Appendix S2A: Problem Solutions for Appendix 2.A

Problem 2.A.1 Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$.

(a) Prove that, for each $K, L \in \mathbb{N}_0$, there is a constant $D_{K,L}$ such that the Fourier coefficients

$$\hat{a}(k,n,\ell) = \tfrac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\ e^{-ik\theta}e^{-i\ell\theta'}\ a\big(e^{i\theta}, n, e^{i\theta'}\big)$$

obey

$$\big|\hat{a}(k,n,\ell)\big| \le D_{K,L}\tfrac{1}{(1+|k|)^K}(1+|n|)^m\tfrac{1}{(1+|\ell|)^L}$$

for all $k, \ell, n \in \mathbb{Z}$.

(b) Let, for each $k, \ell \in \mathbb{Z}$,

$$\hat{\mathcal{A}}(k,\ell) = \sum_{n\in\mathbb{Z}} \tfrac{1}{(2\pi)^2}\int_0^{2\pi} d\theta\ e^{-ik\theta}\int_0^{2\pi} d\theta'\ e^{in(\theta-\theta')}\ a\big(e^{i\theta}, n, e^{i\theta'}\big)e^{i\ell\theta'}$$

be the k^{th} Fourier coefficient of Ψ_a applied to $e_\ell(\theta) = e^{i\ell\theta}$. Prove that for each $K \in \mathbb{N}_0$, there is a constant D_K such that

$$\big|\hat{\mathcal{A}}(k,\ell)\big| \le D_K\tfrac{(1+|\ell|)^m}{(1+|k-\ell|)^K}$$

(c) Prove that if $f \in C^\infty(\partial\mathbb{D})$, then $\Psi_a f \in C^\infty(\partial\mathbb{D})$.

Solution. (a) If $K, L \in \mathbb{N}_0$, then, by repeated integration by parts,

$$\begin{aligned}
\big|k^K\ell^L\hat{a}(k,n,\ell)\big| &= \tfrac{1}{(2\pi)^2}\bigg|\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\ a\big(e^{i\theta}, n, e^{i\theta'}\big)\tfrac{\partial^K}{\partial\theta^K}e^{-ik\theta}\tfrac{\partial^L}{\partial\theta'^L}e^{-i\ell\theta'}\ \bigg| \\
&= \tfrac{1}{(2\pi)^2}\bigg|\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\ e^{-ik\theta}e^{-i\ell\theta'}\tfrac{\partial^K}{\partial\theta^K}\tfrac{\partial^L}{\partial\theta'^L}a\big(e^{i\theta}, n, e^{i\theta'}\big)\bigg| \\
&\le \tfrac{1}{(2\pi)^2}\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\ \big|\tfrac{\partial^K}{\partial\theta^K}\tfrac{\partial^L}{\partial\theta'^L}a\big(e^{i\theta}, n, e^{i\theta'}\big)\big| \\
&\le \tfrac{1}{(2\pi)^2}\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\ C_{0,K,L}(1+|n|)^m \\
&\le C_{0,K,L}(1+|n|)^m
\end{aligned}$$

Adding together four of these bounds, including one with K and L both replaced by zero, one with just K replaced by 0 and one with just L replaced by zero, yields

$$\big(1+|k|^K\big)\big(1+|\ell|^L\big)\big|\hat{a}(k,n,\ell)\big| \le (C_{0,0,0}+C_{0,0,L}+C_{0,K,0}+C_{0,K,L})(1+|n|)^m$$

which implies

$$\big|\hat{a}(k,n,\ell)\big| \le (C_{0,0,0}+C_{0,0,L}+C_{0,K,0}+C_{0,K,L})(1+|n|)^m\big(1+|k|^K\big)^{-1}\big(1+|\ell|^L\big)^{-1}$$

and the desired bound.

(b) We have

$$\hat{\mathcal{A}}(k,\ell) = \sum_{n\in\mathbb{Z}} \hat{a}(k-n,n,n-\ell)$$

So, by part (a)

$$\big|\hat{\mathcal{A}}(k,\ell)\big| \le D_{K,\tilde{L}} \sum_{n\in\mathbb{Z}} \tfrac{(1+|n|)^m}{(1+|k-n|)^K(1+|n-\ell|)^{\tilde{L}}}$$

By Lemma 2.3.4 (Peetre's inequality)

$$\begin{aligned}(1+n^2)^{\frac{m}{2}} &\le 2^{\frac{|m|}{2}}(1+\ell^2)^{\frac{m}{2}}(1+|n-\ell|^2)^{\frac{|m|}{2}}\\ (1+|k-n|^2)^{-\frac{K}{2}} &\le 2^{\frac{K}{2}}(1+|k-\ell|^2)^{-\frac{K}{2}}(1+|n-\ell|^2)^{\frac{K}{2}}\end{aligned}$$

so that, picking $\tilde{L} \ge |m|+K+2$,

$$\big|\hat{\mathcal{A}}(k,\ell)\big| \le \text{const}\,\tfrac{(1+|\ell|)^m}{(1+|k-\ell|)^K}\sum_{n\in\mathbb{Z}}\tfrac{1}{(1+|n-\ell|)^{\tilde{L}-|m|-K}} \le \text{const}\,\tfrac{(1+|\ell|)^m}{(1+|k-\ell|)^K}$$

as desired.

(b) The Fourier coefficient

$$\begin{aligned}\widehat{\Psi_a f}(k) &= \tfrac{1}{(2\pi)^2}\sum_{n\in\mathbb{Z}}\int_0^{2\pi}d\theta\int_0^{2\pi}d\theta'\ e^{-ik\theta}e^{in(\theta-\theta')}a\big(e^{i\theta},n,e^{i\theta'}\big)f\big(e^{i\theta'}\big)\\ &= \tfrac{1}{(2\pi)^2}\sum_{\ell,n\in\mathbb{Z}}\int_0^{2\pi}d\theta\int_0^{2\pi}d\theta'\ e^{-ik\theta}e^{in(\theta-\theta')}a\big(e^{i\theta},n,e^{i\theta'}\big)\hat{f}(\ell)e^{i\ell\theta'}\\ &= \sum_{\ell,n\in\mathbb{Z}}\hat{a}(k-n,n,n-\ell)\hat{f}(\ell)\\ &= \sum_{\ell\in\mathbb{Z}}\hat{\mathcal{A}}(k,\ell)\hat{f}(\ell)\end{aligned}$$

By the bound of part (b) and the bound $\big|\hat{f}(\ell)\big| \le \frac{C_M}{(1+|\ell|)^M}$, which is valid for all $M\in\mathbb{N}_0$ since $f\in C^\infty(\partial\mathbb{D})$, we have, for every $K,M\in\mathbb{N}_0$,

$$\big|\widehat{\Psi_a f}(k)\big| \le \sum_{\ell\in\mathbb{Z}} D_K\,\tfrac{(1+|\ell|)^m}{(1+|k-\ell|)^K}\,\tfrac{C_M}{(1+|\ell|)^M}$$

By Lemma 2.3.4 (Peetre's inequality)

$$(1+k^2)^{\frac{K}{2}} \le 2^{\frac{K}{2}}(1+|k-\ell|^2)^{\frac{K}{2}}(1+\ell^2)^{\frac{K}{2}}$$

so that

$$(1+|k|)^K\big|\widehat{\Psi_a f}(k)\big| \le \text{const}\sum_{\ell\in\mathbb{Z}} \tfrac{1}{(1+|\ell|)^{M-K-m}}$$

Choosing $M \ge m+K+2$, we have that

$$(1+|k|)^K\big|\widehat{\Psi_a f}(k)\big| \le \text{const}\sum_{\ell\in\mathbb{Z}} \tfrac{1}{(1+|\ell|)^2} < \infty \implies \big|\widehat{\Psi_a f}(k)\big| \le \tfrac{E_K}{(1+|k|)^K}$$

Thus, for every $K' \in \mathbb{N}_0$ the series $\sum_{k\in\mathbb{Z}} \widehat{\Psi_a f}(k)\frac{d^{K'}}{d\theta^{K'}}e^{ik\theta}$ converges absolutely and uniformly on $\mathbb{R}$. Hence $\Psi_a f(e^{i\theta}) = \sum_{k\in\mathbb{Z}} \widehat{\Psi_a f}(k)e^{ik\theta}$ is C^∞. ■

Problem 2.A.2 Let $s, p \in \mathbb{R}$. Prove that each of

$$(1+n^2)^{s/2} \qquad (1+|n|)^s \qquad (1+|n|)^s\log^p\big(2+|n|\big) \qquad \sin\log\big(2+|n|\big)$$

satisfy (2.A.1) for any $m \ge s$, in the first two cases, any $m > s$ in the third and any $m \ge 0$ in the last.

Solution. Let s_1, s_2, s_3, s_4, $p \in \mathbb{R}$ and define, for $\xi > 1$,

$$a(\xi) = \xi^{s_1}\,(1+\xi^2)^{s_2/2}\,(1+\xi)^{s_3}\,(2+\xi)^{s_4}\,\log^p(2+\xi)$$

Then, if $m > s_1+s_2+s_3+s_4$ (equality is allowed if $p \le 0$),

$$\big|a(\xi)\big| \le \text{const}\,(1+\xi)^m$$

for all $\xi > 1$. Furthermore

$$\begin{aligned} a'(\xi) =& s_1\xi^{s_1-1}\,(1+\xi^2)^{s_2/2}\,(1+\xi)^{s_3}\,(2+\xi)^{s_4}\,\log^p(2+\xi)\\ &+ s_2\xi^{s_1+1}\,(1+\xi^2)^{-1+s_2/2}\,(1+\xi)^{s_3}\,(2+\xi)^{s_4}\,\log^p(2+\xi)\\ &+ s_3\xi^{s_1}\,(1+\xi^2)^{s_2/2}\,(1+\xi)^{s_3-1}\,(2+\xi)^{s_4}\,\log^p(2+\xi)\\ &+ s_4\xi^{s_1}\,(1+\xi^2)^{s_2/2}\,(1+\xi)^{s_3}\,(2+\xi)^{s_4-1}\,\log^p(2+\xi)\\ &+ p\xi^{s_1}\,(1+\xi^2)^{s_2/2}\,(1+\xi)^{s_3}\,(2+\xi)^{s_4-1}\,\log^{p-1}(2+\xi) \end{aligned}$$

is a finite linear combination of functions of the same form as a, but with $s_1+s_2+s_3+s_4$ reduced by exactly one. Thus, by Remark 2.A.4 and evenness, $a(|n|)$ satisfies (2.A.1). That takes care of all of the given cases except for $\sin\log(2+|n|)$. It is dealt with in the same way, but using

$$a(\xi) = (2+\xi)^s\,\sin\log(2+\xi) + (2+\xi)^s\,\cos\log(2+\xi)$$

and $m \ge s$. ■

Problem 2.A.3 Prove that the linear operator $f \in C^\infty(\partial\mathbb{D}) \mapsto f(0)$, where $f(0)$ is viewed as a constant function on $\partial\mathbb{D}$, is not in $\Psi^m(\partial\mathbb{D})$ for any $m \in \mathbb{R}$.

Solution. Use δ to denote the specified map. Then applying δ to the function $e_\ell(\theta) = e^{i\ell\theta}$ gives the constant function 1 and the 0^{th} Fourier coefficient of δ applied to e_ℓ is 1 for all ℓ. This does not converge to zero as $\ell \to \infty$. So the bound of part (b) of Problem 2.A.1 cannot be satisfied for any $m \in \mathbb{R}$. ■

Problem 2.A.4 Let $a \in S^m(\partial\mathbb{D})$ and $b \in S^{m'}(\partial\mathbb{D})$ and set

$$c\big(e^{i\theta}, n, e^{i\theta'}\big) = \sum_{\ell\in\mathbb{Z}} a\big(e^{i\theta}, n\big)\hat{b}(\ell, n-\ell)e^{i\ell\theta'} \quad \text{where } \hat{b}(\ell, n) = \tfrac{1}{2\pi}\int_0^{2\pi} d\theta\ e^{-i\ell\theta}\ b\big(e^{i\theta}, n\big)$$

Prove that $c \in A^{m+m'}(\partial\mathbb{D})$ and

$$\Psi_a\Psi_b = \Psi_c$$

Solution. We need to bound

$$\big|D_n^\alpha \tfrac{\partial^\beta}{\partial\theta^\beta}\tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}} c\big(e^{i\theta}, n, e^{i\theta'}\big)\big| \le \sum_{\ell\in\mathbb{Z}} \big|\tfrac{\partial^\beta}{\partial\theta^\beta} D_n^\alpha\big[a\big(e^{i\theta}, n\big)\hat{b}(\ell, n-\ell)\big]\ell^{\beta'}\big|$$

Since $b \in S^{m'}(\partial\mathbb{D})$, for each $\gamma, L \in \mathbb{N}_0$, there is a constant $D_{\gamma,L}$ such that

$$\big|D_n^\gamma\hat{b}(\ell, n)\big| = \big|\widehat{D_n^\gamma b}(\ell, n) \le D_{\gamma,L}\tfrac{(1+|n|)^{m'-\gamma}}{(1+|\ell|)^L}$$

for all $\ell, n \in \mathbb{Z}$. Using this, the bound implicit in $a \in S^m(\partial\mathbb{D})$ and the product rule

$$D_n a(n)b(n) = a(n+1)b(n+1) - a(n)b(n) = a(n+1)D_n b(n) + b(n)D_n a(n)$$

gives

$$\begin{aligned}
\big|D_n^\alpha \tfrac{\partial^\beta}{\partial\theta^\beta}\tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}} c\big(e^{i\theta}, n, e^{i\theta'}\big)\big| &\le \text{const} \sum_{\ell\in\mathbb{Z}}\sum_{\gamma=0}^{\alpha}(1+|n|)^{m-\alpha+\gamma}\tfrac{(1+|n-\ell|)^{m'-\gamma}}{(1+|\ell|)^L}|\ell|^{\beta'} \\
&\le \text{const} \sum_{\ell\in\mathbb{Z}}\sum_{\gamma=0}^{\alpha}(1+|n|)^{m+m'-\alpha}\tfrac{1}{(1+|\ell|)^{L-|m'-\gamma|-\beta'}} \\
&\le \text{const}\,(1+|n|)^{m+m'-\alpha}
\end{aligned}$$

if $L > |m'| + \alpha + \beta' + 1$, as desired. The computation showing that $\Psi_a \Psi_b = \Psi_c$ is

$$\begin{aligned}
(\Psi_a \Psi_b f)\big(e^{i\theta}\big) &= \tfrac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^{2\pi} e^{in(\theta-\theta'')} a\big(e^{i\theta}, n\big)(\Psi_b f)\big(e^{i\theta''}\big)\ d\theta'' \\
&= \tfrac{1}{(2\pi)^2} \sum_{\ell,n\in\mathbb{Z}} \int_0^{2\pi} d\theta'' \int_0^{2\pi} d\theta'\ e^{in(\theta-\theta'')} e^{i\ell(\theta''-\theta')} a\big(e^{i\theta}, n\big) b\big(e^{i\theta''}, \ell\big) f\big(e^{i\theta'}\big) \\
&= \tfrac{1}{2\pi} \sum_{\ell,n\in\mathbb{Z}} \int_0^{2\pi} d\theta'\ e^{in\theta} e^{-i\ell\theta'} a\big(e^{i\theta}, n\big) \hat{b}\big(n-\ell, \ell\big) f\big(e^{i\theta'}\big) \\
&= \tfrac{1}{2\pi} \sum_{\ell,n\in\mathbb{Z}} \int_0^{2\pi} d\theta'\ e^{in(\theta-\theta')} e^{i(n-\ell)\theta'} a\big(e^{i\theta}, n\big) \hat{b}\big(n-\ell, \ell\big) f\big(e^{i\theta'}\big) \\
&= \tfrac{1}{2\pi} \sum_{\ell',n\in\mathbb{Z}} \int_0^{2\pi} d\theta'\ e^{in(\theta-\theta')} e^{i\ell'\theta'} a\big(e^{i\theta}, n\big) \hat{b}\big(\ell', n-\ell'\big) f\big(e^{i\theta'}\big) \text{ where } \ell = n - \ell' \\
&= (\Psi_c f)\big(e^{i\theta}\big)
\end{aligned}$$

■

Problem 2.A.5 Let $m, s, s' \in \mathbb{R}$ and $p \in \mathbb{N}_0$ obey $p \geq |s-m|$ and $s' > s - m$. Let $\Sigma = \big\{\ \{\sigma_n\}_{n\in\mathbb{Z}}\ \big|\ \sigma_n \in \mathbb{C} \text{ for all } n \in \mathbb{Z}\ \big\}$ denote the set of two–sided sequences and

$$\Sigma_m = \Big\{\ \{\sigma_n\}_{n\in\mathbb{Z}} \in \Sigma\ \Big|\ \sup_{n\in\mathbb{Z}} \tfrac{|\sigma_n|}{(1+n^2)^{m/2}} < \infty\ \Big\}$$

Given any $\sigma = \{\sigma_n\}_{n\in\mathbb{Z}} \in \Sigma_m$,

$$f \mapsto \sum_{n\in\mathbb{Z}} \sigma_n \hat{f}(n)\ e^{in\theta}$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of f, is a bounded operator from $H^s(\partial\mathbb{D})$ to $H^{s-m}(\partial\mathbb{D})$. If $\varphi \in C^p(\partial\mathbb{D})$, the space of functions on $\partial\mathbb{D}$ whose derivatives of order up to p are continuous, then

$$\Psi_{\varphi\sigma} f = \varphi(e^{i\theta}) \sum_{n\in\mathbb{Z}} \sigma_n \hat{f}(n)\ e^{in\theta}$$

is also a bounded operator from $H^s(\partial\mathbb{D})$ to $H^{s-m}(\partial\mathbb{D})$.

Let $\sigma^{(1)}, \cdots \sigma^{(l)} \in \Sigma_m$ be asymptotically independent in the sense that, given any $\alpha_1, \cdots, \alpha_l \in \mathbb{C}$, not all zero, there is a sequence $\{n_j\}_{j\in\mathbb{N}}$ obeying $\lim_{j\to\infty} |n_j| = \infty$ for which

$$\lim_{j\to\infty} |n_j|^{s'-s} \big|\alpha_1 \sigma^{(1)}_{n_j} + \cdots + \alpha_l \sigma^{(l)}_{n_j}\big| = \infty$$

Prove that if $\varphi_1 \cdots, \varphi_l \in C^p(\partial\mathbb{D})$ and if

$$\sum_{k=1}^{l} \Psi_{\varphi_k \sigma^{(k)}} : H^s(\partial\mathbb{D}) \to H^{s'}(\partial\mathbb{D})$$

is a bounded operator, then $\varphi_1 \cdots, \varphi_l$ are all identically zero.

Proof: We may choose the norm on $H^s(\partial\mathbb{D})$, $s \in \mathbb{R}$, to be given by

$$|f|_s = \Big[\sum_{n\in\mathbb{Z}} \big(1+n^2\big)^s |\hat{f}_n|^2 \Big]^{1/2}$$

for $f \in C^\infty(\partial\mathbb{D})$. If $\varphi_1 \cdots, \varphi_l$ are not all identically zero, then some Fourier coefficient of at least one of them must be nonzero. Let ℓ be the index of such a Fourier coefficient. For reach $n \in \mathbb{Z}$,

$$f_n(\theta) = \big(1+n^2\big)^{-s/2} e^{in\theta} \qquad g_{n,\ell}(\theta) = \big(1+(n+\ell)^2\big)^{s'/2} e^{i(-n-\ell)\theta}$$

are unit vectors in $H^s(\partial\mathbb{D})$ and $H^{-s'}(\partial\mathbb{D})$ respectively. As, by hypothesis, the operator $\sum_{k=1}^{l} \Psi_{\varphi_k \sigma^{(k)}}$ is bounded from $H^s(\partial\mathbb{D})$ to $H^{s'}(\partial\mathbb{D})$,

$$\begin{aligned}\tfrac{1}{2\pi}\sum_{k=1}^{l}\int_0^{2\pi} g_{n,\ell}(\theta)\big(\Psi_{\varphi_k\sigma^{(k)}} f_n\big)(\theta)\, d\theta &= \tfrac{(1+(n+\ell)^2)^{s'/2}}{(1+n^2)^{s/2}} \tfrac{1}{2\pi}\sum_{k=1}^{l}\int_0^{2\pi} e^{i(-n-\ell)\theta}\varphi_k(\theta)\sigma_n^{(k)} e^{in\theta}\, d\theta \\ &= \tfrac{(1+(n+\ell)^2)^{s'/2}}{(1+n^2)^{s/2}} \sum_{k=1}^{l} \hat{\varphi}_{k,\ell}\sigma_n^{(k)}\end{aligned}$$

must be uniformly bounded in n. By the independence hypothesis, there is a sequence $\{n_j\}_{j\in\mathbb{N}}$ obeying $\lim_{j\to\infty} |n_j| = \infty$ for which

$$\lim_{j\to\infty} |n_j|^{s'-s} \Big| \sum_{k=1}^{m} \hat{\varphi}_{k,\ell}\sigma_{n_j}^{(k)} \Big| = \infty$$

As

$$\lim_{j\to\infty} |n_j|^{s-s'} \frac{\big(1+(n_j+\ell)^2\big)^{s'/2}}{\big(1+n_j^2\big)^{s/2}} = \lim_{j\to\infty} \frac{\big(\frac{1}{n_j^2} + (1+\frac{\ell}{n_j})^2\big)^{s'/2}}{\big(\frac{1}{n_j^2}+1\big)^{s/2}} = 1$$

we have the contradiction

$$\liminf_{n\to\infty} \Big| \tfrac{(1+(n+\ell)^2)^{s'/2}}{(1+n^2)^{s/2}} \sum_{k=1}^{m} \hat{\varphi}_{k,\ell}\sigma_n^{(k)} \Big| \ge \lim_{j\to\infty} |n_j|^{s'-s} \Big| \sum_{k=1}^{m} \hat{\varphi}_{k,\ell}\sigma_{n_j}^{(k)} \Big| = \infty$$

■

Problem 2.A.6 Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$. Set

$$p(e^{i\theta}, n) = e^{-in\theta}\Psi_a(e_n)$$

where $e_n(\theta) = e^{in\theta}$. Prove that $p \in S^m(\partial\mathbb{D})$ and $\Psi_a = \Psi_p$.

Solution. In the notation of Problem 2.A.1

$$\begin{aligned} p(e^{i\theta}, n) &= \sum_{\ell\in\mathbb{Z}} \tfrac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} e^{i\ell(\theta-\theta')} a\big(e^{i\theta}, \ell, e^{i\theta'}\big) e^{in\theta'}\ d\theta' = \sum_{k,\ell\in\mathbb{Z}} e^{i(k+\ell-n)\theta}\hat{a}(k, \ell, \ell-n) \\ &= \sum_{k,\ell\in\mathbb{Z}} e^{i(k+\ell)\theta}\hat{a}(k, \ell+n, \ell) \end{aligned}$$

so that

$$(D_n^\alpha p)(e^{i\theta}, n) = \sum_{k,\ell\in\mathbb{Z}} e^{i(k+\ell)\theta}\widehat{D_n^\alpha a}(k, \ell+n, \ell)$$

Hence, by Problem 2.A.1, with a replaced by $D_n^\alpha a$, followed by Lemma 2.3.4 (Peetre's inequality)

$$\begin{aligned} \big|D_n^\alpha \tfrac{\partial^\beta}{\partial\theta^\beta} p\big(e^{i\theta}, n\big)\big| &\le \sum_{k,\ell\in\mathbb{Z}} |k+\ell|^\beta\ \big|\widehat{D_n^\alpha a}(k, \ell+n, \ell)\big| \\ &\le \text{const} \sum_{k,\ell\in\mathbb{Z}} |k+\ell|^\beta\ \tfrac{1}{(1+|k|)^K}(1+|\ell+n|)^{m-\alpha}\tfrac{1}{(1+|\ell|)^L} \\ &\le \text{const} \sum_{k,\ell\in\mathbb{Z}} \tfrac{1}{(1+|k|)^{K-\beta}}(1+|n|)^{m-\alpha}\tfrac{1}{(1+|\ell|)^{L-\beta-|m-\alpha|}} \\ &\le \text{const}\,(1+|n|)^{m-\alpha} \end{aligned}$$

if $K > \beta+1$ and $L > \beta+|m-\alpha|+1$. Thus $p \in S^m(\partial\mathbb{D})$. By construction, $\Psi_p(e_n) = e^{in\theta}p\big(e^{i\theta}, n\big) = \Psi_a(e_n)$ for all $n \in \mathbb{Z}$. By linearity Ψ_p and Ψ_a agree when applied to any finite linear combination of e_n's. Since such finite linear combinations are dense in $H^s(\partial\mathbb{D})$, for any $s \in \mathbb{R}$, and Ψ_p and Ψ_a are both continuous on $H^s(\partial\mathbb{D})$, Ψ_p and Ψ_a agree on $H^s(\partial\mathbb{D})$ and in particular on $C^\infty(\partial\mathbb{D})$. ■

Problem 2.A.7 Let $a(x,\xi,x')$ obey (2.A.8). Let $\mathcal{K}$ be any compact subset of $\mathbb{R}$ and $\chi \in C_0^\infty(\mathbb{R})$ be identically one on $\mathcal{K}$.

(a) Prove that, for each $\alpha,\beta,\gamma,\gamma' \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\gamma,\gamma'}$ such that the partial Fourier transform

$$\tilde{a}_\chi(x,\xi,k) = \int_{\mathbb{R}} dx'\ e^{-ikx'} a(x,\xi,x')\chi(x')$$

obeys

$$\left|\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}\tfrac{\partial^\gamma}{\partial k^\gamma}\tilde a_\chi(x,\xi,k)\right| \le C_{\alpha,\beta,\gamma,\gamma'}\tfrac{(1+|\xi|)^{m-\alpha}}{(1+|k|)^{\gamma'}}$$

(b) Let $\alpha,\beta\in\mathbb{N}_0$ and $\gamma\in\mathbb{R}$ obey $\gamma>m+\beta+1$. Prove that if $f\in C_0(\mathcal{K})$, then

$$\sup_{x\in\mathbb{R}}\left|x^\alpha\tfrac{d^\beta}{dx^\beta}\mathfrak{P}_a f(x)\right| \le \text{const}\,\|f\|_{H^\gamma(\mathbb{R})}$$

Solution. (a) By integration by parts

$$\begin{aligned}
k^{\gamma'}\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}\tfrac{\partial^\gamma}{\partial k^\gamma}\tilde a_\chi(x,\xi,k) &= \int_{\mathbb{R}} dx'\ k^{\gamma'}e^{-ikx'}(-ix')^\gamma\chi(x')\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}a(x,\xi,x')\\
&= \int_{\mathbb{R}} dx'\ (-ix')^\gamma\chi(x')\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}a(x,\xi,x')i^{\gamma'}\tfrac{\partial^{\gamma'}}{\partial x'^{\gamma'}}e^{-ikx'}\\
&= (-i)^{\gamma+\gamma'}\int_{\mathbb{R}} dx'\ e^{-ikx'}\tfrac{\partial^{\gamma'}}{\partial x'^{\gamma'}}\big[(x')^\gamma\chi(x')\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}a(x,\xi,x')\big]
\end{aligned}$$

Since χ has compact support there is an $R>0$ such that $\chi(x')$ vanishes for all $|x'|>R$. Hence, by (2.A.8),

$$\left|k^{\gamma'}\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}\tfrac{\partial^\gamma}{\partial k^\gamma}\tilde a_\chi(x,\xi,k)\right| \le \text{const}\,(1+|\xi|)^{m-\alpha}\int_{|x'|\le R}dx'$$

as desired.

(b) If $f\in C_0(\mathcal{K})$, then

$$\begin{aligned}
\mathfrak{P}_a f(x) &= \tfrac{1}{2\pi}\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}dx'\ e^{i\xi(x-x')}a(x,\xi,x')\chi(x')f(x')\\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}dk\ e^{i\xi x}\tilde a_\chi(x,\xi,\xi-k)\hat f(k)
\end{aligned}$$

so that

$$\begin{aligned}
x^\alpha\tfrac{d^\beta}{dx^\beta}\mathfrak{P}_a f(x) &= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}dk\ x^\alpha e^{i\xi x}\big(i\xi+\tfrac{d}{dx}\big)^\beta\tilde a_\chi(x,\xi,\xi-k)\hat f(k)\\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}dk\ \big(i\xi+\tfrac{d}{dx}\big)^\beta\tilde a_\chi(x,\xi,\xi-k)\hat f(k)\big(-i\tfrac{\partial}{\partial\xi}\big)^\alpha e^{i\xi x}\\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}dk\ e^{i\xi x}\big(i\tfrac{\partial}{\partial\xi}\big)^\alpha\big(i\xi+\tfrac{d}{dx}\big)^\beta\tilde a_\chi(x,\xi,\xi-k)\hat f(k)
\end{aligned}$$

Hence, by part (a) and Lemma 2.3.4 (Peetre's inequality),

$$\begin{aligned}\left|x^\alpha \tfrac{d^\beta}{dx^\beta}\mathfrak{P}_a f(x)\right| &\le \text{const} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dk\ \tfrac{(1+|\xi|)^{m+\beta}}{(1+|\xi-k|)^{\gamma'}}|\hat{f}(k)| \\ &= \text{const} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dk\ \tfrac{(1+|\xi+k|)^{m+\beta}}{(1+|\xi|)^{\gamma'}}|\hat{f}(k)| \\ &\le \text{const} \int_{\mathbb{R}} d\xi\ \tfrac{1}{(1+|\xi|)^{\gamma'-|m+\beta|}} \int_{\mathbb{R}} dk\ (1+k^2)^{\frac{m+\beta}{2}}|\hat{f}(k)| \\ &\le \text{const} \int_{\mathbb{R}} dk\ \tfrac{1}{(1+k^2)^{\frac{\gamma-m-\beta}{2}}}(1+k^2)^{\frac{\gamma}{2}}|\hat{f}(k)|\end{aligned}$$

if we choose $\gamma' > |m+\beta|+1$. The desired bound now follows by Cauchy–Schwarz, provided that $\gamma - m - \beta > 1$ so that $(1+k^2)^{-(\gamma-m-\beta)/2} \in L^2(\mathbb{R})$. ■

Problem 2.A.8 Let $m, s \in \mathbb{R}$ and $b \in C^\infty(\mathbb{R}^3)$. Assume that, for each $\beta, \beta' \in \mathbb{N}_0$, there is a constant $D_{\beta,\beta'}$ such that

$$\int \left|\tfrac{\partial^\beta}{\partial x^\beta}\tfrac{\partial^{\beta'}}{\partial x'^{\beta'}} b(x,\xi,x')\right|\ dx dx' \le D_{\beta,\beta'}(1+|\xi|)^m$$

Prove that there is a constant C such that if $f \in C_0^\infty(\mathbb{R})$, then

$$\|\mathfrak{P}_b f\|_{H^{s-m}(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}$$

Proof: The Fourier transform

$$\hat{b}(p,\xi,k) = \iint dx dx'\ e^{-ipx} e^{-ikx'} b(x,\xi,x')$$

obeys, for all $P, K \in \mathbb{N}_0$,

$$\left|\hat{b}(p,\xi,k)\right| \le C_{P,K}\tfrac{(1+|\xi|)^m}{(1+|p|)^P(1+|k|)^K} \tag{S2A.1}$$

We may express the Fourier transform of $\mathfrak{P}_b f(x)$ in terms of $\hat{b}(p,\xi,k)$ by

$$\widehat{\mathfrak{P}_b f}(p) = \tfrac{1}{(2\pi)^2}\iint d\xi dk\ \hat{b}(p-\xi,\xi,\xi-k)\hat{f}(k) = \tfrac{1}{(2\pi)^2}\int dk\ B(p,k)\hat{f}(k)$$

where

$$B(p,k) = \int d\xi\ \hat{b}(p-\xi,\xi,\xi-k)$$

Since

$$\|\mathfrak{P}_b f\|^2_{H^{s-m}(\mathbb{R})} = \tfrac{1}{2\pi}\int dp\ \big(1+p^2\big)^{s-m}\big|\widehat{\mathfrak{P}_a f}(p)\big|^2 \quad \|f\|^2_{H^s(\mathbb{R})} = \tfrac{1}{2\pi}\int dk\ \big(1+k^2\big)^{s}\big|\hat f(k)\big|^2$$

it suffices, by Proposition 2.3.12, with $\alpha=\beta=1$, $p=q=2$ and $\mu=\nu$ being Lebesgue measure on $X=Y=\mathbb{R}$, to prove that

$$\sup_p \int dk\ \big(1+p^2\big)^{\frac{s-m}{2}}|B(p,k)|\big(1+k^2\big)^{-\frac{s}{2}}\ ,\ \sup_k \int dp\ \big(1+p^2\big)^{\frac{s-m}{2}}|B(p,k)|\big(1+k^2\big)^{-\frac{s}{2}}\ <\infty$$

These bounds follow from

$$\big(1+p^2\big)^{\frac{s-m}{2}}|\hat b(p-\xi,\xi,\xi-k)|\big(1+k^2\big)^{-\frac{s}{2}} \le \tfrac{\text{const}}{(1+|p-\xi|)^2(1+|\xi-k|)^2}$$

which, in turn, follows easily from (S2A.1), with $P=|s-m|+2$, $K=|s|+2$, and the consequences

$$\begin{aligned}\big(1+p^2\big)^{\frac{s-m}{2}} &\le 2^{\frac{|s-m|}{2}}\big(1+|p-\xi|^2\big)^{\frac{|s-m|}{2}}\big(1+\xi^2\big)^{\frac{s-m}{2}}\\ \big(1+k^2\big)^{-\frac{s}{2}} &\le 2^{\frac{|s|}{2}}\big(1+|\xi-k|^2\big)^{\frac{|s|}{2}}\big(1+\xi^2\big)^{-\frac{s}{2}}\end{aligned}$$

of Lemma 2.3.4 (Peetre's inequality). ■

Problem 2.A.9 Prove Lemma 2.A.14.

Proof: Since

$$\begin{aligned}\mathfrak{P}_a f(x) &= \tfrac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ e^{i\xi(x-x')}(x-x')^{m'} b(x,\xi,x')f(x')\\ &= \tfrac{i^{m'}}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ e^{i\xi(x-x')}\tfrac{\partial^{m'}}{\partial\xi^{m'}} b(x,\xi,x')f(x')\end{aligned} \tag{S2A.2}$$

and $\frac{\partial^{m'}}{\partial\xi^{m'}}b(x,\xi,x')\in A^{m-m'}(\mathbb{R})$, parts (a) and (b) follow from parts (a) and (b) of Proposition 2.A.13, respectively. ■

Problem 2.A.10 Prove Proposition 2.A.15.

Proof: Taylor expanding in the third argument about $x'=x$,

$$\begin{aligned}a(x,\xi,x') &= \sum_{k=0}^{\ell} \tfrac{1}{k!}\tfrac{\partial^k}{\partial x'^k}a(x,\xi,x')\big|_{x'=x}\big(x'-x\big)^k + \tfrac{1}{\ell!}\int_x^{x'}\big(x'-t\big)^{\ell}\tfrac{\partial^{\ell+1}a}{\partial x'^{\ell+1}}(x,\xi,t)\ dt\\ &= \sum_{k=0}^{\ell} \tfrac{1}{k!}\tfrac{\partial^k}{\partial x'^k}a(x,\xi,x')\big|_{x'=x}\big(x'-x\big)^k\\ &\qquad + \tfrac{1}{\ell!}(x'-x)^{\ell+1}\int_0^1\big(1-u\big)^{\ell}\tfrac{\partial^{\ell+1}a}{\partial x'^{\ell+1}}\big(x,\xi,ux'+(1-u)x\big)\ du\end{aligned}$$

By (S2A.2), the pseudodifferential operator associated with the k^{th} term is exactly $\mathfrak{P}_{a_k}$. The bounds on R_ℓ are provided by Lemma 2.A.14. ■

Problem 2.A.11 Let Φ be a diffeomorphism of $\mathbb{R}$ and $J(x,x') = \int_0^1 {\Phi^{-1}}'\big(tx+(1-t)x'\big)\,dt$. Prove that, for each $k \in \mathbb{N}_0$,

$$\tfrac{\partial^k}{\partial x'^k} J(x,x')\big|_{x'=x} = \tfrac{1}{k+1}\tfrac{\partial^{k+1}}{\partial x^{k+1}}\Phi^{-1}(x)$$

Solution.

$$\begin{aligned}\tfrac{\partial^k}{\partial x'^k} J(x,x')\big|_{x'=x} &= \int_0^1 \tfrac{\partial^k}{\partial x'^k}{\Phi^{-1}}'\big(tx+(1-t)x'\big)\,dt\Big|_{x'=x}\\ &= \int_0^1 \tfrac{\partial^{k+1}}{\partial x^{k+1}}\Phi^{-1}(x)\,(1-t)^k\,dt\\ &= -\tfrac{\partial^{k+1}}{\partial x^{k+1}}\Phi^{-1}(x)\,\tfrac{(1-t)^{k+1}}{k+1}\Big|_{t=0}^{t=1}\\ &= \tfrac{1}{k+1}\tfrac{\partial^{k+1}}{\partial x^{k+1}}\Phi^{-1}(x)\end{aligned}$$

■

Problem 2.A.12 Let $m \in \mathbb{N}_0$, $q \in C^\infty(\mathbb{R})$ and $a(x,\xi) = q(x)\xi^m \in S^m(\mathbb{R})$. Then

$$\begin{aligned}\mathfrak{P}_a^\Phi f(x) &= \Big(\mathfrak{P}_a(f\circ\Phi)\Big)\big(\Phi^{-1}(x)\big) = (-i)^m q\big(\Phi^{-1}(x)\big)\tfrac{d^m}{dy^m}f\big(\Phi(y)\big)\big|_{y=\Phi^{-1}(x)}\\ &= (-i)^m q\big(\Phi^{-1}(x)\big)\begin{cases} f(x) & \text{if } m=0\\ \Phi'\big(\Phi^{-1}(x)\big)f'(x) & \text{if } m=1\\ \Phi'\big(\Phi^{-1}(x)\big)^2 f''(x) + \Phi''\big(\Phi^{-1}(x)\big)f'(x) & \text{if } m=2\end{cases}\end{aligned}$$

by the chain and product rules. Verify that Proposition 2.A.17 and Proposition 2.A.15 give the same formulae for $m = 0,1,2$, up to an error R that is a smoothing map. That "R is smoothing" means that for each compact $\mathcal{K} \subset \mathbb{R}$ and each $s, s' \in \mathbb{R}$ there is a constant C such that

$$\|Rf\|_{H^{s'}(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}$$

for all $f \in C_0^\infty(\mathcal{K})$.

Solution. By Proposition 2.A.17, $\mathfrak{P}_a^\Phi = \mathfrak{P}_b$ where

$$\begin{aligned}b(x,\xi,x') &= \tfrac{{\Phi^{-1}}'(x')}{J(x,x')}\,a\big(\Phi^{-1}(x), \tfrac{\xi}{J(x,x')}, \Phi^{-1}(x')\big)\\ &= \tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}}\,q\big(\Phi^{-1}(x)\big)\xi^m\end{aligned}$$

By Proposition 2.A.15, for all $\ell \ge m$,

$$\mathfrak{P}_b = \sum_{k=0}^{\ell} \mathfrak{P}_{b_k} + R_\ell$$

where

$$b_k(x,\xi) = \tfrac{i^k}{k!}\tfrac{\partial^k}{\partial \xi^k}\tfrac{\partial^k}{\partial x'^k} b(x,\xi,x')\big|_{x'=x} = i^k\tbinom{m}{k} q\big(\Phi^{-1}(x)\big)\xi^{m-k}\tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}}\Big|_{x'=x}$$

if $k \le m$ and $b_k \equiv 0$ if $k > m$. Thus R_ℓ is independent of ℓ for $\ell > m$ and is smoothing. For $0 \le k \le m$

$$\mathfrak{P}_{b_k} = (-i)^{m-k}(-i)^k\tbinom{m}{k} q\big(\Phi^{-1}(x)\big)\tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}}\Big|_{x'=x}\tfrac{d^{m-k}}{dx^{m-k}}$$

Denote $\mathcal{J}(x) = {\Phi^{-1}}'(x)$. Then

$$\begin{aligned}
k=0 &\implies \tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}} = \tfrac{\mathcal{J}(x')}{J(x,x')^{m+1}}\\
k=1 &\implies \tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}} = \tfrac{\mathcal{J}'(x')}{J(x,x')^{m+1}} - (m+1)\tfrac{\mathcal{J}(x')J_2(x,x')}{J(x,x')^{m+2}}\\
k=2 &\implies \tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}} = \tfrac{\mathcal{J}''(x')}{J(x,x')^{m+1}} - 2(m+1)\tfrac{\mathcal{J}'(x')J_2(x,x')}{J(x,x')^{m+2}} - (m+1)\tfrac{\mathcal{J}(x')J_{22}(x,x')}{J(x,x')^{m+2}}\\
&\qquad\qquad + (m+1)(m+2)\tfrac{\mathcal{J}(x')J_2^2(x,x')}{J(x,x')^{m+3}}
\end{aligned}$$

By Problem 2.A.11,

$$\begin{aligned}
k=0 &\implies \tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}}\Big|_{x'=x} = \tfrac{1}{\mathcal{J}(x)^m}\\
k=1 &\implies \tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}}\Big|_{x'=x} = \tfrac{\mathcal{J}'(x)}{\mathcal{J}(x)^{m+1}} - \tfrac{1}{2}(m+1)\tfrac{\mathcal{J}(x)\mathcal{J}'(x)}{\mathcal{J}(x)^{m+2}} = \tfrac{1-m}{2}\tfrac{\mathcal{J}'(x)}{\mathcal{J}(x)^{m+1}}\\
k=2 &\implies \tfrac{\partial^k}{\partial x'^k}\tfrac{{\Phi^{-1}}'(x')}{J(x,x')^{m+1}}\Big|_{x'=x} = \tfrac{\mathcal{J}''(x)}{\mathcal{J}(x)^{m+1}} - (m+1)\tfrac{\mathcal{J}'(x)^2}{\mathcal{J}(x)^{m+2}} - \tfrac{m+1}{3}\tfrac{\mathcal{J}(x)\mathcal{J}''(x)}{\mathcal{J}(x)^{m+2}}\\
&\qquad\qquad + \tfrac{1}{4}(m+1)(m+2)\tfrac{\mathcal{J}(x)\mathcal{J}'(x)^2}{\mathcal{J}(x)^{m+3}}
\end{aligned}$$

For $m=0$,

$$\sum_{k=0}^{m} \mathfrak{P}_{b_k} = \mathfrak{P}_{b_0} = q\big(\Phi^{-1}(x)\big)$$

For $m=1$,

$$\begin{aligned}
\sum_{k=0}^{1} \mathfrak{P}_{b_k} &= (-i)\, q\big(\Phi^{-1}(x)\big)\tfrac{1}{\mathcal{J}(x)}\tfrac{d}{dx} + (-i) q\big(\Phi^{-1}(x)\big)\big[\tfrac{1-m}{2}\tfrac{\mathcal{J}'(x)}{\mathcal{J}(x)^{m+1}}\big]\\
&= (-i)\, q\big(\Phi^{-1}(x)\big)\Phi'\big(\Phi^{-1}(x)\big)\tfrac{d}{dx}
\end{aligned}$$

For $m = 2$,

$$\begin{aligned}\sum_{k=0}^{2} \mathfrak{P}_{b_k} &= (-i)^2 q\big(\Phi^{-1}(x)\big) \tfrac{1}{\mathcal{J}(x)^2} \tfrac{d^2}{dx^2} + (-i)^2 2q\big(\Phi^{-1}(x)\big) \big[\tfrac{1-2}{2} \tfrac{\mathcal{J}'(x)}{\mathcal{J}(x)^3}\big] \tfrac{d}{dx} \\ &\qquad + (-i)^2 q\big(\Phi^{-1}(x)\big) \big[\tfrac{\mathcal{J}''(x)}{\mathcal{J}(x)^3} - 3\tfrac{\mathcal{J}'(x)^2}{\mathcal{J}(x)^4} - \tfrac{\mathcal{J}(x)\mathcal{J}''(x)}{\mathcal{J}(x)^4} + 3\tfrac{\mathcal{J}(x)\mathcal{J}'(x)^2}{\mathcal{J}(x)^5}\big] \\ &= (-i)^2\, q\big(\Phi^{-1}(x)\big) \Phi'\big(\Phi^{-1}(x)\big)^2 \tfrac{d^2}{dx^2} - (-i)^2 q\big(\Phi^{-1}(x)\big) \tfrac{\mathcal{J}'(x)}{\mathcal{J}(x)^3} \tfrac{d}{dx}\end{aligned}$$

Since $\mathcal{J}(x) = \Phi'\big(\Phi^{-1}(x)\big)^{-1}$

$$\tfrac{\mathcal{J}'(x)}{\mathcal{J}(x)^3} = \Phi'\big(\Phi^{-1}(x)\big)^3\ (-1)\Phi'\big(\Phi^{-1}(x)\big)^{-2} \Phi''\big(\Phi^{-1}(x)\big) {\Phi^{-1}}'(x) = -\Phi''\big(\Phi^{-1}(x)\big)$$

as desired. ■