}\, z < 1$ *and continuous and bounded on* $0 \le \mathrm{Re}\, z \le 1$. *Assume that*

$$\sup_{\mathrm{Re}\, z=0} |F(z)| \le M_0 \qquad \sup_{\mathrm{Re}\, z=1} |F(z)| \le M_1$$

Then

$$\sup_{\operatorname{Re} z=t} |F(z)| \le M_0^{1-t} M_1^t$$

for all $0 \le t \le 1$.

Proof: Define, for $\varepsilon > 0$,

$$F_\varepsilon(z) = e^{\varepsilon z^2} F(z) M_0^{z-1} M_1^{-z}$$

Since $\operatorname{Re}(z^2) = (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2$,

$$\big|F_\varepsilon(z)\big| = e^{\varepsilon(\operatorname{Re} z)^2} e^{-\varepsilon(\operatorname{Im} z)^2} |F(z)| M_0^{\operatorname{Re} z-1} M_1^{-\operatorname{Re} z}$$

Recalling that $|F(z)|$ is bounded, by hypothesis, we have

$$\sup_{\operatorname{Re} z=0} |F_\varepsilon(z)| \le 1 \qquad \sup_{\operatorname{Re} z=1} |F_\varepsilon(z)| \le e^\varepsilon \qquad \lim_{|\operatorname{Im} z|\to\infty} \sup_{0\le \operatorname{Re} z\le 1} |F_\varepsilon(z)| = 0$$

The maximum modulus of $F_\varepsilon(z)$ on the rectangle $\{0 \le \operatorname{Re} z \le 1, |\operatorname{Im} z| \le y\}$ is achieved on the boundary of the rectangle, since $F_\varepsilon(z)$ is holomorphic. Hence

$$\sup_{0\le \operatorname{Re} z\le 1} |F_\varepsilon(z)| = \lim_{y\to\infty} \sup_{\substack{|\operatorname{Im} z|\le y \\ 0\le \operatorname{Re} z\le 1}} |F_\varepsilon(z)| \le \lim_{y\to\infty} \max\left\{ e^\varepsilon \ , \sup_{\substack{|\operatorname{Im} z|=y \\ 0\le \operatorname{Re} z\le 1}} |F_\varepsilon(z)| \right\} = e^\varepsilon$$

Taking the limit $\varepsilon \to 0+$ gives $\left|F(z) M_0^{\operatorname{Re} z-1} M_1^{-\operatorname{Re} z}\right| \le 1$ and hence

$$|F(z)| \le M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$$

which is what we want. ■

Let $s_0 \in \mathbb{R}$ and suppose that A is a continuous linear operator on $H^{s_0}(\mathbb{R}^n)$. For any $s > s_0$, $H^s(\mathbb{R}^m) \subset H^{s_0}(\mathbb{R}^m)$ so that we may also view A as an operator defined on $H^s(\mathbb{R}^m)$, though possibly still taking values in $H^{s_0}(\mathbb{R}^m)$. The next lemma shows that if, in addition, we know that $AH^{s_1}(\mathbb{R}^m) \subset H^{s_1}(\mathbb{R}^m)$ for some $s_1 > s_0$, then necessarily $AH^s(\mathbb{R}^m) \subset H^s(\mathbb{R}^m)$ for all s between s_0 and s_1. The lemma also provides a bound on the operator norm

$$\|A\|_s = \sup_{\substack{f\in H^s(\mathbb{R}^m) \\ f\ne 0}} \frac{\vert\!\vert\!\vert Af \vert\!\vert\!\vert_{s,m}}{\vert\!\vert\!\vert f \vert\!\vert\!\vert_{s,m}}$$

of A, viewed as a map on $H^s(\mathbb{R}^m)$, in terms of its operator norms when viewed as maps on $H^{s_0}(\mathbb{R}^m)$ and on $H^{s_1}(\mathbb{R}^m)$.

Lemma 2.3.2 (Interpolation) *Let* $s_0, s_1 \in \mathbb{R}$, $0 \le t \le 1$ *and* $s = (1-t)s_0 + ts_1$. *Let*

$$\begin{aligned} A : H^{s_0}(\mathbb{R}^m) \to H^{s_0}(\mathbb{R}^m) \\ A : H^{s_1}(\mathbb{R}^m) \to H^{s_1}(\mathbb{R}^m) \end{aligned}$$

be bounded. Then

$$A : H^{s}(\mathbb{R}^m) \to H^{s}(\mathbb{R}^m)$$

is bounded with

$$\|A\|_s \le \|A\|_{s_0}^{1-t} \|A\|_{s_1}^{t}$$

Proof: By Problem 2.3.1, it suffices to prove that

$$\Big| \int_{\mathbb{R}^m} \widehat{Af}(k) \overline{\hat{g}}(k) \ \tfrac{d^m k}{(2\pi)^m} \Big| \le \|A\|_{s_0}^{1-t} \, \|A\|_{s_1}^{t} \, |f|_s \, |g|_{-s}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^m)$. So let $f, g \in \mathcal{S}(\mathbb{R}^m)$ and set, for $z \in \mathbb{C}$, $\sigma(z) = (1-z)s_0 + zs_1$. Observe that $\sigma(t) = s$. Define $f_z \in \mathcal{S}(\mathbb{R}^m)$ by

$$\hat{f}_z(k) = \big(1 + |k|^2\big)^{\frac{s-\sigma(z)}{2}} \hat{f}(k)$$

We shall apply the three lines theorem, Lemma 2.3.1, to

$$F(z) = \int_{\mathbb{R}^m} \big(1 + |k|^2\big)^{\frac{\sigma(z)-s}{2}} \widehat{Af_z}(k) \overline{\hat{g}}(k) \ \tfrac{d^m k}{(2\pi)^m}$$

in order to derive a bound on

$$F(t) = \int_{\mathbb{R}^m} \widehat{Af}(k) \overline{\hat{g}}(k) \ \tfrac{d^m k}{(2\pi)^m}$$

Inserting $1 = \big(1 + |k|^2\big)^{\frac{\operatorname{Re}\sigma(z)}{2}} \big(1 + |k|^2\big)^{-\frac{\operatorname{Re}\sigma(z)}{2}}$ into the integral defining $F(z)$ and applying Cauchy–Schwarz gives

$$\begin{aligned} |F(z)| \ &\le \ \Big[\int_{\mathbb{R}^m} \big(1 + |k|^2\big)^{\operatorname{Re}\sigma(z)} \big|\widehat{Af_z}(k)\big|^2 \ \tfrac{d^m k}{(2\pi)^m} \Big]^{1/2} \\ &\qquad \Big[\int_{\mathbb{R}^m} \big(1 + |k|^2\big)^{-\operatorname{Re}\sigma(z)} \big(1 + |k|^2\big)^{\operatorname{Re}\sigma(z)-s} \big|\hat{g}(k)\big|^2 \ \tfrac{d^m k}{(2\pi)^m} \Big]^{1/2} \\ &\le \ \Big[\int_{\mathbb{R}^m} \big(1 + |k|^2\big)^{\operatorname{Re}\sigma(z)} \big|\widehat{Af_z}(k)\big|^2 \ \tfrac{d^m k}{(2\pi)^m} \Big]^{1/2} \Big[\int_{\mathbb{R}^m} \big(1 + |k|^2\big)^{-s} \big|\hat{g}(k)\big|^2 \ \tfrac{d^m k}{(2\pi)^m} \Big]^{1/2} \\ &= \ \big|Af_z\big|_{\operatorname{Re}\sigma(z)} \, |g|_{-s} \le \|A\|_{\operatorname{Re}\sigma(z)} \, |f_z|_{\operatorname{Re}\sigma(z)} \, |g|_{-s} \le \|A\|_{\operatorname{Re}\sigma(z)} \, |f|_s \, |g|_{-s} \end{aligned}$$

since $\left|\hat{f}_z(k)\right|^2 = \left(1+|k|^2\right)^{s-\operatorname{Re}\sigma(z)}|\hat{f}(k)|^2$. When $\operatorname{Re} z = 0$, $\operatorname{Re}\sigma(z) = s_0$ so that

$$|F(z)| \le \|A\|_{s_0} \, |f|_s \, |g|_{-s}$$

When $\operatorname{Re} z = 1$, $\operatorname{Re}\sigma(z) = s_1$ so that

$$|F(z)| \le \|A\|_{s_1} \, |f|_s \, |g|_{-s}$$

By Problem 2.3.2, $F(z)$ is an entire function of z so that the three lines theorem gives

$$\Big| \int_{\mathbb{R}^m} \widehat{Af}(k)\overline{\hat{g}}(k) \, \tfrac{d^m k}{(2\pi)^m} \Big| = \big|F(t)\big| \le \|A\|_{s_0}^{1-t} \, \|A\|_{s_1}^{t} \, |f|_s \, |g|_{-s}$$

and the Lemma follows. ■

Problem 2.3.1 Let $t < s$ and let A be a bounded linear map on $H^t(\mathbb{R}^m)$. Prove that if there is a constant M such that

$$\Big| \int_{\mathbb{R}^m} \widehat{Af}(k)\overline{\hat{g}}(k) \, \tfrac{d^m k}{(2\pi)^m} \Big| \le M \, |f|_s \, |g|_{-s}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^m)$ then A is a bounded linear map $H^s(\mathbb{R}^m)$ with norm at most M.

Problem 2.3.2 Let $s \in \mathbb{R}$. Let $s \in \mathbb{R}$ and A be a bounded linear map on $H^s(\mathbb{R}^m)$. Let $f, g \in \mathcal{S}(\mathbb{R}^m)$ and set, for $z \in \mathbb{C}$, $\sigma(z) = (1-z)s_0 + zs_1$. Define

$$F(z) = \int_{\mathbb{R}^m} \left(1+|k|^2\right)^{\frac{\sigma(z)-s}{2}} \widehat{Af_z}(k)\overline{\hat{g}}(k) \, \tfrac{d^m k}{(2\pi)^m}$$

where $f_z \in \mathcal{S}(\mathbb{R}^m)$ is determined by

$$\hat{f}_z(k) = \left(1+|k|^2\right)^{\frac{s-\sigma(z)}{2}} \hat{f}(k)$$

(a) Prove that $\frac{f_{z'}-f_z}{z'-z}$ converges in $H^s(\mathbb{R}^m)$ as $z' \to z$.

(b) Prove that $F(z)$ is an entire function of z.

Problem 2.3.3 Let Ω be an open subset of $\mathbb{R}^n$. Let $1 \le p < q < r \le \infty$. Prove that if $f \in L^p(\Omega) \cap L^r(\Omega)$, then $f \in L^q(\Omega)$ and

$$\|f\|_{L^q(\Omega)} \le \|f\|_{L^p(\Omega)}^{1-t} \|f\|_{L^r(\Omega)}^{t}$$

where $0 < t < 1$ is determined by $\frac{1}{q} = \frac{1-t}{p} + \frac{t}{r}$.

Problem 2.3.4 The goal of this problem is to prove the Riesz–Thorin interpolation theorem, which is as follows. Let Ω be an open subset of $\mathbb{R}^n$. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and, for $0 \le t \le 1$, define p_t, q_t by

$$\tfrac{1}{p_t} = \tfrac{1-t}{p_0} + \tfrac{t}{p_1} \qquad \tfrac{1}{q_t} = \tfrac{1-t}{q_0} + \tfrac{t}{q_1}$$

Suppose that T is a linear transformation from $L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ to $L^{q_0}(\Omega) \cap L^{q_1}(\Omega)$ which satisfies

$$\|Tf\|_{L^{q_0}(\Omega)} \le M_0 \|f\|_{L^{p_0}(\Omega)} \qquad \|Tf\|_{L^{q_1}(\Omega)} \le M_1 \|f\|_{L^{p_1}(\Omega)}$$

for all $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$. Then, for each $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ and $0 < t < 1$, $Tf \in L^{q_t}(\Omega)$ and

$$\|Tf\|_{L^{q_t}(\Omega)} \le M_0^{1-t} M_1^t \|f\|_{L^{p_t}(\Omega)}$$

(a) Prove the Riesz–Thorin interpolation theorem in the special case that $p_0 = p_1$.

(b) Denote by

$$\Sigma(\Omega) = \Big\{ \sum_{j=1}^m a_j \chi_{E_j} \ \Big|\ m \in \mathbb{N},\ a_1, \cdots, a_m \in \mathbb{C}, \ E_1, \cdots, E_m \subset \Omega \text{ measureable and of finite measure} \Big\}$$

the set of simple functions on Ω. Prove that if $f \in \Sigma(\Omega)$, then

$$\|Tf\|_{L^{q_t}(\Omega)} = \sup \Big\{ \Big| \int_\Omega (Tf)(x) g(x)\, d^n x \Big| \ \Big|\ g \in \Sigma(\Omega),\ \|g\|_{L^{q'_t}(\Omega)} \le 1 \Big\}$$

where q'_t, the conjugate index to q_t, is determined by $\frac{1}{q_t} + \frac{1}{q'_t} = 1$.

(c) Let $p_0 \ne p_1$ and $0 < t < 1$. Prove that

$$\Big| \int_\Omega (Tf)(x) g(x)\, d^n x \Big| \le M_0^{1-t} M_1^t$$

for all $f, g \in \Sigma(\Omega)$ with $\|f\|_{L^{p_t}(\Omega)}, \|g\|_{L^{q'_t}(\Omega)} \le 1$.

(d) Prove the Riesz–Thorin interpolation theorem.

Problem 2.3.5 Let $\ell, L \in \mathbb{Z}$ and $s \in \mathbb{R}$ with $\ell \le s \le L$. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and assume that $\varphi \in C^{\max\{|\ell|,|L|\}}(\partial\Omega)$. Prove that the map $v \in C^\infty(\partial\Omega) \mapsto \varphi v$ has a unique extension to a bounded, linear map on $H^s(\partial\Omega)$ and there is a constant C, depending only on Ω, $\max\{|\ell|, |L|\}$ and n, such that

$$\|\varphi u\|_{s,\partial\Omega} \le C \|\varphi\|_{C^{|\ell|}(\partial\Omega)}^{\frac{L-s}{L-\ell}} \|\varphi\|_{C^{|L|}(\partial\Omega)}^{\frac{s-\ell}{L-\ell}} \|u\|_{s,\partial\Omega} \le C \|\varphi\|_{C^{\max\{|\ell|,|L|\}}(\partial\Omega)} \|u\|_{s,\partial\Omega}$$

for all $u \in H^s(\partial\Omega)$.

Problem 2.3.6 (Hausdorff–Young Inequality) Let $1 \le p \le 2$ and let $2 \le q \le \infty$ obey $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^q(\mathbb{R}^n)$ and

$$\|\hat{f}\|_{L^q(\mathbb{R}^n, \frac{d^nk}{(2\pi)^n})} \le \|f\|_{L^p(\mathbb{R}^n, d^nx)}$$

Problem 2.3.7 Let $2 < p \le \infty$ and let $1 \le q < 2$ obey $\frac{1}{p} + \frac{1}{q} = 1$. Prove that there does NOT exist a constant C such that

$$\|\hat{f}\|_{L^q(\mathbb{R}^n, \frac{d^nk}{(2\pi)^n})} \le C\|f\|_{L^p(\mathbb{R}^n, d^nx)}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Hint: consider $f(x) = e^{-(a+ib)x^2}$ with $a > 0$.

Since $\big|\hat{f}(k)\hat{g}(k)\big| \le \|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \,\big|\hat{g}(k)\big|$ for almost every $k \in \mathbb{R}^n$, we have

$$\|\hat{f}\hat{g}\|_{L^q(\mathbb{R}^n)} \le \|\hat{f}\|_{L^\infty(\mathbb{R}^n)}\|\hat{g}\|_{L^q(\mathbb{R}^n)}$$

Young's inequality is the "Fourier transform" of this inequality.

Lemma 2.3.3 (Young's Inequality) *Let $1 \le p \le \infty$. If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$$

Proof: First consider $1 < p < \infty$ and set $q = \big(1 - \frac{1}{p}\big)^{-1}$. By Hölder's inequality,

$$\begin{aligned} |f * g(x)| &= \left|\int f(x-y)\, g(y)\; d^ny\right| \\ &\le \int |f(x-y)|^{1/q}\; |f(x-y)|^{1/p}|g(y)|\; d^ny \\ &\le \Big[\int |f(x-y)|\; d^ny\Big]^{1/q}\Big[\int |f(x-y)||g(y)|^p\; d^ny\Big]^{1/p} \\ &= \|f\|_{L^1(\mathbb{R}^n)}^{1/q}\Big[\int |f(x-y)||g(y)|^p\; d^ny\Big]^{1/p} \end{aligned}$$

Hence

$$\begin{aligned} \|f * g\|_{L^p(\mathbb{R}^n)}^p &\le \|f\|_{L^1(\mathbb{R}^n)}^{p/q}\int d^nx \int d^ny\; |f(x-y)||g(y)|^p \\ &= \|f\|_{L^1(\mathbb{R}^n)}^{p/q}\int d^ny \int d^nx\; |f(x-y)||g(y)|^p \\ &= \|f\|_{L^1(\mathbb{R}^n)}^{1+p/q}\int d^ny\; |g(y)|^p = \|f\|_{L^1(\mathbb{R}^n)}^p\|g\|_{L^p(\mathbb{R}^n)}^p \end{aligned}$$

In the remaining cases, the desired inequality follows directly from

$$|f * g(x)| \le \begin{cases} \int |f(x-y)||g(y)| \, d^n y & \text{if } p = 1 \\ \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} & \text{if } p = \infty \end{cases}$$

■

Problem 2.3.8

(a) Let $1 \le p, q, r \le \infty$ obey $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Prove that if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$, then

$$\left| \int f(x) g(x-y) h(y) \, d^n x d^n y \right| \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}$$

(b) Let $1 \le p, q, r \le \infty$ obey $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Prove that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$$

Next we prove that, for appropriate functions φ, the map $u \mapsto \varphi u$ is a bounded map on $H^s(\mathbb{R}^n)$. By way of preparation, we prove

Lemma 2.3.4 (Peetre's Inequality)

$$\big(1 + |k|^2\big)^s \big(1 + |p|^2\big)^{-s} \le 2^{|s|} \big(1 + |k-p|^2\big)^{|s|}$$

for all $k, p \in \mathbb{R}^n$ *and* $s \in \mathbb{R}$

Proof: Since $0 \le (a-b)^2 = a^2 - 2ab + b^2$, we have that $2ab \le a^2 + b^2$ and

$$(1 + |k|^2) \le \big(1 + (|p| + |k-p|)^2\big) \le \big(1 + 2|p|^2 + 2|k-p|^2\big) \le 2\big(1 + |p|^2\big)\big(1 + |k-p|^2\big)$$

If $s \ge 0$, it suffices to take the s^{th} power of this bound. If $s \le 0$, exchange the roles of k and p and then take the $(-s)^{\text{th}}$ power. ■

Lemma 2.3.5 *Let $s \in \mathbb{R}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$. Then $\varphi u \in H^s(\mathbb{R}^n)$ and*

$$\left|\varphi u\right|_s \le 2^{\frac{|s|}{2}} \Big(\int (1+|p|^2)^{\frac{|s|}{2}} \left|\hat{\varphi}(p)\right| \, \tfrac{d^n p}{(2\pi)^n} \Big) \left|u\right|_s$$

Proof: Let $u \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\widehat{\varphi u}(k) = \int \hat{\varphi}(k-p)\hat{u}(p) \, \tfrac{d^n p}{(2\pi)^n}$$

Hence, by Lemma 2.3.4,

$$\begin{aligned} \big(1+|k|^2\big)^{\frac{s}{2}} \big|\widehat{\varphi u}(k)\big| &\le \int \big(1+|k|^2\big)^{\frac{s}{2}} \big(1+|p|^2\big)^{-\frac{s}{2}} \big(1+|p|^2\big)^{\frac{s}{2}} |\hat{\varphi}(k-p)| \, |\hat{u}(p)| \, \tfrac{d^n p}{(2\pi)^n} \\ &\le 2^{\frac{|s|}{2}} \int \big(1+|k-p|^2\big)^{\frac{|s|}{2}} \big(1+|p|^2\big)^{\frac{s}{2}} |\hat{\varphi}(k-p)| \, |\hat{u}(p)| \, \tfrac{d^n p}{(2\pi)^n} \end{aligned}$$

The right hand side is $2^{\frac{|s|}{2}} \frac{1}{(2\pi)^n} (f * g)(k)$ with

$$f(p) = \big(1+|p|^2\big)^{\frac{|s|}{2}} |\hat{\varphi}(p)| \qquad g(p) = \big(1+|p|^2\big)^{\frac{s}{2}} |\hat{u}(p)|$$

Hence by Young's inequality, Lemma 2.3.3,

$$\begin{aligned} \left|\varphi u\right|_s &= \tfrac{1}{(2\pi)^{n/2}} \big\| \big(1+|k|^2\big)^{\frac{s}{2}} |\widehat{\varphi u}(k)| \big\|_{L^2(\mathbb{R}^n)} \le 2^{\frac{|s|}{2}} \tfrac{1}{(2\pi)^{3n/2}} \|f\|_{L^1(\mathbb{R}^n)} \, \|g\|_{L^2(\mathbb{R}^n)} \\ &= 2^{\frac{|s|}{2}} \Big(\int (1+|p|^2)^{\frac{|s|}{2}} \left|\hat{\varphi}(p)\right| \, \tfrac{d^n p}{(2\pi)^n} \Big) \left|u\right|_s \end{aligned}$$

■

We now start the proof of Poincaré's inequality, which bounds the L^2 norm of a function in a region in terms the L^2 norm of the gradient of the function and the values of the function on the boundary of the region. By way of preparation, we prove

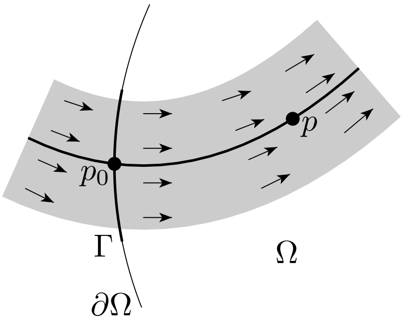
Lemma 2.3.6 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and Γ be an open subset of $\partial\Omega$. Let $p \in \Omega$ and $\gamma(t,p)$, $-1 < t < 2$, be a smooth curve obeying*

- ○ $\gamma(0,p) \in \Gamma$
- ○ $\gamma(1,p) = p$
- ○ $\gamma(t,p) \in \Omega$ *for all* $0 < t < 2$
- ○ γ *meets* Γ *transversely*

Then there exists a neighbourhood N of $\{\ \gamma(t) \ \big|\ 0 \le t \le 1\ \}$ and a C^∞ diffeomorphism $\psi : N \to \{\ y \in \mathbb{R}^{n-1} \ \big|\ |y| < 1\ \} \times (-1,2)$ such that $N \cap \partial\Omega \subset \Gamma$, $\psi(\Gamma) \subset \mathbb{R}^{n-1}$ and $\psi(\Omega \cap N) = \{\ y \in \mathbb{R}^{n-1} \ \big|\ |y| < 1\ \} \times (0,2)$.

Proof: Let $p_0 \in \partial\Omega$ be the point where γ intersects Γ. Let $\gamma(t_q, p)$ be the point on γ that is closest to q. There is a unique such point if q is close enough to γ. We define the vector field

$$v(q) = \tfrac{d\gamma}{dt}(t_q, p)$$



Then we solve the ODE

$$\begin{aligned} \tfrac{dq}{dt} &= v(q) \\ q(y,0) &= \phi^{-1}(y) \end{aligned}$$

where ϕ is a C^∞ diffeomorphism from a neighbourhood of p_0 in Γ to the unit ball in $\mathbb{R}^{n-1}$. Then $N = \{\ q(y,t)\ \big|\ |y| < 1,\ -1 < t < 2\ \}$ is the range of q and ψ is the inverse of q. ■

Proposition 2.3.7 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and Γ be an open subset of $\partial\Omega$. Denote by $d^{n-1}\sigma$ the surface measure on Γ. There is a constant C, which depends only on Ω and Γ, such that, for all $u \in C^\infty(\overline{\Omega})$,*

$$\int_\Omega |u(x)|^2\ dx \le C\Big(\int_\Omega \big|\nabla u(x)\big|^2\ d^n x + \int_\Gamma |u(x)|^2\ d^{n-1}\sigma\Big)$$

If R is the restriction map of Theorem 2.2.2,

$$\|u\|_{L^2(\Omega)} \le C\Big(\sum_{i=1}^n \|\partial_i u\|^2_{L^2(\Omega)} + \|Ru\|^2_{L^2(\partial\Omega)}\Big)^{1/2}$$

for all $u \in H^1(\Omega)$.

Proof: Let $u \in C^\infty(\overline{\Omega})$, $p \in \Omega$ and $\big(N(p), \psi\big)$ be as in Lemma 2.3.6. Then, making the change of variables $x \in N(p) \mapsto \psi(x) = (y,t) \in \{\ y \in \mathbb{R}^{n-1}\ \big|\ |y| < 1\ \} \times (-1,2)$ and denoting the Jacobian $\big|\tfrac{dx}{dq}\big|$,

$$\int_{N(p)\cap\Omega} \big|u(x)\big|^2\ d^n x = \int_0^2 dt \int_{|y|<1} d^{n-1}y\ \big|u \circ \psi^{-1}(y,t)\big|^2\ \big|\tfrac{dx}{dq}\big|$$

By the fundamental theorem of calculus,

$$u \circ \psi^{-1}(y,t) = u \circ \psi^{-1}(y,0) + \int_0^t \tfrac{d}{dt}\big(u \circ \psi^{-1}\big)(y,s)\ ds$$

which implies that, for all $0 \le t \le 2$,

$$\begin{aligned}\big|u\circ\psi^{-1}(y,t)\big|^2 &\le 2\big|u\circ\psi^{-1}(y,0)\big|^2 + 2\left|\int_0^t \tfrac{d}{dt}\big(u\circ\psi^{-1}\big)(y,s)\ ds\right|^2\\ &\le 2\big|u\circ\psi^{-1}(y,0)\big|^2 + 4\int_0^2 \big|\tfrac{d}{dt}\big(u\circ\psi^{-1}\big)(y,s)\big|^2\ ds\end{aligned}$$

Since $\left|\frac{dx}{dq}\right|$ is bounded

$$\int_{N(p)\cap\Omega}\big|u(x)\big|^2\ d^nx \le C'\int_{|y|<1} d^{n-1}y\ \big|u\circ\psi^{-1}(y,0)\big|^2 + C'\int_0^2 ds\int_{|y|<1} d^{n-1}y\ \big|\tfrac{d}{dt}\big(u\circ\psi^{-1}\big)(y,s)\big|^2$$

Applying the chain rule to $\frac{d}{dt}\big(u\circ\psi^{-1}\big)(y,s)$, using the smoothness of ψ^{-1} and changing variables back again

$$\int_{N(p)\cap\Omega}\big|u(x)\big|^2\ d^nx \le C''\int_\Gamma d^{n-1}\sigma\ \big|u(x)\big|^2 + C'\int_\Omega d^nx\ \big|\nabla u(x)\big|^2$$

as in Problem 2.1.21. Since $\overline{\Omega}$ is compact, we can cover it with finitely many of the $N(p)$'s. This gives the first inequality.

By Theorem 2.2.2 with $\ell = 1$, the operator R is bounded from $u \in H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ and hence to $L^2(\partial\Omega)$. By Lemma 2.2.13, $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. So the second inequality follows by continuity. ■

Problem 2.3.9 Let Ω be an open subset of $\mathbb{R}^n$. Prove that if $u \in H_0^2(\Omega)$, then

$$\sum_{i=1}^n \big\|\tfrac{\partial u}{\partial x_i}\big\|_{L^2(\Omega)}^2 \le \sum_{i=1}^n \|u\|_{L^2(\Omega)}\ \big\|\tfrac{\partial^2 u}{\partial x_i^2}\big\|_{L^2(\Omega)}$$

We now prove another Poincaré inequality. Proposition 2.3.7 bounded a function in terms of its gradient and its values on the boundary of the region of interest. This time, the boundary values of the function are replaced by the average value of the function on some region of interest.

Lemma 2.3.8 *Let Ω be a convex open subset of $\mathbb{R}^n$ and S a measurable subset of Ω. If $u \in C^1(\Omega)$, then, for all $x \in \Omega$,*

$$\big|u(x) - (u)_S\big| \le \tfrac{d^n}{n|S|}\int_\Omega \tfrac{1}{|x-y|^{n-1}}|\nabla u(y)|\ d^ny$$

where d is the diameter of Ω, $|S|$ is the measure of S and $(u)_S = \frac{1}{|S|}\int_S u(x)\ d^nx$ is the average value of u on S.

Proof: By the fundamental theorem of calculus,

$$u(x) - u(y) = -\int_0^{|x-y|} \tfrac{d}{dr} u(x + r\omega)\ dr \qquad \omega = \tfrac{y-x}{|y-x|}$$

for all distinct$x, y \in \Omega$. Integrating over S with respect to y,

$$|S|\big[u(x) - (u)_S\big] = -\int_S d^n y \ \int_0^{|x-y|} \tfrac{d}{dr} u(x + r\omega)\ dr$$

If χ_Ω is the characteristic function of Ω, then, since Ω is convex,

$$\begin{aligned}
\big|u(x) - (u)_S\big| &\le \tfrac{1}{|S|} \int_S d^n y \ \int_0^{|x-y|} dr\ |\nabla u(x + r\omega)| \chi_\Omega(x + r\omega) \\
&\le \tfrac{1}{|S|} \int_{S^{n-1}} d^{n-1}\omega \int_0^d d\rho\ \rho^{n-1} \int_0^\rho dr\ |\nabla u(x + r\omega)| \chi_\Omega(x + r\omega) \qquad \text{where } y = x + \rho\omega \\
&= \tfrac{1}{|S|} \int_{S^{n-1}} d^{n-1}\omega \int_0^d dr \int_r^d d\rho\ \rho^{n-1}\ |\nabla u(x + r\omega)| \chi_\Omega(x + r\omega) \\
&\le \tfrac{d^n}{n|S|} \int_{S^{n-1}} d^{n-1}\omega \int_0^d dr\ |\nabla u(x + r\omega)| \chi_\Omega(x + r\omega) \\
&\le \tfrac{d^n}{n|S|} \int_{S^{n-1}} d^{n-1}\omega \int_0^d dr\ |\nabla u(x + r\omega)| \chi_\Omega(x + r\omega) \\
&\le \tfrac{d^n}{n|S|} \int_{S^{n-1}} d^{n-1}\omega \int_0^\infty dr\, r^{n-1}\ \tfrac{1}{r^{n-1}} |\nabla u(x + r\omega)| \chi_\Omega(x + r\omega) \\
&= \tfrac{d^n}{n|S|} \int d^n y\ \tfrac{1}{|x-y|^{n-1}} |\nabla u(y)| \chi_\Omega(y) \qquad \text{where } y = x + r\omega
\end{aligned}$$

■

Lemma 2.3.9 *Let $0 < \mu \le 1$ and $1 \le p \le q \le \infty$ obey $\delta = \frac{1}{p} - \frac{1}{q} < \mu$. Then if Ω is any bounded open subset of $\mathbb{R}^n$, the operator*

$$(Vf)(x) = \int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)}} f(y)\ d^n y$$

maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ and

$$\big\|Vf\big\|_{L^q(\Omega)} \le \big(\tfrac{1-\delta}{\mu-\delta}\big)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_{L^p(\Omega)}$$

Here $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega|$ is the measure of Ω.

Proof: Choose $r \geq 1$ so that $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta$. By Hölder,

$$\begin{aligned}
\big|(Vf)(x)\big| &\leq \int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r(1-1/p)}} \tfrac{1}{|x-y|^{n(1-\mu)r/q}} |f(y)|^{p/q} \, |f(y)|^{p\delta} \, d^n y \\
&\leq \bigg[\int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} d^n y\bigg]^{1-1/p} \bigg[\int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} |f(y)|^p d^n y\bigg]^{1/q} \bigg[\int_\Omega |f(y)|^p \, d^n y\bigg]^\delta
\end{aligned}$$

so that

$$\begin{aligned}
\|Vf\|_{L^q(\Omega)}^q &\leq \bigg[\sup_x \int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} \, d^n y\bigg]^{q-\frac{q}{p}} \bigg[\int_\Omega |f(y)|^p d^n y\bigg]^{q\delta} \int_\Omega d^n x \int_\Omega d^n y \, \tfrac{1}{|x-y|^{n(1-\mu)r}} |f(y)|^p \\
&\leq \bigg[\sup_x \int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} \, d^n y\bigg]^{1+q-\frac{q}{p}} \bigg[\int_\Omega |f(y)|^p \, d^n y\bigg]^{1+q\delta} \\
&= \bigg[\sup_x \int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} \, d^n y\bigg]^{\frac{q}{r}} \bigg[\int_\Omega |f(y)|^p \, d^n y\bigg]^{\frac{q}{p}}
\end{aligned}$$

To complete the proof, it suffices to show that

$$\int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} \, d^n y \leq \big(\tfrac{1-\delta}{\mu-\delta}\big)^{r(1-\delta)} \omega_n^{r(1-\mu)} |\Omega|^{r(\mu-\delta)} = \tfrac{1-\delta}{\mu-\delta} \omega_n^{r(1-\mu)} |\Omega|^{r(\mu-\delta)} \tag{2.3.1}$$

Think of the left hand side as a function of x and Ω with x running over $\mathbb{R}^n$ and Ω running over measurable subsets of $\mathbb{R}^n$ having some prescribed measure V. Since $\frac{1}{|x-y|^{n(1-\mu)r}}$ decreases as $|x-y|$ increases, this function achieves its maximum value when Ω is the ball of volume V centred on x. Then the ball has radius $R = \big(\frac{V}{\omega_n}\big)^{1/n}$ and, since $1 - r(1-\mu) = \frac{\mu-\delta}{1-\delta} > 0$,

$$\int_\Omega \tfrac{1}{|x-y|^{n(1-\mu)r}} \, d^n y = n\omega_n \int_0^R \tfrac{\rho^{n-1}}{\rho^{n(1-\mu)r}} d\rho = n\omega_n \tfrac{R^{n(1-r+r\mu)}}{n(1-r+r\mu)} = \tfrac{1-\delta}{\mu-\delta} \omega_n \big(\tfrac{V}{\omega_n}\big)^{1-r+r\mu}$$

as desired. ■

Proposition 2.3.10 *Let Ω be a convex open subset of $\mathbb{R}^n$ with smooth boundary and S a measurable subset of Ω. If $u \in H^1(\Omega)$, then,*

$$\big\|u - (u)_S\big\|_{L^2(\Omega)} \leq \omega_n^{1-1/n} \tfrac{d^n}{|S|} |\Omega|^{1/n} \|\nabla u\|_{L^2(\Omega)}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$, d is the diameter of Ω, $|S|$ is the measure of S and $(u)_S = \frac{1}{|S|} \int_S u(x) \, d^n x$ is the average value of u on S.

Proof: By Lemma 2.2.13, $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, so it suffices to consider $u \in C^1(\overline{\Omega})$. Then, by Lemma 2.3.8, followed by Lemma 2.3.9, with $f(y) = |\nabla u(y)|$, $\mu = \frac{1}{n}$, $p = q = 2$ and $\delta = 0$,

$$\begin{aligned}\big\|u - (u)_S\big\|_{L^2(\Omega)} &\le \tfrac{d^n}{n|S|}\big\|Vf\big\|_{L^2(\Omega)} \le \tfrac{d^n}{n|S|}\big(\tfrac{1-\delta}{\mu-\delta}\big)^{1-\delta}\omega_n^{1-\mu}|\Omega|^{\mu-\delta}\|\nabla u\|_{L^2(\Omega)}\\ &= \tfrac{d^n}{|S|}\omega_n^{1-1/n}|\Omega|^{1/n}\|\nabla u\|_{L^2(\Omega)}\end{aligned}$$

as desired. ■

Problem 2.3.10 Let Ω be a convex open subset of $\mathbb{R}^n$ with smooth boundary and S a measurable subset of Ω. Prove that if $u \in H^1(\Omega)$, then,

$$\big\|u - (u)_S\big\|_{L^1(\Omega)} \le \tfrac{d^n}{|S|}\omega_n^{1-\frac{1}{n}}|\Omega|^{\frac{1}{n}}\big\|\nabla u\big\|_{L^1(\Omega)} \le \tfrac{d^n}{|S|}\omega_n^{1-\frac{1}{n}}|\Omega|^{\frac{1}{n}+\frac{1}{2}}\big\|\nabla u\big\|_{L^2(\Omega)}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$, d is the diameter of Ω, $|S|$ is the measure of S and $(u)_S = \frac{1}{|S|}\int_S u(x)\, d^n x$ is the average value of u on S.

Lemma 2.3.11 *Let Ω be a convex open subset of $\mathbb{R}^n$ and let $p > n$. There is a constant C, depending only on p, n and Ω, such that for all $u \in C^1(\Omega)$,*

$$\sup_{\substack{x,y\in\Omega\\ x\ne y}} \tfrac{|u(x)-u(y)|}{|x-y|^\varepsilon} \le C\big\|\nabla u\big\|_{L^p(\Omega)} \qquad \textit{where } \varepsilon = 1 - \tfrac{n}{p}$$

Proof: Let $x, y \in \Omega$ and set $R = |x - y|$. Choose any open ball B_R of radius R that contains both x and y. Then, by Lemma 2.3.8, with Ω and S both being $\Omega \cap B_R$, followed by Lemma 2.3.9, with $f(y) = |\nabla u(y)|$, $\mu = \frac{1}{n}$, $q = \infty$ and $\delta = \frac{1}{p}$,

$$\begin{aligned}|u(x) - (u)_{\Omega\cap B_R}| &\le \tfrac{(2R)^n}{n|\Omega\cap B_R|}\big\|Vf\big\|_{L^\infty(\Omega\cap B_R)}\\ &\le \tfrac{(2R)^n}{n|\Omega\cap B_R|}\big(\tfrac{1-\delta}{\mu-\delta}\big)^{1-\delta}\omega_n^{1-\mu}|\Omega\cap B_R|^{\frac{1}{n}-\frac{1}{p}}\|\nabla u\|_{L^2(\Omega\cap B_R)}\\ &\le CR^{1-\frac{n}{p}}\|\nabla u\|_{L^2(\Omega)}\end{aligned}$$

Similarly $|u(y) - (u)_{\Omega\cap B_R}| \le CR^{1-\frac{n}{p}}\|\nabla u\|_{L^2(\Omega)} = C|x-y|^\varepsilon\|\nabla u\|_{L^2(\Omega)}$ and the claimed bound follows by the triangle inequality. ■

The next inequalities, while having no particular connection to Sobolev spaces, provide simple, useful tools for bounding the operator norm of integral operators on L^p spaces.

Proposition 2.3.12 *Let $\langle X, \mu\rangle$ and $\langle Y, \nu\rangle$ be measure spaces and let $k(x, y)$ be a measurable function on $X \times Y$. Set, for $0 < \alpha, \beta \le \infty$,*

$$L_\alpha = \sup_{x\in X} \begin{cases} \big\{ \int_Y |k(x,y)|^\alpha \, d\nu(y) \big\}^{1/\alpha} & \text{if } 0 < \alpha < \infty \\ \sup_{y\in Y} |k(x,y)| & \text{if } \alpha = \infty \end{cases}$$

$$R_\beta = \sup_{y\in Y} \begin{cases} \big\{ \int_X |k(x,y)|^\beta \, d\mu(x) \big\}^{1/\beta} & \text{if } 0 < \beta < \infty \\ \sup_{x\in X} |k(x,y)| & \text{if } \beta = \infty \end{cases}$$

Consider the map

$$(Kf)(x) = \int_Y k(x,y) f(y) \, d\nu(y)$$

(with domain to be specified).

(a) Let $0 < \alpha, \beta < \infty$ and $1 \le p \le q < \infty$ obey $\frac{\alpha}{p} - \frac{\beta}{q} = \alpha - 1$. If $L_\alpha < \infty$ and $R_\beta < \infty$, then K is a bounded linear operator from $L^p(Y, \nu)$ to $L^q(X, \mu)$ with operator norm bounded by

$$\|K\| \le L_\alpha^{\alpha - \frac{\alpha}{p}} R_\beta^{\frac{\beta}{q}}$$

(b) Let $1 \le \alpha \le \infty$ and $\alpha' = \big(1 - \frac{1}{\alpha}\big)^{-1}$ be the dual index to α. If $L_\alpha < \infty$, then K is a bounded linear operator from $L^{\alpha'}(Y, \nu)$ to $L^\infty(X, \mu)$ with operator norm bounded by L_α.

(c) Assume that X is a σ–finite measure and $1 \le \beta \le \infty$. If $R_\beta < \infty$, then K is a bounded linear operator from $L^1(Y, \nu)$ to $L^\beta(X, \mu)$ with operator norm bounded by R_β.

Proof: We prove part (a). The proofs of parts (b) and (c) are Problem 2.3.11 below. Recall that Hölder's inequality states that

$$\int |f(y)g(y)h(y)| \, d\nu(y) \le \Big\{ \int |f(y)|^r \, d\nu(y) \Big\}^{1/r} \Big\{ \int |g(y)|^s \, d\nu(y) \Big\}^{1/s} \Big\{ \int |h(y)|^t \, d\nu(y) \Big\}^{1/t}$$

if $1 \le, r, s, t \le \infty$ and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$. If one or more of r, s, t are infinite, replace the corresponding integrals on the right hand side by essential suprema. Applying this inequality with $r = q$, $s = p' = \big(1 - \frac{1}{p}\big)^{-1}$ and $t = \big(\frac{1}{p} - \frac{1}{q}\big)^{-1}$ gives

$$\begin{aligned} |(Kf)(x)| &\le \int \big[|k(x,y)|^{\beta/q} |f(y)|^{p/q}\big] \, \big[|k(x,y)|^{\alpha - \alpha/p}\big] \, \big[|f(y)|^{p(1/p - 1/q)}\big] \, d\nu(y) \\ &\le \Big\{ \int |k(x,y)|^\beta |f(y)|^p \, d\nu(y) \Big\}^{\frac{1}{q}} \Big\{ \int |k(x,y)|^\alpha \, d\nu(y) \Big\}^{\frac{1}{s}} \Big\{ \int |f(y)|^p \, d\nu(y) \Big\}^{\frac{1}{p} - \frac{1}{q}} \\ &\le L_\alpha^{\alpha - \frac{\alpha}{p}} \Big\{ \int |k(x,y)|^\beta |f(y)|^p \, d\nu(y) \Big\}^{\frac{1}{q}} \|f\|_{L^p}^{1 - \frac{p}{q}} \end{aligned}$$

Hence

$$\begin{aligned}
\|Kf\|_{L^q}^q &\leq L_\alpha^{q(\alpha-\frac{\alpha}{p})}\|f\|_{L^p}^{q-p}\int |k(x,y)|^\beta |f(y)|^p \, d\nu(y)d\mu(x) \\
&= L_\alpha^{q(\alpha-\frac{\alpha}{p})}\|f\|_{L^p}^{q-p}\int |k(x,y)|^\beta \, d\mu(x)\int |f(y)|^p \, d\nu(y) \\
&\leq L_\alpha^{q(\alpha-\frac{\alpha}{p})} R_\beta^\beta \|f\|_{L^p}^{q-p}\|f\|_{L^p}^p
\end{aligned}$$

which is the desired inequality. ■

Problem 2.3.11 Prove parts (b) and (c) of Proposition 2.3.12.