Appendix 2.A: Pseudodifferential Operators

Pseudodifferential operators (ΨDO's) are generalizations of differential operators

$$\begin{aligned} f(x) \mapsto \sum_\alpha a_\alpha(x) \tfrac{\partial^\alpha f}{\partial x^\alpha}(x) &= \sum_\alpha \int e^{i\xi\cdot x} a_\alpha(x) \big(i\xi\big)^\alpha \hat{f}(\xi) \ \tfrac{d^n \xi}{(2\pi)^n} \\ &= \sum_\alpha \int \tfrac{d^n \xi}{(2\pi)^n} \int d^n x' \ e^{i\xi\cdot(x-x')} a_\alpha(x) \big(i\xi\big)^\alpha f(x') \end{aligned}$$

in which the "symbol" $\sum_\alpha a_\alpha(x)\big(i\xi\big)^\alpha$ is allowed to be a much more general function of x and ξ – not just a polynomial in ξ. In this appendix we provide a very basic introduction to such operators. We consider only operators that act on functions whose argument runs over the real line $\mathbb{R}$ or over the unit circle

$$S^1 = \partial\mathbb{D} = \big\{ \ e^{i\theta} \ \big| \ \theta \in \mathbb{R} \ \big\} \subset \mathbb{C}$$

The general theory deals with operators acting on functions whose argument runs over some manifold of dimension $n \in \mathbb{N}$ including, for example, any connected open subset Ω of $\mathbb{R}^n$. For discussions of the general theory see [Ta,Tr2,SR] or [F2, Chapter 8].

Pseudodifferential Operators on S^1

Definition 2.A.1 (Amplitudes and ΨDO's on S^1)
Let $m \in \mathbb{R}$. The space $A^m(\partial\mathbb{D})$ of amplitudes on $\partial\mathbb{D}$ of order m is the set of all functions $a \in C^\infty(\partial\mathbb{D} \times \mathbb{Z} \times \partial\mathbb{D})$ with the property that, for each $\alpha, \beta, \beta' \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\beta'}$ such that

$$\big| D_n^\alpha \tfrac{\partial^\beta}{\partial\theta^\beta} \tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}} a\big(e^{i\theta}, n, e^{i\theta'}\big)\big| \le C_{\alpha,\beta,\beta'}(1+|n|)^{m-\alpha} \tag{2.A.1}$$

for all $\theta, \theta' \in \mathbb{R}$ and $n \in \mathbb{Z}$. Here D_n is the difference operator

$$(D_n g)(n) = g(n+1) - g(n)$$

We associate to the amplitude a the operator $\Psi_a : C^\infty(\partial\mathbb{D}) \to C^\infty(\partial\mathbb{D})$ defined by

$$\Psi_a f(e^{i\theta}) = \tfrac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^{2\pi} e^{in(\theta-\theta')} a\big(e^{i\theta}, n, e^{i\theta'}\big) f\big(e^{i\theta'}\big) \ d\theta' \tag{2.A.2}$$

We denote $\Psi^m(\partial\mathbb{D}) = \big\{ \ \Psi_a \ \big| \ a \in A^m(\partial\mathbb{D}) \ \big\}$.

Problem 2.A.1 Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$.

(a) Prove that, for each $K, L \in \mathbb{N}_0$, there is a constant $D_{K,L}$ such that the Fourier coefficients

$$\hat{a}(k,n,\ell) = \tfrac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \; e^{-ik\theta} e^{-i\ell\theta'} \; a\big(e^{i\theta}, n, e^{i\theta'}\big)$$

obey

$$\big|\hat{a}(k,n,\ell)\big| \le D_{K,L} \tfrac{1}{(1+|k|)^K} (1+|n|)^m \tfrac{1}{(1+|\ell|)^L}$$

for all $k, \ell, n \in \mathbb{Z}$.

(b) Let, for each $k, \ell \in \mathbb{Z}$,

$$\hat{\mathcal{A}}(k,\ell) = \sum_{n\in\mathbb{Z}} \tfrac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \; e^{-ik\theta} \int_0^{2\pi} d\theta' \; e^{in(\theta-\theta')} \; a\big(e^{i\theta}, n, e^{i\theta'}\big) e^{i\ell\theta'}$$

be the k^{th} Fourier coefficient of Ψ_a applied to $e_\ell(\theta) = e^{i\ell\theta}$. Prove that for each $K \in \mathbb{N}_0$, there is a constant D_K such that

$$\big|\hat{\mathcal{A}}(k,\ell)\big| \le D_K \tfrac{(1+|\ell|)^m}{(1+|k-\ell|)^K}$$

(c) Prove that if $f \in C^\infty(\partial\mathbb{D})$, then $\Psi_a f \in C^\infty(\partial\mathbb{D})$.

Remark 2.A.2 The formula (2.A.2) is nonstandard in that it uses a discrete sum $\sum_{n\in\mathbb{Z}}$. Under the standard formalism, viewing S^1 as just another manifold, one works in local coordinates and defines, for f supported in the coordinate patch,

$$\Psi_a f(x) = \int \tfrac{d^n\xi}{(2\pi)^n} \int d^n x' \; e^{i\xi\cdot(x-x')} a(x,\xi,x') f(x')$$

Remark 2.A.3 Let $m \in \mathbb{R}$ and $a \in C^\infty(\partial\mathbb{D} \times \mathbb{R} \times \partial\mathbb{D})$. Assume that, for each $\alpha, \beta, \beta' \in \mathbb{N}_0$, there is a constant $\tilde{C}_{\alpha,\beta,\beta'}$ such that

$$\big|\tfrac{\partial^\alpha}{\partial\xi^\alpha} \tfrac{\partial^\beta}{\partial\theta^\beta} \tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}} a\big(e^{i\theta}, \xi, e^{i\theta'}\big)\big| \le \tilde{C}_{\alpha,\beta,\beta'} (1+|\xi|)^{m-\alpha} \tag{2.A.3}$$

for all $\theta, \xi, \theta' \in \mathbb{R}$. We claim that the restriction of a to $\xi \in \mathbb{Z}$ obeys the condition (2.A.1).

Proof: Define $(D_n g)(\xi) = g(\xi+1) - g(\xi) = \int_\xi^{\xi+1} g'(t)\, dt$ for all $\xi \in \mathbb{R}$. Then

$$\sup_\xi \big(1+|\xi|\big)^m \big|(D_n g)(\xi)\big| \le \text{const}_m \sup_\xi \big(1+|\xi|\big)^m |g'(\xi)|$$

and, by induction,

$$\begin{aligned}\sup_{\xi}\big(1+|\xi|\big)^m\big|(D_n^\alpha g)(\xi)\big| &\le \mathrm{const}_m \sup_{\xi}\big(1+|\xi|\big)^m\big|\tfrac{d}{d\xi}(D_n^{\alpha-1}g)(\xi)\big|\\ &\le \mathrm{const}_{m,\alpha}\sup_{\xi}\big(1+|\xi|\big)^m|g^{(\alpha)}(\xi)|\end{aligned}$$

since D_ξ and $\frac{d}{d\xi}$ commute. Thus

$$\begin{aligned}&\sup_{\theta,\theta',n}\Big|(1+|n|)^{-m+\alpha}D_n^\alpha\tfrac{\partial^\beta}{\partial\theta^\beta}\tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}}a\big(e^{i\theta},n,e^{i\theta'}\big)\Big|\\ &\qquad\le \mathrm{const}_{m-\alpha,\alpha}\sup_{\theta,\theta',\xi}\Big|(1+|\xi|)^{-m+\alpha}\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial\theta^\beta}\tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}}a\big(e^{i\theta},\xi,e^{i\theta'}\big)\Big|\\ &\qquad\le \mathrm{const}_{m-\alpha,\alpha}\,\tilde{C}_{\alpha,\beta,\beta'}\end{aligned}$$

■

Remark 2.A.4 Any linear combination of amplitudes, that is functions satisfying (2.A.1), is again an amplitude. Furthermore if, for each $\beta,\beta'\in\mathbb{N}_0$ and $n\in\mathbb{Z}$,

$$\sup_{\theta,\theta'}\big|\tfrac{\partial^\beta}{\partial\theta^\beta}\tfrac{\partial^{\beta'}}{\partial\theta'^{\beta'}}a\big(e^{i\theta},n,e^{i\theta'}\big)\big|<\infty \tag{2.A.4}$$

and if $a\big(e^{i\theta},n,e^{i\theta'}\big)$ vanishes for all $|n|$ sufficiently large, then $a\in A^m(\partial\mathbb{D})$ for all $m\in\mathbb{R}$. Thus to verify (2.A.1), it suffices to verify (2.A.4) and to find an extension $a\big(e^{i\theta},\xi,e^{i\theta'}\big)$ that satisfies (2.A.3) for $|\xi|$ sufficiently large.

Example 2.A.5 If $a\big(e^{i\theta},n,e^{i\theta'}\big)=b\big(e^{i\theta}\big)n^m c\big(e^{i\theta'}\big)$ for some $m\in\mathbb{N}_0$ and $b,c\in C^\infty(\partial\mathbb{D})$, then $a\in A^m(\partial\mathbb{D})$ and

$$\begin{aligned}\Psi_a f(e^{i\theta}) &= \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} e^{in(\theta-\theta')}b\big(e^{i\theta}\big)n^m c\big(e^{i\theta'}\big)f\big(e^{i\theta'}\big)\,d\theta'\\ &= b\big(e^{i\theta}\big)(-i)^m\tfrac{d^m}{d\theta^m}\tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} e^{in(\theta-\theta')}c\big(e^{i\theta'}\big)f\big(e^{i\theta'}\big)\,d\theta'\\ &= b\big(e^{i\theta}\big)(-i)^m\tfrac{d^m}{d\theta^m}c\big(e^{i\theta}\big)f\big(e^{i\theta}\big)\end{aligned}$$

So, in this case, Ψ_a is the differential operator $b\big(e^{i\theta}\big)(-i)^m\frac{d^m}{d\theta^m}c\big(e^{i\theta}\big)$. Because we are allowing, for pedagogical purposes, the amplitude $a\big(e^{i\theta},n,e^{i\theta'}\big)$ to depend on both θ and θ', any pseudodifferential operator Ψ_a has many different amplitudes. In the example under consideration,

$$\begin{aligned}\Psi_a f(e^{i\theta}) &= b\big(e^{i\theta}\big)(-i)^m\tfrac{d}{d\theta^m}c\big(e^{i\theta}\big)f\big(e^{i\theta}\big)=(-i)^m b\big(e^{i\theta}\big)\sum_{\ell=0}^m\tbinom{m}{\ell}\tfrac{d^{m-\ell}c}{d\theta^{m-\ell}}\big(e^{i\theta}\big)\tfrac{d^\ell f}{d\theta^\ell}\big(e^{i\theta}\big)\\ &= \Psi_{\tilde{a}}f(e^{i\theta})\end{aligned}$$

where

$$\tilde{a}\big(e^{i\theta}, n, e^{i\theta'}\big) = b\big(e^{i\theta}\big) \sum_{\ell=0}^{m} \tbinom{m}{\ell}(-i)^{m-\ell} \tfrac{d^{m-\ell}}{d\theta^{m-\ell}} c\big(e^{i\theta}\big) n^{\ell}$$

happens to be independent of θ'.

Remark 2.A.6 Amplitudes that are independent of θ' are called symbols and are usually denoted $a\big(e^{i\theta}, n\big)$. For a symbol

$$\Psi_a f(e^{i\theta}) = \tfrac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^{2\pi} e^{in(\theta-\theta')} a\big(e^{i\theta}, n\big) f\big(e^{i\theta'}\big)\ d\theta' = \sum_{n\in\mathbb{Z}} e^{in\theta} a\big(e^{i\theta}, n\big) \hat{f}_n \tag{2.A.5}$$

where

$$\hat{f}(n) = \tfrac{1}{2\pi} \int_0^{2\pi} e^{-in\theta'} f\big(e^{i\theta'}\big)\ d\theta'$$

is the n^{th} Fourier coefficient of f. In particular, if $f\big(e^{i\theta}\big) = e_n\big(e^{i\theta}\big) \equiv e^{in\theta}$

$$\Psi_a e_n(e^{i\theta}) = e^{in\theta} a\big(e^{i\theta}, n\big) \implies a\big(e^{i\theta}, n\big) = e^{-in\theta} \Psi_a e_n(e^{i\theta})$$

so the symbol of a pseudodifferential operator is uniquely determined. The space of symbols of order m on $\partial\mathbb{D}$ is denoted $S^m(\partial\mathbb{D})$.

Example 2.A.7 Suppose that the symbol $a\big(e^{i\theta}, n\big)$ is "positive homogeneous" of degree m, in the sense that

$$a\big(e^{i\theta}, tn\big) = t^m a\big(e^{i\theta}, n\big)$$

for all $t > 0$. Then

$$\begin{aligned} n \in \mathbb{N} &\implies a\big(e^{i\theta}, n\big) = n^m a\big(e^{i\theta}, 1\big) \\ -n \in \mathbb{N} &\implies a\big(e^{i\theta}, n\big) = |n|^m a\big(e^{i\theta}, -1\big) \\ & \quad\ a\big(e^{i\theta}, 0\big) = 0 \qquad \text{if } m \neq 0 \end{aligned}$$

If we define the projection operators

$$\mathcal{P}_+ e^{in\theta} = \begin{cases} e^{in\theta} & \text{if } n > 0 \\ 0 & \text{if } n \le 0 \end{cases} \quad \mathcal{P}_- e^{in\theta} = \begin{cases} e^{in\theta} & \text{if } n < 0 \\ 0 & \text{if } n \ge 0 \end{cases} \quad \mathcal{P}_0 e^{in\theta} = \begin{cases} e^{in\theta} & \text{if } n = 0 \\ 0 & \text{if } n \ne 0 \end{cases}$$

then, if $a\big(e^{i\theta}, n\big)$ is positive homogeneous of degree $m \in \mathbb{N}_0$,

$$\Psi_a = a\big(e^{i\theta}, 1\big)\mathcal{P}_+ \tfrac{d^m}{d\theta^m} + (-1)^m a\big(e^{i\theta}, -1\big)\mathcal{P}_- \tfrac{d^m}{d\theta^m} + \begin{cases} a\big(e^{i\theta}, 0\big)\mathcal{P}_0 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

While (linear combinations) of positive homogeneous symbols are very common, there are lots of other possibilities. For example,

$$\big(1+n^2\big)^{s/2} \qquad (1+|n|)^s \qquad (1+|n|)^s \log^p\big(1+|n|\big) \qquad \sin\log\big(1+|n|\big)$$

with $s, p \in \mathbb{R}$.

Problem 2.A.2 Let $s, p \in \mathbb{R}$. Prove that each of

$$(1+n^2)^{s/2} \qquad (1+|n|)^s \qquad (1+|n|)^s \log^p\big(2+|n|\big) \qquad \sin\log\big(2+|n|\big)$$

satisfy (2.A.1) for any $m \ge s$, in the first two cases, any $m > s$ in the third and any $m \ge 0$ in the last.

Problem 2.A.3 Prove that the linear operator $f \in C^\infty(\partial\mathbb{D}) \mapsto f(0)$, where $f(0)$ is viewed as a constant function on $\partial\mathbb{D}$, is not in $\Psi^m(\partial\mathbb{D})$ for any $m \in \mathbb{R}$.

Problem 2.A.4 Let $a \in S^m(\partial\mathbb{D})$ and $b \in S^{m'}(\partial\mathbb{D})$ and set

$$c\big(e^{i\theta}, n, e^{i\theta'}\big) = \sum_{\ell\in\mathbb{Z}} a\big(e^{i\theta}, n\big)\hat{b}(\ell, n-\ell)e^{i\ell\theta'} \quad \text{where } \hat{b}(\ell, n) = \tfrac{1}{2\pi}\int_0^{2\pi} d\theta\ e^{-i\ell\theta}\ b\big(e^{i\theta}, n\big)$$

Prove that $c \in A^{m+m'}(\partial\mathbb{D})$ and

$$\Psi_a\Psi_b = \Psi_c$$

Proposition 2.A.8 *Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$.*

(a) For every $s \in \mathbb{R}$, the operator Ψ_a defined by (2.A.2) has a unique extension to a bounded linear operator from $H^s(\partial\mathbb{D})$ to $H^{s-m}(\partial\mathbb{D})$.

(b) Suppose further that a has a zero of order $m' \in \mathbb{N}$ at $\theta = \theta'$, in the sense that $a\big(e^{i\theta}, n, e^{i\theta'}\big) = \big[e^{i\theta} - e^{i\theta'}\big]^{m'} b\big(e^{i\theta}, n, e^{i\theta'}\big)$ with $b \in A^m(\partial\mathbb{D})$. Then, for every $s \in \mathbb{R}$, the operator Ψ_a defined by (2.A.2) has a unique extension to a bounded linear operator from $H^s(\partial\mathbb{D})$ to $H^{s-m+m'}(\partial\mathbb{D})$.

Proof: (a) In the notation of Problem 2.A.1, the k^{th} Fourier coefficient of $\Psi_a f(e^{i\theta})$ is given by

$$\widehat{\Psi_a f}(k) = \sum_{\ell\in\mathbb{Z}} \mathcal{A}(k,\ell)\hat{f}(\ell)$$

where $\hat{f}(\ell)$ is the ℓ^{th} Fourier coefficient of f. Since

$$\|\Psi_a f\|^2_{H^{s-m}(\partial\mathbb{D})} = \sum_{k\in\mathbb{Z}} (1+k^2)^{s-m}\big|\widehat{\Psi_a f}(k)\big|^2 \qquad \|f\|^2_{H^s(\partial\mathbb{D})} = \sum_{\ell\in\mathbb{Z}} (1+\ell^2)^s\big|\hat{f}(\ell)\big|^2$$

it suffices, by Proposition 2.3.12, with $\alpha = \beta = 1$, $p = q = 2$ and $\mu = \nu$ being the counting measure on $X = Y = \mathbb{Z}$, to prove that

$$\sup_k \sum_\ell (1+k^2)^{\frac{s-m}{2}}|\hat{\mathcal{A}}(k,\ell)|(1+\ell^2)^{-\frac{s}{2}}\ ,\ \sup_\ell \sum_k (1+k^2)^{\frac{s-m}{2}}|\hat{\mathcal{A}}(k,\ell)|(1+\ell^2)^{-\frac{s}{2}}\ < \infty$$

Since $\sum_{\ell\in\mathbb{Z}}\frac{1}{(1+|k-\ell|)^2} = \sum_{k\in\mathbb{Z}}\frac{1}{(1+|k-\ell|)^2} = \sum_{n\in\mathbb{Z}}\frac{1}{(1+|n|)^2} < \infty$ these two bounds will follow easily if we can prove that

$$(1+k^2)^{\frac{s-m}{2}}\big|\hat{\mathcal{A}}(k,\ell)\big|(1+\ell^2)^{-\frac{s}{2}} \le \tfrac{\text{const}}{(1+|k-\ell|)^2}$$

But this bound, in turn, follows easily from part (b) of Problem 2.A.1, with $K=|s-m|+2$ and the consequence

$$(1+k^2)^{\frac{s-m}{2}} \le 2^{\frac{|s-m|}{2}}(1+|k-\ell|^2)^{\frac{|s-m|}{2}}(1+\ell^2)^{\frac{s-m}{2}}$$

of Lemma 2.3.4 (Peetre's inequality).

(b) Multiplication by $(-1)^{m'}e^{-im'\theta}$ preserves the bounds (2.A.1), so we may equally well assume that

$$a\big(e^{i\theta},n,e^{i\theta'}\big) = \big[e^{-i(\theta-\theta')}-1\big]^{m'} b\big(e^{i\theta},n,e^{i\theta'}\big)$$

Since

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{in(\theta-\theta')}\big[e^{-i(\theta-\theta')}-1\big]c\big(e^{i\theta},n,e^{i\theta'}\big) &= \sum_{n\in\mathbb{Z}}\big[e^{i(n-1)(\theta-\theta')}-e^{in(\theta-\theta')}\big]c\big(e^{i\theta},n,e^{i\theta'}\big)\\ &= \sum_{n\in\mathbb{Z}} e^{in(\theta-\theta')}\big[c\big(e^{i\theta},n+1,e^{i\theta'}\big)-c\big(e^{i\theta},n,e^{i\theta'}\big)\big]\\ &= \sum_{n\in\mathbb{Z}} e^{in(\theta-\theta')}(D_nc)\big(e^{i\theta},n,e^{i\theta'}\big)\end{aligned}$$

we have that

$$\Psi_a f(e^{i\theta}) = \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} e^{in(\theta-\theta')}(D_n^{m'}b)\big(e^{i\theta},n,e^{i\theta'}\big)f\big(e^{i\theta'}\big)\,d\theta' \tag{2.A.6}$$

It now suffices to apply the result of part (a) with the amplitude $a\in A^m(\partial\mathbb{D})$ replaced by $D_n^{m'}b\in A^{m-m'}(\partial\mathbb{D})$ and m replaced by $m-m'$. ■

Problem 2.A.5 Let $m,s,s'\in\mathbb{R}$ and $p\in\mathbb{N}_0$ obey $p\ge|s-m|$ and $s'>s-m$. Let $\Sigma=\big\{\ \{\sigma_n\}_{n\in\mathbb{Z}}\ \big|\ \sigma_n\in\mathbb{C}\text{ for all }n\in\mathbb{Z}\ \big\}$ denote the set of two–sided sequences and

$$\Sigma_m = \Big\{\ \{\sigma_n\}_{n\in\mathbb{Z}}\in\Sigma\ \Big|\ \sup_{n\in\mathbb{Z}}\tfrac{|\sigma_n|}{(1+n^2)^{m/2}}<\infty\ \Big\}$$

Given any $\sigma=\{\sigma_n\}_{n\in\mathbb{Z}}\in\Sigma_m$,

$$f\mapsto\sum_{n\in\mathbb{Z}}\sigma_n\hat{f}(n)\ e^{in\theta}$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of f, is a bounded operator from $H^s(\partial\mathbb{D})$ to $H^{s-m}(\partial\mathbb{D})$. If $\varphi \in C^p(\partial\mathbb{D})$, the space of functions on $\partial\mathbb{D}$ whose derivatives of order up to p are continuous, then

$$\Psi_{\varphi\sigma} f = \varphi(e^{i\theta}) \sum_{n\in\mathbb{Z}} \sigma_n \hat{f}(n)\; e^{in\theta}$$

is also a bounded operator from $H^s(\partial\mathbb{D})$ to $H^{s-m}(\partial\mathbb{D})$.

Let $\sigma^{(1)}, \cdots \sigma^{(l)} \in \Sigma_m$ be asymptotically independent in the sense that, given any $\alpha_1, \cdots, \alpha_l \in \mathbb{C}$, not all zero, there is a sequence $\{n_j\}_{j\in\mathbb{N}}$ obeying $\lim_{j\to\infty} |n_j| = \infty$ for which

$$\lim_{j\to\infty} |n_j|^{s'-s} \big|\alpha_1 \sigma^{(1)}_{n_j} + \cdots + \alpha_l \sigma^{(l)}_{n_j}\big| = \infty$$

Prove that if $\varphi_1 \cdots, \varphi_l \in C^p(\partial\mathbb{D})$ and if

$$\sum_{k=1}^{l} \Psi_{\varphi_k \sigma^{(k)}} : H^s(\partial\mathbb{D}) \to H^{s'}(\partial\mathbb{D})$$

is a bounded operator, then $\varphi_1 \cdots, \varphi_l$ are all identically zero.

Problem 2.A.6 Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$. Set

$$p(e^{i\theta}, n) = e^{-in\theta} \Psi_a(e_n)$$

where $e_n(\theta) = e^{in\theta}$. Prove that $p \in S^m(\partial\mathbb{D})$ and $\Psi_a = \Psi_p$.

Problem 2.A.6 shows that every amplitude $a \in A^m(\partial\mathbb{D})$ has a corresponding (θ' independent) symbol in $S^m(\partial\mathbb{D})$. We next provide an algorithm for explicitly computing that symbol, at least approximately to arbitrary degree of accuracy, in a certain sense. By way of preparation, we develop an expansion for $g(e^{i\theta})$ about $\theta = 0$ that is similar to a Taylor expansion, but preserves the periodicity of every term. If g were analytic at $z = 1$, we could simply substitute $z = e^{i\theta}$ into

$$g(z) = \sum_{k=0}^{\infty} \tfrac{1}{k!} g^{(k)}(1) \left(z - 1\right)^k$$

to obtain

$$g(e^{i\theta}) = \sum_{k=0}^{\infty} \tfrac{1}{k!} g^{(k)}(1) \left(e^{i\theta} - 1\right)^k$$

Even if g is only defined on $\partial\mathbb{D}$, we can get a similar expansion. Define

$$\partial_z g(e^{i\theta}) = -ie^{-i\theta} \tfrac{d}{d\theta} g(e^{i\theta})$$

Starting with

$$g(e^{i\theta}) = g(1) + \int_0^\theta \partial_z g(e^{it})\ ie^{it}dt$$

and repeadly applying

$$\begin{aligned}\tfrac{1}{\ell!}\big(e^{i\theta} - e^{it}\big)^\ell \partial_z^{\ell+1} g(e^{it})\ ie^{it} = -\tfrac{d}{dt}\Big[\tfrac{1}{(\ell+1)!}\big(e^{i\theta} - e^{it}\big)^{\ell+1}\partial_z^{\ell+1} g(e^{it})\Big] \\ + \tfrac{1}{(\ell+1)!}\big(e^{i\theta} - e^{it}\big)^{\ell+1}\partial_z^{\ell+2} g(e^{it})\ ie^{it}\end{aligned}$$

gives

$$g(e^{i\theta}) = \sum_{k=0}^{\ell} \tfrac{1}{k!}\partial_z^k g(1)\ \big(e^{i\theta} - 1\big)^k + \tfrac{1}{\ell!}\int_0^\theta \big(e^{i\theta} - e^{it}\big)^\ell \partial_z^{\ell+1} g(e^{it})\ ie^{it}dt$$

Similarly, for any expansion point $e^{i\tilde{\theta}}$,

$$g(e^{i\theta}) = \sum_{k=0}^{\ell} \tfrac{1}{k!}\partial_z^k g(e^{i\tilde{\theta}})\ \big(e^{i\theta} - e^{i\tilde{\theta}}\big)^k + \tfrac{1}{\ell!}\int_{\tilde{\theta}}^\theta \big(e^{i\theta} - e^{it}\big)^\ell \partial_z^{\ell+1} g(e^{it})\ ie^{it}dt \tag{2.A.7}$$

Proposition 2.A.9 *Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$. Define, for each $k \in \mathbb{N}_0$*

$$a_k(e^{i\theta}, n) = \tfrac{1}{k!}e^{ik\theta}\partial_{z'}^k D_n^k a(e^{i\theta}, n, e^{i\theta'})\big|_{e^{i\theta'}=e^{i\theta}}$$

where $\partial_{z'} = -ie^{-i\theta'}\frac{\partial}{\partial\theta'}$. Then, for all $s \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$

$$\Psi_a = \sum_{k=0}^{\ell} \Psi_{a_k} + R_\ell$$

where R_ℓ is a bounded linear operator from $H^s(\partial\mathbb{D})$ to $H^{s-m+\ell+1}(\partial\mathbb{D})$.

Proof: By (2.A.7), with $\theta \to \theta'$ and $\tilde{\theta} \to \theta$,

$$\begin{aligned}a(e^{i\theta}, n, e^{i\theta'}) = \sum_{k=0}^{\ell} \tfrac{1}{k!}e^{ik\theta}\partial_{z'}^k a(e^{i\theta}, n, e^{i\theta'})\big|_{e^{i\theta'}=e^{i\theta}}\ \big(e^{i(\theta'-\theta)} - 1\big)^k \\ + \tfrac{1}{\ell!}\int_\theta^{\theta'} \big(e^{i\theta'} - e^{it}\big)^\ell \partial_{z'}^{\ell+1} a(e^{i\theta}, n, e^{it})\ ie^{it}dt\end{aligned}$$

By (2.A.6), the pseudodifferential operator associated with the k^{th} term is exactly Ψ_{a_k}. By part (b) of Proposition 2.A.8, the pseudodifferential operator associated with the remainder term is a bounded operator from $H^s(\partial\mathbb{D})$ to $H^{s-m+\ell+1}(\partial\mathbb{D})$. ■

Remark 2.A.10 The conclusion of Proposition 2.A.9 is generally written

$$\Psi_a = \sum_{k=0}^{\infty} \Psi_{a_k}$$

where the infinite sum is to be interpretted "asymptotically". The definition of "asymptotically" here is precisely the statement of Proposition 2.A.9. That is, for each $\ell \in \mathbb{N}_0$, $\Psi_a - \sum_{k=0}^{\ell} \Psi_{a_k}$ is a bounded linear operator from $H^s(\partial\mathbb{D})$ to $H^{s-m+\ell+1}(\partial\mathbb{D})$ for all s.

Pseudodifferential Operators on $\mathbb{R}^1$

Definition 2.A.11 (Amplitudes and ΨDO's on $\mathbb{R}^1$)
Let $m \in \mathbb{R}$. The space $A^m(\mathbb{R})$ of amplitudes on $\mathbb{R}$ of order m is the set of all $a \in C^\infty(\mathbb{R}^3)$ with the property that, for each $\alpha, \beta, \beta' \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\beta'}$ such that

$$\left|\tfrac{\partial^\alpha}{\partial\xi^\alpha} \tfrac{\partial^\beta}{\partial x^\beta} \tfrac{\partial^{\beta'}}{\partial x'^{\beta'}} a(x,\xi,x')\right| \le C_{\alpha,\beta,\beta'}(1+|\xi|)^{m-\alpha} \tag{2.A.8}$$

for all $x, \xi, x' \in \mathbb{R}$. We associate to the amplitude a the pseudodifferential operator $\mathfrak{P}_a : C_0^\infty(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined by

$$\mathfrak{P}_a f(x) = \tfrac{1}{2\pi} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ e^{i\xi(x-x')} a(x,\xi,x') f(x') \tag{2.A.9}$$

We denote $\Psi^m(\mathbb{R}) = \left\{\ \mathfrak{P}_a\ \middle|\ a \in A^m(\mathbb{R})\ \right\}$. The space $S^m(\mathbb{R})$ of symbols on $\mathbb{R}$ of order m is the set of all $a \in C^\infty(\mathbb{R}^2)$ for which $a(x,\xi) \in A^m(\mathbb{R})$.

Problem 2.A.7 Let $a(x,\xi,x')$ obey (2.A.8). Let $\mathcal{K}$ be any compact subset of $\mathbb{R}$ and $\chi \in C_0^\infty(\mathbb{R})$ be identically one on $\mathcal{K}$.

(a) Prove that, for each $\alpha, \beta, \gamma, \gamma' \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\gamma,\gamma'}$ such that the partial Fourier transform

$$\tilde{a}_\chi(x,\xi,k) = \int_{\mathbb{R}} dx'\ e^{-ikx'} a(x,\xi,x')\chi(x')$$

obeys

$$\left|\tfrac{\partial^\alpha}{\partial\xi^\alpha} \tfrac{\partial^\beta}{\partial x^\beta} \tfrac{\partial^\gamma}{\partial k^\gamma} \tilde{a}_\chi(x,\xi,k)\right| \le C_{\alpha,\beta,\gamma,\gamma'} \tfrac{(1+|\xi|)^{m-\alpha}}{(1+|k|)^{\gamma'}}$$

(b) Let $\alpha, \beta \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$ obey $\gamma > m+\beta+1$. Prove that if $f \in C_0(\mathcal{K})$, then

$$\sup_{x\in\mathbb{R}} \left|x^\alpha \tfrac{d^\beta}{dx^\beta} \mathfrak{P}_a f(x)\right| \le \text{const}\, \|f\|_{H^\gamma(\mathbb{R})}$$

Remark 2.A.12 The hypothesis (2.A.8) is stronger than the standard one. Normally one assumes that, for each $\alpha, \beta, \beta' \in \mathbb{N}_0$ and each compact $\mathcal{K} \subset \mathbb{R}^2$, there is a constant $C_{\alpha,\beta,\beta',\mathcal{K}}$ such that

$$\left|\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}\tfrac{\partial^{\beta'}}{\partial x'^{\beta'}}a(x,\xi,x')\right| \le C_{\alpha,\beta,\beta',\mathcal{K}}(1+|\xi|)^{m-\alpha}$$

for all $\xi \in \mathbb{R}$ and $(x,x') \in \mathcal{K}$.

Proposition 2.A.13 *Let $m \in \mathbb{R}$ and $a \in A^m(\mathbb{R})$. Let $\mathcal{K}$ be any compact subset of $\mathbb{R}$.*

(a) Let $s \in \mathbb{R}$. There is a constant C, depending on $m, s, \mathcal{K}$ and a, but independent of f such that, if $f \in C_0^\infty(\mathcal{K})$, then

$$\|\mathfrak{P}_a f\|_{H^{s-m}(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}$$

(b) Let $\alpha, \beta \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$ obey $\gamma > m+\beta+1$. There is a constant C, depending on $m, \alpha, \beta, \gamma, \mathcal{K}$ and a, but independent of f such that, if $f \in C_0(\mathcal{K})$, then

$$\sup_{x\in\mathbb{R}}\left|x^\alpha \tfrac{d^\beta}{dx^\beta}\mathfrak{P}_a f(x)\right| \le C\|f\|_{H^\gamma(\mathbb{R})}$$

Proof: The proof of part (b) is Problem 2.A.7. For part (a), let $\chi(x') \in C_0^\infty(\mathbb{R})$ be identically one on $\mathcal{K}$. Then if $f \in C_0^\infty(\mathcal{K})$,

$$\begin{aligned}
\mathfrak{P}_a f(x) &= \tfrac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ e^{i\xi(x-x')}a(x,\xi,x')\chi(x')f(x') \\
&= \tfrac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ \tfrac{a(x,\xi,x')}{1+|x-x'|^2}\chi(x')f(x')\big(1-\tfrac{\partial^2}{\partial\xi^2}\big)e^{i\xi(x-x')} \\
&= \tfrac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ e^{i\xi(x-x')}\big(1-\tfrac{\partial^2}{\partial\xi^2}\big)\tfrac{a(x,\xi,x')}{1+|x-x'|^2}\chi(x')f(x') \\
&= \tfrac{1}{2\pi}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dx'\ e^{i\xi(x-x')}b(x,\xi,x')f(x') \\
&= \mathfrak{P}_b f(x)
\end{aligned}$$

where

$$b(x,\xi,x') = \big(1-\tfrac{\partial^2}{\partial\xi^2}\big)\tfrac{a(x,\xi,x')}{1+|x-x'|^2}\chi(x')$$

has compact support in x' and obeys

$$\left|\tfrac{\partial^\alpha}{\partial\xi^\alpha}\tfrac{\partial^\beta}{\partial x^\beta}\tfrac{\partial^{\beta'}}{\partial x'^{\beta'}}b(x,\xi,x')\right| \le D_{\alpha,\beta,\beta'}\tfrac{(1+|\xi|)^{m-\alpha}}{1+x^2}$$

Part (a) now follows by Problem 2.A.8. ■

Problem 2.A.8 Let $m, s \in \mathbb{R}$ and $b \in C^\infty(\mathbb{R}^3)$. Assume that, for each $\beta, \beta' \in \mathbb{N}_0$, there is a constant $D_{\beta,\beta'}$ such that

$$\int \left| \tfrac{\partial^\beta}{\partial x^\beta} \tfrac{\partial^{\beta'}}{\partial x'^{\beta'}} b(x,\xi,x') \right| \, dx dx' \le D_{\beta,\beta'} (1+|\xi|)^m$$

Prove that there is a constant C such that if $f \in C_0^\infty(\mathbb{R})$, then

$$\|\mathfrak{P}_b f\|_{H^{s-m}(\mathbb{R})} \le C \|f\|_{H^s(\mathbb{R})}$$

Lemma 2.A.14 *Let $m \in \mathbb{R}$ and $a \in A^m(\mathbb{R})$. Suppose further that a has a zero of order $m' \in \mathbb{N}$ at $x = x'$, in the sense that $a(x,\xi,x') = [x-x']^{m'} b(x,\xi,x')$ with $b \in A^m(\mathbb{R})$. Let $\mathcal{K}$ be any compact subset of $\mathbb{R}$.*

(a) Let $s \in \mathbb{R}$. There is a constant C such that, if $f \in C_0^\infty(\mathcal{K})$, then

$$\|\mathfrak{P}_a f\|_{H^{s-m+m'}(\mathbb{R})} \le C \|f\|_{H^s(\mathbb{R})}$$

(b) For each $\alpha, \beta \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$ obeying $\gamma > m - m' + \beta + 1$, there is a constant C such that, if $f \in C_0(\mathcal{K})$, then

$$\sup_{x \in \mathbb{R}} \left| x^\alpha \tfrac{d^\beta}{dx^\beta} \mathfrak{P}_a f(x) \right| \le C \|f\|_{H^\gamma(\mathbb{R})}$$

Proof: The proof is Problem 2.A.9. ■

Problem 2.A.9 Prove Lemma 2.A.14.

Proposition 2.A.15 *Let $m \in \mathbb{R}$ and $a \in A^m(\mathbb{R})$. Define, for each $k \in \mathbb{N}_0$*

$$a_k(x,\xi) = (-i)^k \tfrac{1}{k!} \tfrac{\partial^k}{\partial \xi^k} \tfrac{\partial^k}{\partial x'^k} a(x,\xi,x') \Big|_{x'=x}$$

Then, for all $\ell \in \mathbb{N}_0$,

$$\mathfrak{P}_a = \sum_{k=0}^{\ell} \mathfrak{P}_{a_k} + R_\ell$$

where R_ℓ obeys the following bounds. Let $\mathcal{K}$ be any compact subset of $\mathbb{R}$. If $s \in \mathbb{R}$, then there is a constant C such that

$$\|R_\ell f\|_{H^{s-m+\ell+1}(\mathbb{R})} \le C \|f\|_{H^s(\mathbb{R})}$$

for all $f \in C_0^\infty(\mathcal{K})$. If $\alpha, \beta \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$ obey $\gamma > m - \ell + \beta$, then there is a constant C such that

$$\sup_{x \in \mathbb{R}} \left| x^\alpha \tfrac{d^\beta}{dx^\beta} R_\ell f(x) \right| \le C \|f\|_{H^\gamma(\mathbb{R})}$$

for all $f \in C_0(\mathcal{K})$.

Proof: The proof is Problem 2.A.10. ■

Problem 2.A.10 Prove Proposition 2.A.15.

We now use the Poisson summation formula to write a ΨDO on S^1 as the "periodization" of a ΨDO on $\mathbb{R}^1$. Let $m \in \mathbb{R}$ and $a \in C^\infty(\partial\mathbb{D}\times\mathbb{R}\times\partial\mathbb{D})$ satisfy the bounds (2.A.3). Let $\chi \in C_0^\infty(\mathbb{R})$ obey

$$\sum_{\ell\in\mathbb{Z}} \chi(\theta - 2\pi\ell) = 1 \qquad \text{for all } \theta \in \mathbb{R}$$

Lemma 2.A.16 *Let $a \in A^m(\partial\mathbb{D})$ be the restriction to $\xi \in \mathbb{Z}$ of a C^∞ function $a\big(e^{i\theta}, \xi, e^{i\theta'}\big)$ that obeys (2.A.3). For all $f \in C^\infty(\partial\mathbb{D})$*

$$(\Psi_a f)(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \big(\mathfrak{P}_{\tilde a}\, \chi\tilde f\,\big)(\theta + 2\pi n)$$

where

$$\tilde f(x) = f(e^{ix}) \qquad \tilde a(x,\xi,x') = a\big(e^{ix},\xi,e^{ix'}\big)$$

Proof: Since $e^{in(\theta-\theta')}a\big(e^{i\theta}, n, e^{i\theta'}\big)f\big(e^{i\theta'}\big)$ has period 2π in θ'

$$\begin{aligned}\int_0^{2\pi} e^{in(\theta-\theta')}a\big(e^{i\theta}, n, e^{i\theta'}\big)f\big(e^{i\theta'}\big)\,d\theta' &= \sum_{\ell\in\mathbb{Z}}\int_0^{2\pi} e^{in(\theta-\theta')}a\big(e^{i\theta}, n, e^{i\theta'}\big)f\big(e^{i\theta'}\big)\chi(\theta'-2\pi\ell)\,d\theta'\\ &= \sum_{\ell\in\mathbb{Z}}\int_{-2\pi\ell}^{-2\pi(\ell-1)} e^{in(\theta-\theta')}a\big(e^{i\theta}, n, e^{i\theta'}\big)f\big(e^{i\theta'}\big)\chi(\theta')\,d\theta'\\ &= \int_{\mathbb{R}} e^{in(\theta-\theta')}a\big(e^{i\theta}, n, e^{i\theta'}\big)f\big(e^{i\theta'}\big)\chi(\theta')\,d\theta'\end{aligned}$$

so that

$$\Psi_a f = \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} e^{in(\theta-\theta')}a\big(e^{i\theta}, n, e^{i\theta'}\big)\chi(\theta')f\big(e^{i\theta'}\big)\,d\theta'$$

Then by the Poisson summation formula [DM]

$$\sum_{n\in\mathbb{Z}} H(n) = \sum_{n\in\mathbb{Z}}\int e^{2\pi in\xi}H(\xi)\,d\xi$$

for all $H \in \mathcal{S}(\mathbb{R})$. For any fixed $\theta \in \mathbb{R}$, applying this with

$$H(\xi) = \tfrac{1}{2\pi}\int e^{i\xi(\theta-\theta')}a\big(e^{i\theta}, \xi, e^{i\theta'}\big)\chi(\theta')f\big(e^{i\theta'}\big)\,d\theta'$$

gives

$$\begin{aligned}(\Psi_a f)(e^{i\theta}) &= \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int e^{in(\theta-\theta')}a\big(e^{i\theta},n,e^{i\theta'}\big)\chi(\theta')f\big(e^{i\theta'}\big)\ d\theta' \\ &= \tfrac{1}{2\pi}\sum_{n\in\mathbb{Z}}\iint e^{i\xi(\theta+2\pi n-\theta')}a\big(e^{i\theta},\xi,e^{i\theta'}\big)\chi(\theta')f\big(e^{i\theta'}\big)\ d\theta'\ d\xi \\ &= \sum_{n\in\mathbb{Z}}\big(\mathfrak{P}_{\tilde a}\,\chi\tilde f\,\big)(\theta+2\pi n)\end{aligned}$$

■

Changes of Coordinates

We now explore the effect of changes of coordinates on ΨDO's. Think of $\mathbb{R}$ as a manifold having a single coordinate patch with the coordinate named y. Suppose that we are given a ΨDO $\mathfrak{P}_a$ that maps functions of y to functions of y. Let Φ be a diffeomorphism of $\mathbb{R}$ and view Φ as giving a change of coordinates $x=\Phi(y)$. If f is a function of x, the corresponding function of y is $f\big(\Phi(y)\big)$, which is mapped by $\mathfrak{P}_a$ to

$$\Big(\mathfrak{P}_a(f\circ\Phi)\Big)(y) = \tfrac{1}{2\pi}\int d\xi\int dy'\ e^{i\xi(y-y')}a(y,\xi,y')\ f\big(\Phi(y')\big)$$

In x–coordinates this image is

$$\begin{aligned}\mathfrak{P}_a^{\Phi}f(x) &= \Big(\mathfrak{P}_a(f\circ\Phi)\Big)\big(\Phi^{-1}(x)\big) \\ &= \tfrac{1}{2\pi}\int d\xi\int dy'\ e^{i\xi(\Phi^{-1}(x)-y')}a\big(\Phi^{-1}(x),\xi,y'\big)\ f\big(\Phi(y')\big) \\ &= \tfrac{1}{2\pi}\int d\xi\int dx'\ {\Phi^{-1}}'(x')\ e^{i\xi(\Phi^{-1}(x)-\Phi^{-1}(x'))}a\big(\Phi^{-1}(x),\xi,\Phi^{-1}(x')\big)\ f(x')\end{aligned}$$

after the substitution $y'=\Phi^{-1}(x')$. Since

$$\Phi^{-1}(x)-\Phi^{-1}(x') = \int_0^1 \tfrac{d}{dt}\Phi^{-1}\big(tx+(1-t)x'\big)\ dt = (x-x')J(x,x')$$

where

$$J(x,x') = \int_0^1 {\Phi^{-1}}'\big(tx+(1-t)x'\big)\ dt$$

we have, making the second substitution $\xi\to\frac{\xi}{J(x,x')}$,

$$\begin{aligned}\mathfrak{P}_a^{\Phi}f(x) &= \tfrac{1}{2\pi}\int d\xi\int dx'\ {\Phi^{-1}}'(x')\ e^{i\xi J(x,x')(x-x')}a\big(\Phi^{-1}(x),\xi,\Phi^{-1}(x')\big)\ f(x') \\ &= \tfrac{1}{2\pi}\int d\xi\int dx'\ \tfrac{{\Phi^{-1}}'(x')}{J(x,x')}\ e^{i\xi(x-x')}a\big(\Phi^{-1}(x),\tfrac{\xi}{J(x,x')},\Phi^{-1}(x')\big)\ f(x') \\ &= \mathfrak{P}_b f(x)\end{aligned}$$

where

$$b(x,\xi,x') = \tfrac{\Phi^{-1\prime}(x')}{J(x,x')}\ a\big(\Phi^{-1}(x),\tfrac{\xi}{J(x,x')},\Phi^{-1}(x')\big)$$

Conditions on Φ that are sufficient to ensure that $b \in A^m(\mathbb{R})$ are provided in

Proposition 2.A.17 *Let* $m \in \mathbb{R}$, $a \in A^m(\mathbb{R})$ *and* Φ *be a diffeomorphism of* $\mathbb{R}$ *with* $\inf_{y\in\mathbb{R}}|\Phi'(y)| > 0$ *and, for each* $\alpha \in \mathbb{N}$, $\sup_{y\in\mathbb{R}}|\Phi^{(\alpha)}(y)| < \infty$. *Set*

$$b(x,\xi,x') = \tfrac{\Phi^{-1\prime}(x')}{J(x,x')}\ a\big(\Phi^{-1}(x),\tfrac{\xi}{J(x,x')},\Phi^{-1}(x')\big)$$

where $J(x,x') = \int_0^1 \Phi^{-1\prime}\big(tx+(1-t)x'\big)\ dt$. *Then* $b \in A^m(\mathbb{R})$ *and*

$$\mathfrak{P}_a^\Phi f \equiv \Big(\mathfrak{P}_a(f\circ\Phi)\Big)\circ\Phi^{-1} = \mathfrak{P}_b f$$

Furthermore

$$\sigma(x,\xi) = a\big(\Phi^{-1}(x)\ ,\ \Phi'\big(\Phi^{-1}(x)\big)\xi\ ,\ \Phi^{-1}(x)\big) \in S^m(\mathbb{R})$$

and, if $\mathcal{K}$ *is a compact subset of* $\mathbb{R}$ *and* $s \in \mathbb{R}$, *then there is a constant* C *such that*

$$\|\mathfrak{P}_a^\Phi f - \mathfrak{P}_\sigma f\|_{H^{s-m+1}(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}$$

for all $f \in C_0^\infty(\mathcal{K})$.

Proof: The hypotheses on Φ are sufficient to ensure that $\Phi^{-1\prime}(x) = \frac{1}{\Phi'(\Phi^{-1}(x))}$ is bounded away from zero and that all derivatives of Φ^{-1} (of order at least one) are bounded. As Φ is a diffeomorphism, $\Phi'(y)$ and $\Phi^{-1\prime}(x)$ have the same sign, say$(-1)^\varepsilon$, for all $x, y \in \mathbb{R}$. If $(-1)^\varepsilon\Phi^{-1\prime}(x) \ge \mu > 0$ for all $x \in \mathbb{R}$, then

$$(-1)^\varepsilon J(x,x') = \int_0^1 (-1)^\varepsilon\Phi^{-1\prime}\big(tx+(1-t)x'\big)\ dt \ge \int_0^1 \mu\ dt = \mu$$

for all $x, x' \in \mathbb{R}$. Thus $J(x,x')$ is also bounded away from zero. Similarly, $J(x,x')$ has bounded derivatives. The product, quotient and chain rules now give that $b \in A^m(\mathbb{R})$ and $\sigma \in S^m(\mathbb{R})$. The only aspect of the bounds that requires any care is the tracking of the ξ dependence of x and x' derivatives. The basic argument that deals with this is illustrated in

$$\begin{aligned}\big|\tfrac{\partial}{\partial x}a\big(y,\tfrac{\xi}{J(x,x')},y'\big)\big| &= \big|\tfrac{\partial a}{\partial\xi}\big(y,\tfrac{\xi}{J(x,x')},y'\big)\ \xi\tfrac{J_x(x,x')}{J(x,x')^2}\big|\\ &\le C_{1,0,0}\big(1+\big|\tfrac{\xi}{J(x,x')}\big|\big)^{m-1}\big|\xi\tfrac{J_x(x,x')}{J(x,x')^2}\big|\\ &\le \text{const}\ \big(1+|\xi|\big)^{m-1}|\xi|\\ &\le \text{const}\ \big(1+|\xi|\big)^{m}\end{aligned}$$

The bound on $\mathfrak{P}_a^\Phi f - \mathfrak{P}_\sigma f$ follows immediately from Proposition 2.A.15 with $\ell = 0$ and the observation that

$$J(x,x) = \Phi^{-1\prime}(x) = \tfrac{1}{\Phi'(\Phi^{-1}(x))}$$

■

Problem 2.A.11 Let Φ be a diffeomorphism of $\mathbb{R}$ and $J(x,x') = \int_0^1 \Phi^{-1\prime}\big(tx+(1-t)x'\big)\,dt$. Prove that, for each $k \in \mathbb{N}_0$,

$$\tfrac{\partial^k}{\partial x'^k} J(x,x')\big|_{x'=x} = \tfrac{1}{k+1}\tfrac{\partial^{k+1}}{\partial x^{k+1}}\Phi^{-1}(x)$$

Problem 2.A.12 Let $m \in \mathbb{N}_0$, $q \in C^\infty(\mathbb{R})$ and $a(x,\xi) = q(x)\xi^m \in S^m(\mathbb{R})$. Then

$$\begin{aligned}\mathfrak{P}_a^\Phi f(x) &= \Big(\mathfrak{P}_a(f\circ\Phi)\Big)\big(\Phi^{-1}(x)\big) = (-i)^m q\big(\Phi^{-1}(x)\big)\tfrac{d^m}{dy^m} f\big(\Phi(y)\big)\big|_{y=\Phi^{-1}(x)}\\ &= (-i)^m q\big(\Phi^{-1}(x)\big)\begin{cases} f(x) & \text{if } m=0\\ \Phi'\big(\Phi^{-1}(x)\big)f'(x) & \text{if } m=1\\ \Phi'\big(\Phi^{-1}(x)\big)^2 f''(x) + \Phi''\big(\Phi^{-1}(x)\big)f'(x) & \text{if } m=2\end{cases}\end{aligned}$$

by the chain and product rules. Verify that Proposition 2.A.17 and Proposition 2.A.15 give the same formulae for $m = 0, 1, 2$, up to an error R that is a smoothing map. That "R is smoothing" means that for each compact $\mathcal{K} \subset \mathbb{R}$ and each $s, s' \in \mathbb{R}$ there is a constant C such that

$$\|Rf\|_{H^{s'}(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}$$

for all $f \in C_0^\infty(\mathcal{K})$.

Here is an analog of Proposition 2.A.17 for $\partial\mathbb{D}$.

Proposition 2.A.18 *Let $m \in \mathbb{R}$ and $a \in A^m(\partial\mathbb{D})$ be the restriction to $\xi \in \mathbb{Z}$ of a C^∞ function $a\big(e^{i\theta},\xi,e^{i\theta'}\big)$ that obeys (2.A.3). Let ϕ be any C^∞ diffeomorphism of $\partial\mathbb{D}$. Define, for each $k \in \mathbb{N}_0$, $\sigma_k(e^{i\theta},n) \in S^{m-k}(\partial\mathbb{D})$ by (2.A.10) below. In particular,*

$$\sigma_0(e^{i\theta},n) = a\Big(\phi^{-1}(e^{i\theta})\ ,\ \Phi'\big(\Phi^{-1}(\theta)\big)\,n\ ,\ \phi^{-1}(e^{i\theta})\Big)$$

where Φ is the C^∞ diffeomorphism of $\mathbb{R}$ (unique up to translations by integer multiples of 2π) such that $\phi(e^{i\theta}) = e^{i\Phi(\theta)}$ for all $\theta \in \mathbb{R}$. Then, for all $\ell \in \mathbb{N}_0$,

$$\Psi_a^\phi f \equiv \Big(\Psi_a(f\circ\phi)\Big)\circ\phi^{-1} = \sum_{k=0}^{\ell}\Psi_{\sigma_k} f + R_\ell f$$

where R_ℓ is a bounded map from $H^s(\partial\mathbb{D})$ to $H^{s-m+\ell+1}(\partial\mathbb{D})$ for each $s \in \mathbb{R}$.

Proof: Observe that $\Phi(\theta+2\pi n) = \Phi(\theta) + \operatorname{sgn}\Phi'\ 2\pi n$ for all $n \in \mathbb{Z}$ so that Φ' is periodic of period 2π and Φ automatically satisfies the hypotheses of Proposition 2.A.17. Let $f \in C^\infty(\partial\mathbb{D})$ and set, as in Lemma 2.A.16,

$$\tilde f(x) = f(e^{ix}) \qquad \tilde a(x,\xi,x') = a\big(e^{ix},\xi,e^{ix'}\big) \in A^m(\mathbb{R})$$

Since $\phi^{-1}(e^{i\theta}) = e^{i\Phi^{-1}(\theta)}$ and $\tilde f\circ\Phi(x) = \tilde f\big(\Phi(x)\big) = f\big(e^{i\Phi(x)}\big) = f\circ\phi\big(e^{ix}\big)$

$$\begin{aligned}
\Big(\Psi_a\big(f\circ\phi\big)\Big)\circ\phi^{-1}(e^{i\theta}) &= \Psi_a\big(f\circ\phi\big)\big(e^{i\Phi^{-1}(\theta)}\big)\\
&= \sum_{n\in\mathbb{Z}} \big(\mathfrak{P}_{\tilde a}\ \chi(\tilde f\circ\Phi)\big)(\Phi^{-1}(\theta)+2\pi n) \text{ by Lemma 2.A.16}\\
&= \sum_{n\in\mathbb{Z}} \big(\mathfrak{P}_{\tilde a}\ \chi(\tilde f\circ\Phi)\big)\big(\Phi^{-1}(\theta+2\pi n)\big)
\end{aligned}$$

In the last step, we made a change of summation index $n \to \mathrm{sgn}\Phi'\, n$. Defining $\tilde\chi = \chi\circ\Phi^{-1}$, we have, by Proposition 2.A.17,

$$\Big(\Psi_a\big(f\circ\phi\big)\Big)\circ\phi^{-1}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \big(\mathfrak{P}^{\Phi}_{\tilde a}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n) = \sum_{n\in\mathbb{Z}} \big(\mathfrak{P}_{\tilde b}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n)$$

where

$$\begin{aligned}
\tilde b(x,\xi,x') &= \tfrac{\mathcal{J}(x')}{J(x,x')}\ \tilde a\big(\Phi^{-1}(x), \tfrac{\xi}{J(x,x')}, \Phi^{-1}(x')\big) \in A^m(\mathbb{R})\\
\mathcal{J}(x') &= {\Phi^{-1}}'(x')\\
J(x,x') &= \int_0^1 {\Phi^{-1}}'\big(tx+(1-t)x'\big)\ dt
\end{aligned}$$

Set, for each $k\in\mathbb{N}_0$,

$$\begin{aligned}
\tilde b_k(x,\xi) &= (-i)^k \tfrac{1}{k!}\tfrac{\partial^k}{\partial\xi^k}\tfrac{\partial^k}{\partial x'^k}\tilde b(x,\xi,x')\big|_{x'=x} \in S^{m-k}(\mathbb{R})\\
&= (-i)^k \tfrac{1}{k!}\tfrac{\partial^k}{\partial\xi^k}\tfrac{\partial^k}{\partial x'^k}\tfrac{\mathcal{J}(x')}{J(x,x')}\ \tilde a\big(\Phi^{-1}(x), \tfrac{\xi}{J(x,x')}, \Phi^{-1}(x')\big)\Big|_{x'=x}
\end{aligned}$$

While $J(x,x')$ is not periodic in x and x', $J(x,x)$ and, for all $k\in\mathbb{N}$, $\frac{\partial^k}{\partial x'^k}J(x,x')\big|_{x'=x} = \frac{1}{k+1}\mathcal{J}^{(k)}(x)$ (see Problem 2.A.11) are periodic of period 2π. So there is, for each $k\in\mathbb{N}_0$, a $\sigma_k\in C^\infty(\partial\mathbb{D},\mathbb{R})$ such that

$$\sigma_k\big(e^{i\theta},\xi\big) = \tilde b_k(\theta,\xi) \tag{2.A.10}$$

By Proposition 2.A.15, for each $\ell'\in\mathbb{N}_0$,

$$\big(\Psi^{\phi}_a f\big)(e^{i\theta}) = \sum_{n\in\mathbb{Z}}\sum_{k=0}^{\ell'} \big(\mathfrak{P}_{\tilde b_k}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n) + \sum_{n\in\mathbb{Z}} \big(\tilde R_{\ell'}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n)$$

where $\tilde R_{\ell'}$ obeys the bounds specified in Proposition 2.A.15. Since

$$\sum_{j\in\mathbb{Z}}\tilde\chi(\theta-2\pi j) = \sum_{j\in\mathbb{Z}}\chi\big(\Phi^{-1}(\theta-2\pi j)\big) = \sum_{j\in\mathbb{Z}}\chi\big(\Phi^{-1}(\theta) - \mathrm{sgn}\Phi'\ 2\pi j\big) = 1$$

Lemma 2.A.16 gives, for each $k \in \mathbb{N}_0$,

$$\sum_{n\in\mathbb{Z}} \big(\mathfrak{P}_{\tilde b_k}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n) = \big(\Psi_{\sigma_k} f\big)(e^{i\theta})$$

so that

$$\big(\Psi_a^\phi f\big)(e^{i\theta}) = \sum_{k=0}^{\ell} \big(\Psi_{\sigma_k} f\big)(e^{i\theta}) + \big(R_\ell f\big)(e^{i\theta})$$

where

$$\big(R_\ell f\big)(e^{i\theta}) = \sum_{k=\ell+1}^{\ell'} \big(\Psi_{\sigma_k} f\big)(e^{i\theta}) + \sum_{n\in\mathbb{Z}} \big(\tilde R_{\ell'}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n)$$

Since $\sigma_k \in S^{m-k}(\partial\mathbb{D})$, Ψ_{σ_k} is a bounded linear operator from $H^s(\partial\mathbb{D})$ to $H^{s-m+k}(\partial\mathbb{D})$ for all $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Thus each Ψ_{σ_k} with $k \geq \ell+1$ obeys the bound prescribed for R_ℓ and it suffices to find, for each $s \in \mathbb{R}$, a finite $\ell' \geq \ell+1$ such that $\sum_{n\in\mathbb{Z}} \big(\tilde R_{\ell'}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n)$ is also a bounded operator from from $H^s(\partial\mathbb{D})$ to $H^{s-m+\ell+1}(\partial\mathbb{D})$.

By the second bound of Proposition 2.A.15,

$$\sup_{x\in\mathbb{R}} \big| x^\alpha \tfrac{d^\beta}{dx^\beta} \big(\tilde R_{\ell'}\tilde\chi\tilde f\big)(x)\big| \leq C\|\tilde\chi\tilde f\|_{H^\gamma(\mathbb{R})}$$

if $\gamma > m-\ell'+\beta$. In particular,

$$\sup_{x\in\mathbb{R}} \big| \tfrac{d^\beta}{d\theta^\beta} \big(\tilde R_{\ell'}\tilde\chi\tilde f\big)(\theta+2\pi n)\big| \leq \text{const}\ \tfrac{1}{1+n^2}\|\tilde\chi\tilde f\|_{H^s(\mathbb{R})}$$

for all $0 \leq \theta \leq 2\pi$ and $n \in \mathbb{Z}$, provided $s > m-\ell'+\beta$. Consequently

$$\Big\| \sum_{n\in\mathbb{Z}} \big(\tilde R_{\ell'}\ \tilde\chi\tilde f\,\big)(\theta+2\pi n)\Big\|_{H^{s-m+\ell+1}(\partial\mathbb{D})} \leq \text{const}\ \|\tilde\chi\tilde f\|_{H^s(\mathbb{R})}$$

provided $\beta \geq s-m+\ell+1$ and $s > m-\ell'+\beta$. Both of these inequalities may be satisfied by first picking a $\beta \in \mathbb{N}_0$ obeying $\beta \geq s-m+\ell+1$ and then picking $\ell' > \max\{\ell+1, m+\beta-s\}$. Hence it remains only to prove that $\|\tilde\chi\tilde f\|_{H^s(\mathbb{R})} \leq \text{const}\ \|f\|_{H^s(\partial\mathbb{D})}$.

Recall that we defined

$$\tilde f(\theta) = f\big(e^{i\theta}\big) = \sum_{n\in\mathbb{Z}} \hat f(n) e^{in\theta}$$

so that the Fourier tansform

$$\widehat{\tilde\chi\tilde f}(k) = \int_{\mathbb{R}} d\theta\ e^{-ik\theta}\tilde\chi(\theta)\tilde f(\theta) = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} d\theta\ e^{-ik\theta}\tilde\chi(\theta)\ e^{in\theta}\hat f(n) = \sum_{n\in\mathbb{Z}} \hat{\tilde\chi}(k-n)\hat f(n)$$

Since

$$\|\tilde{\chi}\tilde{f}\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} \tfrac{dk}{2\pi}\ (1+k^2)^s \big|\widehat{\tilde{\chi}\tilde{f}}(k)\big|^2 \qquad \|f\|^2_{H^s(\partial\mathbb{D})} = \sum_{n\in\mathbb{Z}} (1+n^2)^s \big|\hat{f}(n)\big|^2$$

it suffices, by Proposition 2.3.12, with $\alpha = \beta = 1$, $p = q = 2$, μ being the counting measure on $X = \mathbb{Z}$ and ν being Lebesgue measure on $Y = \mathbb{R}$, to prove that

$$\sup_k \sum_n (1+k^2)^{\frac{s}{2}} |\hat{\tilde{\chi}}(k-n)| (1+n^2)^{-\frac{s}{2}}\ ,\ \sup_n \int_{\mathbb{R}} dk\ (1+k^2)^{\frac{s}{2}} |\hat{\tilde{\chi}}(k-n)| (1+n^2)^{-\frac{s}{2}}\ < \infty$$

Since

$$\sup_{k\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \tfrac{1}{(1+|k-n|)^2} = \sup_{0\le k\le 1} \sum_{n\in\mathbb{Z}} \tfrac{1}{(1+|k-n|)^2} < \infty \text{ and } \int dk\ \tfrac{1}{(1+|k-n|)^2} = \int dk \tfrac{1}{(1+|k|)^2} < \infty$$

these two bounds will follow easily if we can prove that

$$(1+k^2)^{\frac{s}{2}} |\hat{\tilde{\chi}}(k-n)| (1+n^2)^{-\frac{s}{2}} \le \tfrac{\text{const}}{(1+|k-n|)^2}$$

As $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ there is, for each $K \in \mathbb{N}$, a constant such that $\big|\hat{\tilde{\chi}}(k-n)\big| \le \frac{D_K}{(1+|k-n|^2)^{K/2}}$ for all $k \in \mathbb{R}$ and $n \in \mathbb{N}$. Choosing $K = |s-m| + 2$ and using Lemma 2.3.4 (Peetre's inequality) to prove

$$(1+k^2)^{\frac{s}{2}} \le 2^{\frac{|s|}{2}} (1+|k-n|^2)^{\frac{|s-m|}{2}} (1+n^2)^{\frac{s}{2}}$$

does the job. ■

Example 2.A.19 Let ϕ be any C^∞ diffeomorphism of $\partial\mathbb{D}$ and Φ be a C^∞ diffeomorphism of $\mathbb{R}$ (unique up to translations by integer multiples of 2π) such that $\phi(e^{i\theta}) = e^{i\Phi(\theta)}$ for all $\theta \in \mathbb{R}$. Denote by ς_ϕ the sign of Φ'. If $\varsigma_\phi = +1$, then ϕ is orientation preserving. If $\varsigma_\phi = -1$, then ϕ is orientation reversing. Let $\sigma(x) \in C^\infty(\mathbb{R})$ take the value $+1$ for all $x \ge 1$, take the value -1 for all $x \le -1$ and take the value 0 at $x = 0$. The restriction of σ to integer arguments is

$$\text{sgn}(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -1 & \text{if } n < 0 \end{cases}$$

In the notation of Example 2.A.7

$$\Psi_\sigma = \mathcal{P}_+ - \mathcal{P}_-$$

To apply Proposition 2.A.18, with $a\big(e^{i\theta},\xi,e^{i\theta'}\big)=\sigma(\xi)$, we first compute

$$\begin{aligned}\tilde a(x,\xi,x') &= a\big(e^{ix},\xi,e^{ix'}\big)=\sigma(\xi)\\ \tilde b(x,\xi,x') &= \tfrac{\mathcal{J}(x')}{J(x,x')}\ \tilde a\big(\Phi^{-1}(x),\tfrac{\xi}{J(x,x')},\Phi^{-1}(x')\big)=\tfrac{\mathcal{J}(x')}{J(x,x')}\ \sigma\big(\tfrac{\xi}{J(x,x')}\big)\\ \tilde b_k(x,\xi) &= (-i)^k\tfrac{1}{k!}\tfrac{\partial^k}{\partial\xi^k}\tfrac{\partial^k}{\partial x'^k}\tilde b(x,\xi,x')\big|_{x'=x}\end{aligned}$$

where

$$\mathcal{J}(x')=\Phi^{-1\prime}(x')\qquad J(x,x')=\int_0^1\Phi^{-1\prime}\big(tx+(1-t)x'\big)\ dt$$

In particular

$$\sigma_0\big(e^{i\theta},\xi\big)=\tilde b_0(\theta,\xi)=\tilde b(\theta,\xi,\theta')\big|_{\theta'=\theta}=\sigma\big(\tfrac{\xi}{\mathcal{J}(\theta)}\big)$$

Since $\mathcal{J}(\theta)$ is bounded away from zero and has sign ς_ϕ, $\sigma_0\big(e^{i\theta},\xi\big)-\varsigma_\phi\sigma(\xi)$ has compact support in ξ. So, by Remark 2.A.4,

$$\sigma_0\big(e^{i\theta},\xi\big)-\varsigma_\phi\sigma(\xi)\in A^\infty(\partial\mathbb{D})\equiv\bigcap_{m'\in\mathbb{R}}A^{m'}(\partial\mathbb{D})$$

and $\Psi_{\sigma_0}-\varsigma_\phi\Psi_\sigma=\Psi_{\sigma_0-\varsigma_\phi\sigma}$ is a bounded map from $H^s(\partial\mathbb{D})$ to $H^{s'}(\partial\mathbb{D})$ for all $s,s'\in\mathbb{R}$. Since $\frac{d^k}{d\xi^k}\sigma(\xi)=0$ for all $k\in\mathbb{N}$ and $|\xi|\ge1$ and since $J(x,x')$ is bounded away from zero, $\tilde b_k(x,\xi)$ has compact support in ξ for all $k\in\mathbb{N}$. Again, by Remark 2.A.4, for all $k\in\mathbb{N}$,

$$\sigma_k\big(e^{i\theta},\xi\big)=\tilde b_k(\theta,\xi)\in A^\infty(\partial\mathbb{D})$$

So, by Proposition 2.A.18,

$$\Psi^\phi_\sigma=\varsigma_\phi\Psi_\sigma+R$$

where R is a bounded map from $H^s(\partial\mathbb{D})$ to $H^{s'}(\partial\mathbb{D})$ for all $s,s'\in\mathbb{R}$.

Example 2.A.20 In the notation of Example 2.A.19, for odd $m\in\mathbb{N}$,

$$\Psi_{|n|^m}f=\Psi_{n^m}\Psi_\sigma f=(-i)^m\tfrac{d^m}{d\theta^m}\big(\mathcal{P}_+-\mathcal{P}_-\big)f$$

For any $m\in\mathbb{N}$,

$$\begin{aligned}\Psi^\phi_{n^m}f(e^{i\theta}) &= \Big(\Psi_{n^m}\big(f\circ\phi\big)\Big)\big(\phi^{-1}(e^{i\theta})\big)=(-i)^m\tfrac{d^m}{d\vartheta^m}f\big(\phi(e^{i\vartheta})\big)\big|_{e^{i\vartheta}=\phi^{-1}(e^{i\theta})}\\ &= (-i)^m\tfrac{d^m}{d\vartheta^m}f\big(e^{i\Phi(\vartheta)}\big)\big|_{\vartheta=\Phi^{-1}(\theta)}\end{aligned}$$

In particular, for $m=1$,

$$\Psi^\phi_{n^m}f(e^{i\theta})=(-i)\Phi'\big(\Phi^{-1}(\theta)\big)\tfrac{d}{d\theta}f\big(e^{i\theta}\big)$$

by the chain and product rules and

$$\begin{aligned}\Psi^\phi_{|n|}f(e^{i\theta}) &= \Psi^\phi_n\Psi^\phi_\sigma f(e^{i\theta})=(-i)\Phi'\big(\Phi^{-1}(\theta)\big)\tfrac{d}{d\theta}\Psi^\phi_\sigma f(e^{i\theta})\\ &= (-i)\varsigma_\phi\Phi'\big(\Phi^{-1}(\theta)\big)\tfrac{d}{d\theta}\Psi_\sigma f(e^{i\theta})+\tilde Rf\\ &= (-i)\varsigma_\phi\Phi'\big(\Phi^{-1}(\theta)\big)\tfrac{d}{d\theta}\big(\mathcal{P}_+-\mathcal{P}_-\big)f(e^{i\theta})+\tilde Rf\end{aligned}$$

where $\tilde R$ is a bounded map from $H^s(\partial\mathbb{D})$ to $H^{s'}(\partial\mathbb{D})$ for all $s,s'\in\mathbb{R}$.