Appendix S4: Problem Solutions for §4

Problem 4.1 Set $q_1 = -\frac{\Delta\gamma_1^{1/2}}{\gamma_1^{1/2}}$, $q_2 = -\frac{\Delta\gamma_2^{1/2}}{\gamma_2^{1/2}}$ and $v = \log\big(\frac{\gamma_1}{\gamma_2}\big)$. Show that

$$\nabla\cdot\big((\gamma_1\gamma_2)^{\frac{1}{2}}\nabla v\big) = 2(\gamma_1\gamma_2)^{\frac{1}{2}}(q_2 - q_1)$$

Solution. Write $\mu_1 = \gamma_1^{1/2}$ and $\mu_2 = \gamma_2^{1/2}$. In terms of the μ_i's, $q_1 = -\frac{\Delta\mu_1}{\mu_1}$, $q_2 = -\frac{\Delta\mu_2}{\mu_2}$, $v = 2\log\big(\frac{\mu_1}{\mu_2}\big)$ and we are to show that $\nabla\cdot\big(\mu_1\mu_2\nabla v\big) = 2\mu_1\mu_2(q_2 - q_1)$. Subtracting

$$\begin{aligned}\nabla\cdot\big(\mu_1\mu_2\nabla\log(\mu_1)\big) &= \nabla\cdot\big(\mu_2\nabla\mu_1\big) = \nabla\mu_2\cdot\nabla\mu_1 + \mu_2\Delta\mu_1\\ \nabla\cdot\big(\mu_1\mu_2\nabla\log(\mu_2)\big) &= \nabla\cdot\big(\mu_1\nabla\mu_2\big) = \nabla\mu_1\cdot\nabla\mu_2 + \mu_1\Delta\mu_2\end{aligned}$$

gives

$$\tfrac{1}{2}\nabla\cdot\big(\mu_1\mu_2\nabla v\big) = \mu_2\Delta\mu_1 - \mu_1\Delta\mu_2 = \mu_1\mu_2\big(-q_1 + q_2\big)$$

as desired. ■

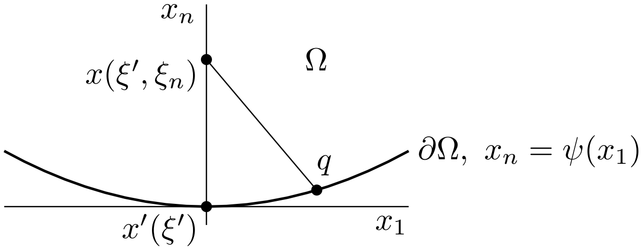
Problem 4.1.1 Let $p \in \partial\Omega$. Let $x'(\xi')$ be a C^∞ parametrization of a neighbourhood of p in $\partial\Omega$ with $x'(0) = p$. Denote by $\hat{n}(x')$ the unit outward normal to $\partial\Omega$ at $x' \in \partial\Omega$. Define $x(\xi',\xi_n) = x'(\xi') - \xi_n\hat{n}\big(x'(\xi')\big)$.

(a) Prove that $x(\xi',\xi_n)$ is a C^∞ diffeomorphism from a neighbourhood of $0 \in \mathbb{R}^n$ to a neighbourhood of $p \in \mathbb{R}^n$.

(b) Prove that, for all sufficiently small (ξ',ξ_n), $x'(\xi')$ is the point of $\partial\Omega$ that is nearest $x(\xi',\xi_n)$, so that the distance from $x(\xi',\xi_n)$ to $\partial\Omega$ is $|\xi_n|$.

Solution. (a) That $x(\xi',\xi_n)$ is C^∞ is obvious, because both $\hat{n}(x')$ and $x'(\xi')$ are. Hence it suffices to prove that the Jacobian of the map $(\xi',\xi_n) \mapsto x(\xi',\xi_n)$ does not vanish at the origin. The Jacobian is the determinant of the Hessian matrix. The columns of the Hessian matrix are the n vectors $\frac{\partial}{\partial\xi'_j}x(0) = \frac{\partial}{\partial\xi'_j}x'(0)$, $1 \le j \le n-1$ and $\frac{\partial}{\partial\xi_n}x(0) = -\hat{n}(p)$. The first $n-1$ of these vectors are independent and tangential to $\partial\Omega$ at p, since $x'(\xi')$ is assumed to be a parametrization of $\partial\Omega$ near p. Hence the first $n-1$ of these vectors are also orthogonal to the n^{th}. So the n vectors are independent and the Jacobian does not vanish.

(b) Fix any sufficiently small $(\xi',\xi_n) \in \mathbb{R}^n$ and any $q \in \partial\Omega$, different from $x'(\xi')$ and sufficiently near p. We must show that the distance from $x(\xi',\xi_n)$ to $x'(\xi')$ is strictly less than the distance from $x(\xi',\xi_n)$ to q. By rotating and translating, in the original x space, we may assume that $x'(\xi')$ is at the origin, that $x(\xi',\xi_n)$ lies on the positive x_n axis, $\partial\Omega$ is tangential to the hyperplane $x_n=0$ at $x'(\xi')$ and that the components numbered 2 through $n-1$, inclusive, of q vanish. Then the equation of the intersection of $\partial\Omega$ with the x_1–x_n plane (which contains q) near the origin is of the form $x_n=\psi(x_1)$ with $\psi(0)=\psi'(0)=0$. If the x_n–coordinate of $x(\xi',\xi_n)$ is y and the x_1–coordinate of q is x, then the distance from $x(\xi',\xi_n)$ to $x'(\xi')$ is y and the distance from $x(\xi',\xi_n)$ to q is $\sqrt{x^2+(y-\psi(x))^2}$. Note that



$$\tfrac{d}{dx}\big(x^2+(y-\psi(x))^2\big)=2x-2(y-\psi(x))\psi'(x)=2\Big[1-(y-\psi(x))\tfrac{\psi'(x)}{x}\Big]x$$

Since $\psi'(0)=0$, $\frac{\psi'(x)}{x}$ is bounded. So for all y and x sufficiently small $\big|(y-\psi(x))\frac{\psi'(x)}{x}\big|\le\frac{1}{2}$. Thus the derivative of the distance function $\sqrt{x^2+(y-\psi(x))^2}$ vanishes at $x=0$ and has the same sign as x. So the distance is minimized at $x=0$ and the distance from $x(\xi',\xi_n)$ to $x'(\xi')$ is strictly less than the distance from $x(\xi',\xi_n)$ to q. ■

Problem 4.1.2 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Suppose that

$$\begin{aligned} L_\gamma u &= F \qquad \text{in } \Omega \\ u\big|_{\partial\Omega} &= f \end{aligned}$$

Let $\mathcal{U}$ and $\mathcal{U}'$ be open neighbourhoods in $\mathbb{R}^n$ of some $p\in\partial\Omega$ with $\overline{\mathcal{U}}\subset\mathcal{U}'$. Prove that, given any $m\in\mathbb{N}$ there exists a C, independent of p, such that

$$\|u\|_{m,\Omega\cap\mathcal{U}}\le C\Big\{\|F\|_{m-2,\Omega\cap\mathcal{U}'}+\|f\|_{m-\frac{1}{2},\partial\Omega\cap\mathcal{U}'}+\|u\|_{L^2(\Omega\cap\mathcal{U}')}\Big\}$$

Solution. We first consider the case $m=1$. Let $\chi,\tilde\chi\in C_0^\infty(\mathcal{U}')$ denote smooth cutoff functions with χ identically one on $\mathcal{U}$ and $\tilde\chi$ identically one on the support of χ. Let w be the solution to

$$\begin{aligned} L_\gamma w &= \tilde\chi F \qquad \text{in } \Omega \\ w\big|_{\partial\Omega} &= \tilde\chi f \end{aligned}$$

From the standard energy estimate of Theorem 3.3.5, we get

$$\begin{aligned}\|w\|_{L^2(\Omega)} \le \|w\|_{1,\Omega} &\le C\big(\|\tilde\chi F\|_{-1,\Omega} + \|\tilde\chi f\|_{\frac12,\partial\Omega}\big)\\ &\le C\big(\|F\|_{-1,\Omega\cap\mathcal{U}'} + \|f\|_{\frac12,\partial\Omega\cap\mathcal{U}'}\big)\end{aligned}\tag{S4.1}$$

by part (a) of Lemma 2.1.16 and Problem 2.3.5. We have used that $\tilde\chi$ vanishes outside of $\mathcal{U}'$. The constant C and those that will follow depend on γ, Ω and the particular choices of χ and $\tilde\chi$, but are independent of p. Since $\tilde\chi(x) = 1$ on the support of χ,

$$\begin{aligned}\int_\Omega \gamma\nabla(u-w)\nabla(\chi^2 v)\ d^nx &= -\int_\Omega L_\gamma(u-w)(\chi^2 v)\ d^nx\\ &= -\int_\Omega F\,(1-\tilde\chi)\,(\chi^2 v)\ d^nx = 0\end{aligned}\tag{S4.2}$$

for any $v \in H^1(\Omega)$ with $\chi^2 v = 0$ on $\partial\Omega$. Similarly, the function $u - w$ satisfies

$$\chi^2(u-w) = \chi^2(1-\tilde\chi)f = 0 \qquad \text{on } \partial\Omega$$

By inserting $v = \overline{u-w}$ into (S4.2) and expanding $\nabla(\chi^2 v) = \chi^2\nabla v + 2\chi v\nabla\chi$ we get

$$\int_\Omega \gamma\chi^2|\nabla(u-w)|^2\ d^nx = -\int_\Omega \gamma\nabla(u-w)2\chi\nabla\chi\ (\,\overline{u-w}\,)\ d^nx$$

By the Cauchy–Schwarz inequality, followed by $AB = (\alpha A)(\frac{1}{\alpha}B) \le \frac12(\alpha A)^2 + \frac12(\frac{1}{\alpha}B)^2$ with $\alpha^2 = \frac{\min_\Omega\gamma}{2\max_\Omega\gamma}$,

$$\begin{aligned}\int_\Omega \gamma\chi^2|\nabla(u-w)|^2\ d^nx &\le 2\max_\Omega\gamma\ \Big(\int_\Omega \chi^2|\nabla(u-w)|^2\ d^nx\Big)^{1/2}\Big(\int_\Omega |\nabla\chi|^2|u-w|^2\ d^nx\Big)^{1/2}\\ &\le \tfrac12\min_\Omega\gamma\int_\Omega \chi^2|\nabla(u-w)|^2\ d^nx + C\int_\Omega |\nabla\chi|^2|u-w|^2\ d^nx\\ &\le \tfrac12\min_\Omega\gamma\int_\Omega \chi^2|\nabla(u-w)|^2\ d^nx + C\|u-w\|^2_{L^2(\Omega\cap\mathcal{U}')}\\ &\le \tfrac12\int_\Omega \gamma\chi^2|\nabla(u-w)|^2\ d^nx + C\|u-w\|^2_{L^2(\Omega\cap\mathcal{U}')}\end{aligned}$$

Subtracting the first term on the right hand side from the left hand side, we get

$$\int_\Omega \gamma\chi^2|\nabla(u-w)|^2\ d^nx \le C\|u-w\|^2_{L^2(\Omega\cap\mathcal{U}')}$$

which, since $\chi = 1$ on $\Omega\cap\mathcal{U}$ and γ is bounded away from zero, immediately yields

$$\begin{aligned}\|u-w\|^2_{1,\Omega\cap\mathcal{U}} &= \int_{\Omega\cap\mathcal{U}} |\nabla(u-w)|^2\ d^nx + \int_{\Omega\cap\mathcal{U}} |u-w|^2\ d^nx\\ &\le \tfrac{1}{\min_\Omega\gamma}\int_\Omega \gamma\chi^2|\nabla(u-w)|^2\ d^nx + \int_{\Omega\cap\mathcal{U}} |u-w|^2\ d^nx\\ &\le C\|u-w\|^2_{L^2(\Omega\cap\mathcal{U}')}\end{aligned}$$

or

$$\|u-w\|_{1,\Omega\cap\mathcal{U}} \le C\|u-w\|_{L^2(\Omega\cap\mathcal{U}')} \tag{S4.3}$$

A combination of (S4.1) and (S4.3) yields

$$\begin{aligned}
\|u\|_{1,\Omega\cap\mathcal{U}} &\le \|w\|_{1,\Omega} + \|u-w\|_{1,\Omega\cap\mathcal{U}} \\
&\le C\big\{\|F\|_{-1,\Omega\cap\mathcal{U}'} + \|f\|_{\frac{1}{2},\partial\Omega\cap\mathcal{U}'} + \|u-w\|_{L^2(\Omega\cap\mathcal{U}')}\big\} \\
&\le C\big\{\|F\|_{-1,\Omega\cap\mathcal{U}'} + \|f\|_{\frac{1}{2},\partial\Omega\cap\mathcal{U}'} + \|w\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega\cap\mathcal{U}')}\big\} \\
&\le C\big\{\|F\|_{-1,\Omega\cap\mathcal{U}'} + \|f\|_{\frac{1}{2},\partial\Omega\cap\mathcal{U}'} + \|u\|_{L^2(\Omega\cap\mathcal{U}')}\big\}
\end{aligned} \tag{S4.4}$$

which is exactly the desired inequality for the case $m=1$.

We now turn to the case $m \ge 2$. Let

$$\mathcal{U}_0 = \mathcal{U} \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_{m-1} \subset \mathcal{U}_m = \mathcal{U}'$$

be open subsets of $\mathbb{R}^n$ with $\overline{\mathcal{U}_{i-1}} \subset \mathcal{U}_i$ for all $1 \le i \le m$ and with the support of χ contained in $\mathcal{U}_1$. As in (3.3.4), the function χu satisfies

$$\begin{aligned}
L_\gamma(\chi u) &= \chi F + 2\gamma\nabla\chi\cdot\nabla u + uL_\gamma\chi \\
\chi u\big|_{\partial\Omega} &= \chi f
\end{aligned} \tag{S4.5}$$

As the right-hand side of (S4.5) vanishes outside $\mathcal{U}_1$, the standard elliptic estimate of Proposition 3.3.10 gives

$$\begin{aligned}
\|u\|_{m,\Omega\cap\mathcal{U}} \le \|\chi u\|_{m,\Omega} &\le C\big\{\|\chi F\|_{m-2,\Omega\cap\mathcal{U}_1} + \|\gamma\nabla\chi\cdot\nabla u\|_{m-2,\Omega\cap\mathcal{U}_1} \\
&\qquad + \|uL_\gamma\chi\|_{m-2,\Omega\cap\mathcal{U}_1} + \|\chi f\|_{m-\frac{1}{2},\partial\Omega\cap\mathcal{U}_1}\big\} \\
&\le C\big\{\|F\|_{m-2,\Omega\cap\mathcal{U}_1} + \|f\|_{m-\frac{1}{2},\partial\Omega\cap\mathcal{U}_1} + \|u\|_{m-1,\Omega\cap\mathcal{U}_1}\big\}
\end{aligned}$$

by part (a) of Lemma 2.1.16 and Problem 2.3.5. Similarly, for each $j=1,\cdots,m-1$

$$\begin{aligned}
\|u\|_{m-j+1,\Omega\cap\mathcal{U}_{j-1}} &\le C\big\{\|F\|_{m-j-1,\Omega\cap\mathcal{U}_j} + \|f\|_{m-j+\frac{1}{2},\partial\Omega\cap\mathcal{U}_j} + \|u\|_{m-j,\Omega\cap\mathcal{U}_j}\big\} \\
&\le C\big\{\|F\|_{m-2,\Omega\cap\mathcal{U}'} + \|f\|_{m-\frac{1}{2},\partial\Omega\cap\mathcal{U}'} + \|u\|_{m-j,\Omega\cap\mathcal{U}_j}\big\}
\end{aligned}$$

Iterating,

$$\|u\|_{m,\Omega\cap\mathcal{U}_0} \le C\big\{\|F\|_{m-2,\Omega\cap\mathcal{U}'} + \|f\|_{m-\frac{1}{2},\partial\Omega\cap\mathcal{U}'} + \|u\|_{1,\Omega\cap\mathcal{U}_{m-1}}\big\}$$

Applying (S4.4) with $\mathcal{U}$ replaced by $\mathcal{U}_{m-1}$ gives the desired bound. ■

Problem 4.1.3 In this problem we construct harmonic functions on $\mathbb{R}^n_+$ that are concentrated near the origin. We denote $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Let

$$P_{x_n}(x') = \tfrac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \tfrac{x_n}{(x_n^2 + |x'|^2)^{n/2}}$$

be the Poisson kernel.

(a) Prove that $\Delta P_{x_n}(x') = 0$ for all $x_n > 0$. Here Δ is the Laplacian on $\mathbb{R}^n$.

(b) Prove that $\int_{\mathbb{R}^{n-1}} P_{x_n}(x')\ d^{n-1}x' = 1$ for all $x_n > 0$.

(c) Let $\varphi(x')$ be a bounded continuous function on $\mathbb{R}^{n-1}$. Prove that

$$\Phi(x) = \int_{\mathbb{R}^{n-1}} P_{x_n}(x' - y')\varphi(y')\ d^{n-1}y'$$

obeys $\Delta\Phi(x) = 0$ in $\mathbb{R}^n_+$ and

$$\lim_{\substack{x \to (z', 0) \\ x_n > 0}} \Phi(x) = \varphi(z')$$

(d) Now suppose that $\varphi(x') = \partial^\alpha \psi(x')$ with $\psi \in C^{|\alpha|}(\mathbb{R}^{n-1})$ supported in the ball of radius 1 centred on the origin. Prove that there is a constant, which depends only on $|\alpha|$ and n, such that

$$|\Phi(x)| \le \tfrac{C}{1+|x|^{|\alpha|+n-1}}$$

Solution. (a) For any $1 \le j \le n$

$$\begin{aligned}
\tfrac{\partial}{\partial x_j} \tfrac{1}{(x_n^2+|x'|^2)^{n/2}} &= -n \tfrac{x_j}{(x_n^2+|x'|^2)^{1+n/2}} \\
\tfrac{\partial^2}{\partial x_j^2} \tfrac{1}{(x_n^2+|x'|^2)^{n/2}} &= -n \tfrac{1}{(x_n^2+|x'|^2)^{1+n/2}} + n(n+2) \tfrac{x_j^2}{(x_n^2+|x'|^2)^{2+n/2}}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{j=1}^n \tfrac{\partial^2}{\partial x_j^2} \tfrac{1}{(x_n^2+|x'|^2)^{n/2}} &= -n^2 \tfrac{1}{(x_n^2+|x'|^2)^{1+n/2}} + n(n+2) \tfrac{x_n^2+|x'|^2}{(x_n^2+|x'|^2)^{2+n/2}} \\
&= 2n \tfrac{1}{(x_n^2+|x'|^2)^{1+n/2}}
\end{aligned}$$

and

$$\begin{aligned}
\Delta \tfrac{x_n}{(x_n^2+|x'|^2)^{n/2}} &= 2\tfrac{\partial}{\partial x_n} \tfrac{1}{(x_n^2+|x'|^2)^{n/2}} + x_n \sum_{j=1}^n \tfrac{\partial^2}{\partial x_j^2} \tfrac{1}{(x_n^2+|x'|^2)^{n/2}} \\
&= -2n \tfrac{x_n}{(x_n^2+|x'|^2)^{1+n/2}} + x_n 2n \tfrac{1}{(x_n^2+|x'|^2)^{1+n/2}} \\
&= 0
\end{aligned}$$

(b) Making the scaling $x' = x_n y'$

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} P_{x_n}(x')\ d^{n-1}x' &= \tfrac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \tfrac{x_n}{(x_n^2+|x'|^2)^{n/2}}\ d^{n-1}x' \\ &= \tfrac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \tfrac{1}{(1+|y'|^2)^{n/2}}\ d^{n-1}y' \\ &= \Omega_{n-1} \tfrac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_0^\infty \tfrac{r^{n-2}}{(1+r^2)^{n/2}}\ dr\end{aligned}$$

where $\Omega_{n-1} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n}{2}-\frac{1}{2})}$ is the surface area of the unit sphere in $\mathbb{R}^{n-1}$. Substituting $r = \tan\theta$,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} P_{x_n}(x')\ d^{n-1}x' &= \tfrac{2\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2}-\frac{1}{2})} \int_0^{\pi/2} \tfrac{\tan^{n-2}\theta}{(1+\tan^2\theta)^{n/2}}\ \sec^2\theta\, d\theta \\ &= \tfrac{2\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2}-\frac{1}{2})} \int_0^{\pi/2} \sin^{n-2}\theta\ d\theta\end{aligned}$$

In the solution to Problem 1.2.2, we saw that

$$\int_0^{\pi/2} \sin^{n-2}\theta\ d\theta = \tfrac{\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2})} \tfrac{\sqrt{\pi}}{2}$$

which gives the desired result.

(c) Since

$$\begin{aligned}\Phi(x) - \varphi(z') &= \int_{\mathbb{R}^{n-1}} P_{x_n}(x'-y')\big[\varphi(y') - \varphi(z')\big]\ d^{n-1}y' \\ &= \int_{\mathbb{R}^{n-1}} P_{x_n}(y')\big[\varphi(y'+x') - \varphi(z')\big]\ d^{n-1}y' \\ &= \tfrac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \tfrac{x_n}{(x_n^2+|y'|^2)^{n/2}} \big[\varphi(y'+x') - \varphi(z')\big]\ d^{n-1}y' \\ &= \tfrac{\Gamma(\frac{n}{2})}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \tfrac{1}{(1+|y'|^2)^{n/2}} \big[\varphi(x_n y'+x') - \varphi(z')\big]\ d^{n-1}y'\end{aligned}$$

The integrand is bounded, for all x, by the integrable function $\frac{2\|\varphi\|_{L^\infty}}{(1+|y'|^2)^{n/2}}$ and converges, pointwise in y', to zero as $x \to (z',0)$. So the right hand side converges to zero as $x \to (z',0)$ by the Lebesgue dominated convergence theorem. This proves

$$\lim_{\substack{x\to(z',0)\\ x_n>0}} \Phi(x) = \varphi(z')$$

The only technicality to be dealt with in the argument

$$\begin{aligned}\Delta\Phi(x) &= \Delta\int_{\mathbb{R}^{n-1}} P_{x_n}(x'-y')\varphi(y')\ d^{n-1}y' \\ &= \int_{\mathbb{R}^{n-1}} \Delta_x P_{x_n}(x'-y')\varphi(y')\ d^{n-1}y' \\ &= 0\end{aligned}$$

is the interchange of order of Δ and $\int$. It is justified since $P_{x_n}(x')$ and all of its derivatives are all L^1 in x', uniformly for x_n in any compact subset of $(0,\infty)$.

(d) By part (b)

$$\begin{aligned}|\Phi(x)| &\le \int_{\mathbb{R}^{n-1}} P_{x_n}(x'-y')\,|\varphi(y')|\ d^{n-1}y' \\ &\le \|\varphi\|_{L^\infty(\mathbb{R}^{n-1})}\int_{\mathbb{R}^{n-1}} P_{x_n}(x'-y')\ d^{n-1}y' \\ &= \|\varphi\|_{L^\infty(\mathbb{R}^{n-1})}\end{aligned}$$

By induction

$$(\partial^\alpha P)_{x_n}(x') = \tfrac{x_n Q_\alpha(x',x_n)}{(x_n^2+|x'|^2)^{|\alpha|+n/2}}$$

with $Q_\alpha(x',x_n)$ a polynomial in x' and x_n that is homogeneous of degree $|\alpha|$. The coefficient of each term in $Q_\alpha(x',x_n)$ depends only n, α and the exponents of that term. So, for $|x|\ge 2$ and $|y'|\le 1$,

$$|x|^{n+|\alpha|-1}\big|(\partial^\alpha P)_{x_n}(x'-y')\big| \le \big(2|x-y'|\big)^{n+|\alpha|-1}\big|(\partial^\alpha P)_{x_n}(x'-y')\big| \le C$$

and, by integration by parts,

$$\begin{aligned}|x|^{n+|\alpha|-1}\,|\Phi(x)| &= |x|^{n+|\alpha|-1}\left|\int_{\mathbb{R}^{n-1}} P_{x_n}(x'-y')\,\partial^\alpha_{y'}\psi(y')\ d^{n-1}y'\right| \\ &= |x|^{n+|\alpha|-1}\left|\int_{\mathbb{R}^{n-1}} (\partial^\alpha P)_{x_n}(x'-y')\,\psi(y')\ d^{n-1}y'\right| \\ &\le C\int_{\mathbb{R}^{n-1}} |\psi(y')|\ d^{n-1}y'\Big| \\ &= C\|\psi\|_{L^1(\mathbb{R}^{n-1})}\end{aligned}$$

Considering $|x|\le 2$ and $|x|\ge 2$ separately, we see that

$$\big[1+|x|^{n+|\alpha|-1}\big]\,|\Phi(x)| \le \big[1+2^{n+|\alpha|-1}\big]\big[\|\varphi\|_{L^\infty(\mathbb{R}^{n-1})}+C\|\psi\|_{L^1(\mathbb{R}^{n-1})}\big]$$

as desired. ■

Problem 4.2.1 Let $w \in L^2_\delta$ be any weak solution to $\Delta w + 2\zeta \cdot \nabla w = 0$. Let

$$\chi \in C^\infty([0,\infty)) \qquad \text{with} \qquad \operatorname{supp}\chi \subset [0,1) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \chi(|k|^2)\, \tfrac{d^n k}{(2\pi)^n} = 1$$

and

$$w_\varepsilon(x) = \beta(\varepsilon x) w(x) \qquad \text{where} \qquad \beta(x) = \int_{\mathbb{R}^n} e^{ik\cdot x}\chi(|k|^2)\, \tfrac{d^n k}{(2\pi)^n}$$

(a) Prove that if the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ vanishes in $N_\varepsilon\big(\mathcal{M}(s)\big) = \big\{\ k \ \big|\ \operatorname{dist}(k, \mathcal{M}(s)) \le \varepsilon\ \big\}$, then there is a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\beta(\varepsilon x)\varphi(x) = \Delta\psi - 2\bar\zeta \cdot \nabla\psi$$

(b) Prove that $\hat w_\varepsilon(k)$ is supported in $N_\varepsilon\big(\mathcal{M}(s)\big)$.

Solution. (a) The Fourier transform of $\beta(\varepsilon x)$ is $\frac{1}{\varepsilon^n}\hat\beta\big(\frac{k}{\varepsilon}\big) = \frac{1}{\varepsilon^n}\chi\big(\big|\frac{k}{\varepsilon}\big|^2\big)$ and vanishes unless $|k| \le \varepsilon'$ for some $\varepsilon' < \varepsilon$. Hence the Fourier transform of $\varphi_\varepsilon(x) = \beta(\varepsilon x)\varphi(x)$, which is the convolution of the Fourier transform of $\beta(\varepsilon x)$ with the Fourier transform of φ, is supported within a distance ε' of the support of the Fourier transform of φ. In particular $\hat\varphi_\varepsilon(k)$ vanishes on an open neighbourhood of $\mathcal{M}(s)$, which is the zero set of $-|k|^2 - 2i\bar\zeta\cdot k$. Furthermore, since $\big| -|k|^2 - 2i\bar\zeta\cdot k\big| \to \infty$ as $|k| \to \infty$, $-|k|^2 - 2i\bar\zeta\cdot k$ is bounded away from zero on the support of $\hat\varphi_\varepsilon(k)$. Consequently,

$$\hat\psi(k) \equiv \tfrac{\hat\varphi_\varepsilon(k)}{-|k|^2 - 2i\bar\zeta\cdot k} \in \mathcal{S}(\mathbb{R}^n)$$

and $\beta(\varepsilon x)\varphi(x) = \Delta\psi - 2\bar\zeta\cdot\nabla\psi$ as desired.

(b) If the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ vanishes in $N_\varepsilon\big(\mathcal{M}(s)\big)$, then, using the notation of part (a),

$$\langle \varphi, w_\varepsilon\rangle = \langle \varphi_\varepsilon, w\rangle = \big\langle \Delta\psi - 2\bar\zeta\cdot\nabla\psi, w\big\rangle = 0$$

since w is a weak solution of $\Delta w + 2\zeta\cdot\nabla w = 0$. As the Fourier transforms of such φ's are dense in $L^2\Big(\mathbb{R}^n \setminus N_\varepsilon\big(\mathcal{M}(s)\big)\Big)$, $\hat w_\varepsilon(k)$ is supported in $N_\varepsilon\big(\mathcal{M}(s)\big)$. ■

Problem 4.2.2 Let $s \in \mathbb{R}$. Let $\mathcal{O}$ and $\mathcal{O}'$ be open subsets of $\mathbb{R}^n$ and $\mathcal{C}$ be a compact subset of $\mathcal{O}'$. Let Ψ be a smooth diffeomorphism from $\mathcal{O}$ to $\mathcal{O}'$. Prove that there is a constant C, depending only on Ψ, s, $\mathcal{O}$ and $\mathcal{C}$, such that

$$\big|u \circ \Psi^{-1}\big|_{s,n} \le C|u|_{s,n}$$

for all $u \in C_0^\infty(\mathcal{C})$.

Solution. For $s \in \mathbb{N}_0$, this follows from part (b) of Lemma 2.1.23, with $\varphi \in C_0^\infty(\mathcal{O})$ chosen to be identically one on $\mathcal{C}$. Next consider $s \in \mathbb{Z}$ with $s < 0$. Again let $\varphi \in C_0^\infty(\mathcal{O}')$ be identically one on $\Psi(\mathcal{C})$. Then, making the change of variables $x = \Psi(y)$,

$$\big\langle v, \varphi(u \circ \Psi^{-1})\big\rangle_{L^2(\mathcal{O}')} = \langle (D\Psi)(\varphi v \circ \Psi), u\rangle_{L^2(\mathcal{O})}$$

for all $v \in C_0^\infty(\mathcal{O}')$. By Problem 2.1.16.b and part (b) of Lemma 2.1.23, again,

$$\Big| \big\langle v, \varphi(u \circ \Psi^{-1})\big\rangle_{L^2(\mathcal{O}')} \Big| \le \|(D\Psi)(\varphi v \circ \Psi)\|_{-s,n} \; \|u\|_{s,n} \le \text{const}\, \|v\|_{-s,n} \; \|u\|_{s,n}$$

which implies $\|u \circ \Psi^{-1}\|_{s,n} \le \text{const}\, \|u\|_{s,n}$ by Problem 2.1.16.b again. Finally, for general $s \in \mathbb{R}^n$, just use interpolation (Lemma 2.3.2). ■

Problem 4.2.3

(a) Prove that

$$\int_{|x|\le R} \tfrac{e^{-ik\cdot x}}{-x_2+ix_1} d^2x = \tfrac{2i}{-k_2+ik_1} \int_0^\pi d\theta \; \big[e^{-i|k|R\cos\theta} - 1\big]$$

(b) Prove that

$$\bigg| \int_{|x|\le R} \tfrac{e^{-ik\cdot x}}{-x_2+ix_1} d^2x \bigg| \le \tfrac{4\pi}{|-k_2+ik_1|} \qquad \text{and} \qquad \lim_{R\to\infty} \int_{|x|\le R} \tfrac{e^{-ik\cdot x}}{-x_2+ix_1} d^2x = \tfrac{-2\pi i}{-k_2+ik_1}$$

for all $k \ne 0$.

Solution. (a) Let $k = (|k|\cos\varphi, |k|\sin\varphi)$. Then $k \cdot x = |k|\big(x_1\cos\varphi + x_2\sin\varphi\big)$. Make the change of variables

$$\begin{aligned} y_1 &= x_1\cos\varphi + x_2\sin\varphi & x_1 &= y_1\cos\varphi - y_2\sin\varphi \\ y_2 &= -x_1\sin\varphi + x_2\cos\varphi & x_2 &= y_1\sin\varphi + y_2\cos\varphi \end{aligned}$$

Then

$$k \cdot x = |k| y_1 \quad \text{and} \quad -x_2 + ix_1 = ie^{i\varphi}(y_1 + iy_2)$$

so that

$$\int_{|x|\le R} \tfrac{e^{-ik\cdot x}}{-x_2+ix_1} d^2x = -ie^{-i\varphi} \int_{|y|\le R} \tfrac{e^{-i|k|y_1}}{y_1+iy_2} d^2y$$

Scaling, $u = |k|y$, and then going to polar coordinates,

$$\begin{aligned}\int_{|x|\le R} \tfrac{e^{-ik\cdot x}}{-x_2+ix_1} d^2x &= -ie^{-i\varphi}\tfrac{1}{|k|}\int_{|u|\le R|k|} \tfrac{e^{-iu_1}}{u_1+iu_2} d^2u = \tfrac{1}{-k_2+ik_1}\int_0^{2\pi} d\theta \int_0^{R|k|} dr\ r\tfrac{e^{-ir\cos\theta}}{re^{i\theta}} \\ &= \tfrac{1}{-k_2+ik_1}\int_{-\pi}^{\pi} d\theta\ \tfrac{e^{-i|k|R\cos\theta}-1}{-i\cos\theta e^{i\theta}} \\ &= \tfrac{i}{-k_2+ik_1}\int_{-\pi}^{\pi} d\theta\ \left[e^{-i|k|R\cos\theta}-1\right]\left[1-i\tan\theta\right]\end{aligned}$$

Since $\left[e^{-i|k|R\cos\theta}-1\right]\tan\theta$ is odd under $\theta\to-\theta$ and $\left[e^{-i|k|R\cos\theta}-1\right]$ is even under $\theta\to-\theta$,

$$\int_{|x|\le R} \tfrac{e^{-ik\cdot x}}{-x_2+ix_1} d^2x = \tfrac{2i}{-k_2+ik_1}\int_0^{\pi} d\theta\ \left[e^{-i|k|R\cos\theta}-1\right]$$

(b) The inequality is obvious, since $\left|e^{-i|k|R\cos\theta}-1\right|\le 2$. To complete the proof, it suffices to prove that

$$\lim_{R\to\infty}\int_0^{\pi} d\theta\ e^{-i|k|R\cos\theta} = 0$$

To do so, let $\varepsilon > 0$ and set $I_\varepsilon = \left\{\ 0\le\theta\le\pi\ \middle|\ 0\le\theta\le\varepsilon,\ \pi-\varepsilon\le\theta\le\pi\ \right\}$ and $J_\varepsilon = \left\{\ \theta\ \middle|\ \varepsilon\le\theta\le\pi-\varepsilon\ \right\}$. Then

$$\left|\int_{I_\varepsilon} d\theta\ e^{-i|k|R\cos\theta}\right| \le 2\varepsilon$$

and

$$\begin{aligned}\lim_{R\to\infty}\int_{J_\varepsilon} d\theta\ e^{-i|k|R\cos\theta} &= \lim_{R\to\infty}\left[\tfrac{e^{-i|k|R\cos\theta}}{-i|k|R\sin\theta}\Big|_{\varepsilon}^{\pi-\varepsilon} - \int_{J_\varepsilon} d\theta\ e^{-i|k|R\cos\theta}\tfrac{d}{d\theta}\tfrac{1}{-i|k|R\sin\theta}\right] \\ &= 0\end{aligned}$$

since $\sin\theta$ is bounded away from 0 on J_ε. This shows that

$$\limsup_{R\to\infty}\left|\int_0^{\pi} d\theta\ e^{-i|k|R\cos\theta}\right| \le 2\varepsilon$$

for all $\varepsilon > 0$ so that the limit is zero. ■

Problem 4.3.1 Prove that if the spaces $\mathcal{C}_{q_j}$ are both graphs of the corresponding Dirichlet–to Neumann–data maps Λ_{q_j}, then one has the estimates

$$\frac{\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac{1}{2},-\frac{1}{2}}}{\sqrt{1+\|\Lambda_{q_1}\|^2_{\frac{1}{2},-\frac{1}{2}}}\sqrt{1+\|\Lambda_{q_2}\|^2_{\frac{1}{2},-\frac{1}{2}}}} \le \operatorname{dist}\left(\mathcal{C}_{q_1},\mathcal{C}_{q_2}\right) \le \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac{1}{2},-\frac{1}{2}}$$

Solution. If the space $\mathcal{C}_q$ is the graph of the Dirichlet– to Neumann–data map Λ_q, then

$$\mathcal{C}_q = \Big\{(f,g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) \ \Big|\ g = \Lambda_q f \ \Big\}$$

For any $(f,g) = (f, \Lambda_{q_1} f) \in \mathcal{C}_{q_1}$

$$\begin{aligned}\inf_{(\tilde f,\tilde g)\in\mathcal{C}_{q_2}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}} &\le \frac{\|(f,\Lambda_{q_1}f)-(f,\Lambda_{q_2}f)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,\Lambda_{q_1}f)\|_{H^{1/2}\oplus H^{-1/2}}} \\ &= \frac{\|(\Lambda_{q_1}-\Lambda_{q_2})f\|_{-1/2,\partial\Omega}}{\sqrt{\|f\|^2_{1/2,\Omega}+\|\Lambda_{q_1}f\|^2_{-1/2,\Omega}}} \\ &\le \frac{\|(\Lambda_{q_1}-\Lambda_{q_2})f\|_{-1/2,\partial\Omega}}{\|f\|_{1/2,\Omega}} \\ &\le \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12}\end{aligned}$$

Similarly, for any $(f,g) = (f, \Lambda_{q_2} f) \in \mathcal{C}_{q_2}$

$$\inf_{(\tilde f,\tilde g)\in\mathcal{C}_{q_1}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}} \le \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12}$$

so that

$$\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) \le \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12}$$

which is the desired right hand inequality.

We now prove the left hand inequality. For any $(f,g) = (f,\Lambda_{q_1}f) \in \mathcal{C}_{q_1}$ we express

$$\mathcal{C}_{q_2} \ni (\tilde f,\tilde g) = (\tilde f, \Lambda_{q_2}\tilde f) = (f,\Lambda_{q_1}f) + (\phi,\Lambda_{q_2}\phi) + (0,[\Lambda_{q_2}-\Lambda_{q_1}]f)$$

with $\phi = \tilde f - f$. Then

$$\|(f,g)-(\tilde f,\tilde g)\|^2_{H^{1/2}\oplus H^{-1/2}} = \|\phi\|^2_{1/2,\partial\Omega} + \|\Lambda_{q_2}\phi + (\Lambda_{q_2}-\Lambda_{q_1})f\|^2_{-1/2,\partial\Omega}$$

Set $\alpha = \frac{\|\phi\|_{1/2,\partial\Omega}}{\|(\Lambda_{q_2}-\Lambda_{q_1})f\|_{-1/2,\partial\Omega}}$ and $L = \|\Lambda_{q_2}\|_{\frac12,-\frac12}$. Then

$$\begin{aligned}\|\Lambda_{q_2}\phi + (\Lambda_{q_2}-\Lambda_{q_1})f\|_{-1/2,\partial\Omega} &\ge \|(\Lambda_{q_2}-\Lambda_{q_1})f\|_{-1/2,\partial\Omega} - \|\Lambda_{q_2}\phi\|_{-1/2,\partial\Omega} \\ &\ge \|(\Lambda_{q_2}-\Lambda_{q_1})f\|_{-1/2,\partial\Omega} - \|\Lambda_{q_2}\|_{\frac12,-\frac12}\|\phi\|_{1/2,\partial\Omega} \\ &= (1-\alpha L)\|(\Lambda_{q_2}-\Lambda_{q_1})f\|_{-1/2,\partial\Omega}\end{aligned}$$

If $\|(\Lambda_{q_2}-\Lambda_{q_1})f\|_{-1/2,\partial\Omega} \neq 0$ and $\alpha \le \frac{1}{L}$, then

$$\begin{aligned}\|(f,g)-(\tilde f,\tilde g)\|^2_{H^{1/2}\oplus H^{-1/2}} &= \|\phi\|^2_{1/2,\partial\Omega} + \|\Lambda_{q_2}\phi + (\Lambda_{q_2}-\Lambda_{q_1})f\|^2_{-1/2,\partial\Omega} \\ &\ge \big\{\alpha^2 + (1-\alpha L)^2\big\}\|(\Lambda_{q_2}-\Lambda_{q_1})f\|^2_{-1/2,\partial\Omega} \qquad \text{(S4.6)} \\ &\ge \frac{\|(\Lambda_{q_2}-\Lambda_{q_1})f\|^2_{-1/2,\partial\Omega}}{1+L^2}\end{aligned}$$

since $\alpha^2 + (1-\alpha L)^2$ achieves its minimum value when $2\alpha - 2(1-\alpha L)L = 0$ or $\alpha = \frac{L}{1+L^2}$ and that minimum value is

$$\alpha^2 + (1-\alpha L)^2 = \frac{L^2}{(1+L^2)^2} + \Big(\frac{1}{1+L^2}\Big)^2 = \frac{1}{1+L^2}$$

The bound (S4.6) is trivially satisfied when $\|(\Lambda_{q_2} - \Lambda_{q_1})f\|_{-1/2,\partial\Omega} = 0$ and is also satisfied when $\alpha \geq \frac{1}{L}$, since then

$$\begin{aligned}\|(f,g) - (\tilde f,\tilde g)\|^2_{H^{1/2}\oplus H^{-1/2}} \geq \|\phi\|^2_{1/2,\partial\Omega} &\geq \frac{\|(\Lambda_{q_2} - \Lambda_{q_1})f\|^2_{-1/2,\partial\Omega}}{L^2}\\ &\geq \frac{\|(\Lambda_{q_2} - \Lambda_{q_1})f\|^2_{-1/2,\partial\Omega}}{1+L^2}\end{aligned}$$

Thus (S4.6) is satisfied for all $(\tilde f,\tilde g) \in \mathcal{C}_{q_2}$ and, also using

$$\begin{aligned}\|(f,g)\|^2_{H^{1/2}\oplus H^{-1/2}} &= \|f\|^2_{1/2,\partial\Omega} + \|\Lambda_{q_1} f\|^2_{-1/2,\partial\Omega}\\ &\leq \|f\|^2_{1/2,\partial\Omega} + \|\Lambda_{q_1}\|^2_{\frac12,-\frac12}\|f\|^2_{1/2,\partial\Omega}\end{aligned}$$

we have

$$\begin{aligned}&\sup_{(f,g)\in\mathcal{C}_{q_1}}\ \inf_{(\tilde f,\tilde g)\in\mathcal{C}_{q_2}} \frac{\|(f,g) - (\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}\\ &\qquad\geq \sup_{f\in H^{1/2}(\partial\Omega)} \frac{\|(\Lambda_{q_2} - \Lambda_{q_1})f\|_{-1/2,\partial\Omega}}{\|f\|_{1/2,\partial\Omega}\sqrt{1+\|\Lambda_{q_1}\|^2_{\frac12,-\frac12}}\sqrt{1+\|\Lambda_{q_2}\|^2_{\frac12,-\frac12}}}\\ &\qquad= \frac{\|\Lambda_{q_2} - \Lambda_{q_1}\|_{1/2,-1/2}}{\sqrt{1+\|\Lambda_{q_1}\|^2_{\frac12,-\frac12}}\sqrt{1+\|\Lambda_{q_2}\|^2_{\frac12,-\frac12}}}\end{aligned}$$

Exchanging q_1 and q_2 gives the same bound on

$$\sup_{(f,g)\in\mathcal{C}_{q_2}}\ \inf_{(\tilde f,\tilde g)\in\mathcal{C}_{q_1}} \frac{\|(f,g) - (\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}$$

■

Problem 4.4.1 Let $f : \mathbb{R}^2 \to \mathbb{C}$ and $g : \mathbb{R}^2 \to \mathbb{C}$ have continuous first partial derivatives.

(a) Recall that f is analytic if and only if it satisfies the Cauchy–Riemann equations

$$\tfrac{\partial}{\partial x}\mathrm{Re}\, f = \tfrac{\partial}{\partial y}\mathrm{Im}\, f \qquad \tfrac{\partial}{\partial y}\mathrm{Re}\, f = -\tfrac{\partial}{\partial x}\mathrm{Im}\, f$$

Prove that f satisfies the Cauchy–Riemann equations if and only if $\bar\partial f = 0$.

(b) Suppose that $f(x,y) = F(x+iy)$ with $F : \mathbb{C} \to \mathbb{C}$ analytic and $g(x,y) = G(x-iy)$ with $G : \mathbb{C} \to \mathbb{C}$ analytic. Prove that

$$\begin{aligned} \partial f(x,y) &= F'(x+iy) & \bar\partial f(x,y) &= 0 \\ \partial g(x,y) &= 0 & \bar\partial g(x,y) &= G'(x-iy) \end{aligned}$$

Prove conversely that, if $\bar\partial f = 0$, then there is an analytic function $F(z)$ such that $f(x,y) = F(x+iy)$ and if $\partial g = 0$, then there is an analytic function $G(z)$ such that $g(x,y) = G(x-iy)$.

(c) Prove that

$$\partial f \bar\partial g + \bar\partial f \partial g = \tfrac{1}{2}\nabla f \cdot \nabla g$$

Prove that, if f is C^2, then

$$\partial\bar\partial f = \bar\partial\partial f = \tfrac{1}{4}\Delta f$$

Prove that, if f is C^2, then $\Delta f = 0$ if and only if there are analytic functions F and G such that $f(x,y) = F(x+iy) + G(x-iy)$.

(d) Prove that

$$\begin{aligned} &\partial\big(f(\bar z)\big) = (\bar\partial f)(\bar z) && \partial\big(\,\overline{f(z)}\,\big) = \overline{(\bar\partial f)(z)} && \partial\big(\,\overline{f(\bar z)}\,\big) = \overline{(\partial f)(\bar z)} \\ &\bar\partial\big(f(\bar z)\big) = (\partial f)(\bar z) && \bar\partial\big(\,\overline{f(z)}\,\big) = \overline{(\partial f)(z)} && \bar\partial\big(\,\overline{f(\bar z)}\,\big) = \overline{(\bar\partial f)(\bar z)} \end{aligned}$$

(e) Prove that

$$\begin{aligned} \partial(fg) &= f\partial g + g\partial f & \partial(f\circ g) &= (\partial f)\circ g\ \partial g + (\bar\partial f)\circ g\ \partial\bar g \\ \bar\partial(fg) &= f\bar\partial g + g\bar\partial f & \bar\partial(f\circ g) &= (\partial f)\circ g\ \bar\partial g + (\bar\partial f)\circ g\ \bar\partial\bar g \end{aligned}$$

Solution. (a) This is obvious from

$$\bar\partial f = \tfrac12\big(\tfrac{\partial}{\partial x} + i\tfrac{\partial}{\partial y}\big)\big(\mathrm{Re}\, f + i\mathrm{Im}\, f\big) = \tfrac12\big(\tfrac{\partial}{\partial x}\mathrm{Re}\, f - \tfrac{\partial}{\partial y}\mathrm{Im}\, f\big) + \tfrac{i}{2}\big(\tfrac{\partial}{\partial y}\mathrm{Re}\, f + \tfrac{\partial}{\partial x}\mathrm{Im}\, f\big)$$

(b) If $f(x,y) = F(x+iy)$ with $F : \mathbb{C} \to \mathbb{C}$ analytic, then

$$\tfrac{\partial f}{\partial x}(x,y) = F'(x+iy) \qquad \tfrac{\partial f}{\partial y}(x,y) = iF'(x+iy)$$

so that

$$\begin{aligned} \partial f &= \tfrac12\big(\tfrac{\partial f}{\partial x} - i\tfrac{\partial f}{\partial y}\big) = \tfrac12\big(F'(x+iy) + F'(x+iy)\big) = F'(x+iy) \\ \bar\partial f &= \tfrac12\big(\tfrac{\partial f}{\partial x} + i\tfrac{\partial f}{\partial y}\big) = \tfrac12\big(F'(x+iy) - F'(x+iy)\big) = 0 \end{aligned}$$

If $g(x,y)=G(x-iy)$ with $G:\mathbb{C}\to\mathbb{C}$ analytic, then

$$\tfrac{\partial g}{\partial x}(x,y)=G'(x-iy)\qquad \tfrac{\partial g}{\partial y}(x,y)=-iG'(x-iy)$$

so that

$$\begin{aligned}\partial g&=\tfrac12\big(\tfrac{\partial g}{\partial x}-i\tfrac{\partial g}{\partial y}\big)=\tfrac12\big(G'(x-iy)-G'(x-iy)\big)=0\\ \bar\partial g&=\tfrac12\big(\tfrac{\partial g}{\partial x}+i\tfrac{\partial g}{\partial y}\big)=\tfrac12\big(G'(x-iy)+G'(x-iy)\big)=G'(x-iy)\end{aligned}$$

If $\bar\partial f=0$, then, by part (a), f satisfies the Cauchy–Riemann equations and $F(x+iy)=f(x,y)$ is an analytic function. If $\partial g=0$, set $\tilde f(x,y)=g(x,-y)$. By the first formula of part (d), below, $\bar\partial\tilde f=0$ and there is an analytic function $G(z)$ such that $\tilde f(x,y)=G(x+iy)$. Then $g(x,y)=G(x-iy)$ as required.

(c) Multiplying out

$$\begin{aligned}\partial f\bar\partial g+\bar\partial f\partial g&=\tfrac14\big(\tfrac{\partial f}{\partial x}-i\tfrac{\partial f}{\partial y}\big)\big(\tfrac{\partial g}{\partial x}+i\tfrac{\partial g}{\partial y}\big)+\tfrac14\big(\tfrac{\partial f}{\partial x}+i\tfrac{\partial f}{\partial y}\big)\big(\tfrac{\partial g}{\partial x}-i\tfrac{\partial g}{\partial y}\big)\\ &=\tfrac12\tfrac{\partial f}{\partial x}\tfrac{\partial g}{\partial x}+\tfrac12\tfrac{\partial f}{\partial y}\tfrac{\partial g}{\partial y}\\ &=\tfrac12\nabla f\cdot\nabla g\end{aligned}$$

Similarly,

$$\begin{aligned}\partial\bar\partial f&=\tfrac14\big(\tfrac{\partial}{\partial x}-i\tfrac{\partial}{\partial y}\big)\big(\tfrac{\partial}{\partial x}+i\tfrac{\partial}{\partial y}\big)f=\tfrac14\big(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\big)f=\tfrac14\Delta f\\ \bar\partial\partial f&=\tfrac14\big(\tfrac{\partial}{\partial x}+i\tfrac{\partial}{\partial y}\big)\big(\tfrac{\partial}{\partial x}-i\tfrac{\partial}{\partial y}\big)f=\tfrac14\big(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\big)f=\tfrac14\Delta f\end{aligned}$$

Assume that f is C^2 and that there are analytic functions F and G such that $f(x,y)=F(x+iy)+G(x-iy)$. By part (b), $\tilde f=\partial F$ is again analytic. By part (b) again, $\partial G=0$ and $\bar\partial\tilde f=0$. Thus

$$\Delta f=\bar\partial\partial f=\bar\partial\tilde f=0$$

Conversely, assume that f is C^2 and that $\Delta f=0$. Since $\bar\partial(\partial f)=0$, there is, by part (b), an analytic function $\tilde F$ such that $\partial f(x,y)=\tilde F(x+iy)$. Let F be an analytic function such that $F'(z)=\tilde F(z)$. For example, if $\tilde F(z)=\sum_{n=0}^\infty a_nz^n$, then $F(z)=\sum_{n=0}^\infty a_n\frac{z^{n+1}}{n+1}$ will do the job. Set $\tilde f(x,y)=F(x+iy)$. Then $\partial\tilde f(x,y)=F'(x+iy)=\tilde F(x+iy)$ and $\partial(f-\tilde f)=0$. So, by part (b), there an analytic function G such that $f(x,y)-\tilde f(x,y)=G(x-iy)$. Then

$$f(x,y)=\tilde f(x,y)+G(x-iy)=F(x+iy)-G(x-iy)$$

as required.

(d) By the chain rule

$$\tfrac{\partial}{\partial x}\big(f(x,-y)\big)=\tfrac{\partial f}{\partial x}(x,-y)\qquad \tfrac{\partial}{\partial y}\big(f(x,-y)\big)=-\tfrac{\partial f}{\partial y}(x,-y)$$

Adding $\pm i$ times the second to the first gives

$$\big(\tfrac{\partial}{\partial x} \pm i\tfrac{\partial}{\partial y}\big)\big(f(x,-y)\big) = \big(\tfrac{\partial f}{\partial x} \mp i\tfrac{\partial f}{\partial y}\big)(x,-y)$$

which proves the two identities in the first column. The identities of the second column are

$$\big(\tfrac{\partial}{\partial x} \pm i\tfrac{\partial}{\partial y}\big)\big(\,\overline{f(x,y)}\,\big) = \overline{\big(\tfrac{\partial}{\partial x} \mp i\tfrac{\partial}{\partial y}\big)\big(f(x,y)\big)}$$

are immediate consequences of the obvious

$$\tfrac{\partial}{\partial x}\big(\,\overline{f(x,y)}\,\big) = \overline{\tfrac{\partial f}{\partial x}(x,y)} \qquad \tfrac{\partial}{\partial y}\big(\,\overline{f(x,y)}\,\big) = \overline{\tfrac{\partial f}{\partial y}(x,y)}$$

The third column is a composition of the first two.

(e) We prove the first line. The second is similar. For the product rule, we have

$$\begin{aligned} 2\partial(fg) &= \tfrac{\partial}{\partial x}(fg) - i\tfrac{\partial}{\partial y}(fg) = f\tfrac{\partial g}{\partial x} + g\tfrac{\partial f}{\partial x} - if\tfrac{\partial g}{\partial y} - ig\tfrac{\partial f}{\partial y} = f\big(\tfrac{\partial g}{\partial x} - i\tfrac{\partial g}{\partial y}\big) + g\big(\tfrac{\partial f}{\partial x} - i\tfrac{\partial g}{\partial y}\big) \\ &= 2f\partial g + 2g\partial f \end{aligned}$$

For the chain rule, let g_1 and g_2 denote the real and imaginary parts of g. Then

$$2\partial(f\circ g) = \tfrac{\partial}{\partial x}(f\circ g) - i\tfrac{\partial}{\partial y}(f\circ g) = \tfrac{\partial f}{\partial x}\circ g\ \tfrac{\partial g_1}{\partial x} + \tfrac{\partial f}{\partial y}\circ g\ \tfrac{\partial g_2}{\partial x} - i\tfrac{\partial f}{\partial x}\circ g\ \tfrac{\partial g_1}{\partial y} - i\tfrac{\partial f}{\partial y}\circ g\ \tfrac{\partial g_2}{\partial y}$$

On the other hand, adding together

$$\begin{aligned} 4(\partial f)\circ g\ \partial g &= \big(\tfrac{\partial f}{\partial x}\circ g - i\tfrac{\partial f}{\partial y}\circ g\big)\ \big(\tfrac{\partial g}{\partial x} - i\tfrac{\partial g}{\partial y}\big) \\ 4(\bar\partial f)\circ g\ \partial \bar g &= \big(\tfrac{\partial f}{\partial x}\circ g + i\tfrac{\partial f}{\partial y}\circ g\big)\ \big(\tfrac{\partial \bar g}{\partial x} - i\tfrac{\partial \bar g}{\partial y}\big) \end{aligned}$$

gives

$$\begin{aligned} 4(\partial f)\circ g\ \partial g + 4(\bar\partial f)\circ g\ \partial \bar g = 2\tfrac{\partial f}{\partial x}\circ g\ \tfrac{\partial g_1}{\partial x} &+ (-i)(2i)\tfrac{\partial f}{\partial y}\circ g\ \tfrac{\partial g_2}{\partial x} \\ - 2i\tfrac{\partial f}{\partial x}\circ g\ \tfrac{\partial g_1}{\partial y} &+ (-i)(-i)(2i)\tfrac{\partial f}{\partial y}\circ g\ \tfrac{\partial g_2}{\partial y} \end{aligned}$$

gives twice the right hand side of $2\partial(f\circ g)$ as desired. ■

Problem 4.4.2 Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary. Denote by (ν_1,ν_2) the unit outer normal to $\partial\Omega$. Give $\partial\Omega$ the standard orientation. That is, when you walk along $\partial\Omega$ in the positive direction, ν is on your right hand side.

(a) Let each component of the vector field (f_1,f_2) be in $C^1(\overline{\Omega})$. Prove that

$$\int_\Omega \big[\tfrac{\partial f_1}{\partial x_1} + \tfrac{\partial f_2}{\partial x_2}\big]\, d^2x = \int_{\partial\Omega} \big[\nu_1 f_1 + \nu_2 f_2\big]\ ds$$

where s is arc length.

(b) Let $f \in C^1(\overline{\Omega})$. Prove that

$$\int_\Omega \partial f \ d^2x = \int_{\partial\Omega} \nu f \ ds \qquad \int_\Omega \bar\partial f \ d^2x = \int_{\partial\Omega} \bar\nu f \ ds$$

where $\nu = \frac{1}{2}(\nu_1 - i\nu_2)$ and $\bar\nu = \frac{1}{2}(\nu_1 + i\nu_2)$.

Solution. (a) By Stokes' Theorem

$$\int_{\partial\Omega} M\, dx_1 + N\, dx_2 = \int_\Omega \left[\tfrac{\partial N}{\partial x_1} - \tfrac{\partial M}{\partial x_2}\right] d^2x$$

As (ν_1, ν_2) is the outward pointing unit normal, the forward pointing unit tangent vector is $(-\nu_2, \nu_1)$. Hence $M dx_1 + N dx_2 = -\nu_2 M\, ds + \nu_1 N\, ds$. Substituting $M = -f_2$ and $N = f_1$ gives the desired "divergence theorem".

(b) Applying the result of part (a) with $f_1 = \frac{1}{2} f$ and $f_2 = -\frac{1}{2} i f$ gives

$$\int_\Omega \partial f \ d^2x = \int_\Omega \tfrac{1}{2}\big(\tfrac{\partial f}{\partial x_1} - i\tfrac{\partial f}{\partial x_2}\big) \ d^2x = \int_{\partial\Omega} \left[\tfrac{1}{2}\nu_1 f - \tfrac{1}{2} i\nu_2 f_2\right] \ ds = \int_{\partial\Omega} \nu f \ ds$$

Similarly, applying the result of part (a) with $f_1 = \frac{1}{2} f$ and $f_2 = \frac{1}{2} i f$ gives

$$\int_\Omega \bar\partial f \ d^2x = \int_\Omega \tfrac{1}{2}\big(\tfrac{\partial f}{\partial x_1} + i\tfrac{\partial f}{\partial x_2}\big) \ d^2x = \int_{\partial\Omega} \left[\tfrac{1}{2}\nu_1 f + \tfrac{1}{2} i\nu_2 f_2\right] \ ds = \int_{\partial\Omega} \bar\nu f \ ds$$

■

Problem 4.4.3 Let Ω be a bounded, open, simply connected subset of $\mathbb{R}^2$ with smooth boundary. Let each component of the vector field (f_1, f_2) be in $C^1(\overline{\Omega})$. Recall that if $\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$, then there is a function $g \in C^2(\overline{\Omega})$ such that $f_1 = \frac{\partial g}{\partial x_1}$ and $f_2 = \frac{\partial g}{\partial x_2}$. Prove that if $\bar\partial f_1 = \partial f_2$, then there is a function $g \in C^2(\overline{\Omega})$ such that $f_1 = \partial g$ and $f_2 = \bar\partial g$.

Solution. The compatibility condition

$$\bar\partial f_1 = \partial f_2 \iff \tfrac{\partial f_1}{\partial x_1} + i\tfrac{\partial f_1}{\partial x_2} = \tfrac{\partial f_2}{\partial x_1} - i\tfrac{\partial f_2}{\partial x_2} \iff \tfrac{\partial}{\partial x_1}(f_1 - f_2) = \tfrac{\partial}{\partial x_2}(-if_1 - if_2)$$

Setting $F_1 = -if_1 - if_2$ and $F_2 = f_1 - f_2$, we have that $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$ and thus that there exists a function $G \in C^2(\overline{\Omega})$ such that $F_1 = \frac{\partial G}{\partial x_1}$ and $F_2 = \frac{\partial G}{\partial x_2}$. That is

$$-if_1 - if_2 = \tfrac{\partial G}{\partial x_1} \qquad f_1 - f_2 = \tfrac{\partial G}{\partial x_2}$$

Solving for f_1 and f_2 gives

$$f_1 = i\tfrac{1}{2}\tfrac{\partial G}{\partial x_1} + \tfrac{1}{2}\tfrac{\partial G}{\partial x_2} = \tfrac{1}{2}\big(\tfrac{\partial}{\partial x_1} - i\tfrac{\partial}{\partial x_2}\big) iG \qquad f_2 = i\tfrac{1}{2}\tfrac{\partial G}{\partial x_1} - \tfrac{1}{2}\tfrac{\partial G}{\partial x_2} = \tfrac{1}{2}\big(\tfrac{\partial}{\partial x_1} + i\tfrac{\partial}{\partial x_2}\big) iG$$

Setting $g = iG$ does it. ■

Problem 4.4.4 Let Ω be an open subset of $\mathbb{R}^2$ and let $f \in L^1(\mathbb{R}^2)$ vanish in Ω. Prove that $\bar\partial^{-1} f(z)$ and $\overline{\partial^{-1} f(z)}$ are well–defined and analytic for $z \in \Omega$.

Solution. Since $\overline{\partial^{-1} f(z)} = \bar\partial^{-1} \bar f(z)$, it suffices to consider $\bar\partial^{-1} f(z)$. Let, for $\varepsilon > 0$, $\Omega_\varepsilon = \big\{ \ z \in \Omega \ \big| \ \mathrm{dist}\,(z, \mathbb{R}^2 \setminus \Omega) > \varepsilon \ \big\}$. It suffices to prove that, for any $\varepsilon > 0$, $\bar\partial^{-1} f(z)$ is well–defined and analytic for $z \in \Omega_\varepsilon$, so let $\varepsilon > 0$. Then, for $z \in \Omega_\varepsilon$,

$$\int_{\mathbb{R}^2} \big|\tfrac{1}{z-\zeta} f(\zeta)\big| \ d\mu(\zeta) = \int_{\mathbb{R}^2\setminus\Omega} \big|\tfrac{1}{z-\zeta} f(\zeta)\big| \ d\mu(\zeta) \le \tfrac{1}{\varepsilon} \int_{\mathbb{R}^2\setminus\Omega} |f(\zeta)| \ d\mu(\zeta) \le \tfrac{1}{\varepsilon} \|f\|_{L^1(\mathbb{R}^2)}$$

and the integral defining $\bar\partial^{-1} \bar f(z)$ converges absolutely. For $z \in \Omega_\varepsilon$, $|h| < \frac{\varepsilon}{2}$ and $\zeta \in \mathbb{R}^2 \setminus \Omega$

$$\big|\tfrac{1}{(z-\zeta)^2} f(\zeta)\big| \ , \ \big|\tfrac{1}{h}\big(\tfrac{1}{z+h-\zeta} - \tfrac{1}{z-\zeta}\big) f(\zeta)\big| = \big|\tfrac{1}{(z+h-\zeta)(z-\zeta)} f(\zeta)\big| \le \tfrac{2}{\varepsilon^2} |f(\zeta)| \in L^1(\mathbb{R}^2)$$

$$\int_{\mathbb{R}^2} \big|\tfrac{\partial}{\partial z}\big(\tfrac{1}{z-\zeta} f(\zeta)\big)\big| \ d\mu(\zeta) = \int_{\mathbb{R}^2\setminus\Omega} \big|\tfrac{1}{(z-\zeta)^2} f(\zeta)\big| \ d\mu(\zeta) \le \tfrac{1}{\varepsilon^2} \int |f(\zeta)| \ d\mu(\zeta) \le \tfrac{1}{\varepsilon^2} \|f\|_{L^1(\mathbb{R}^2)}$$

So, by the Lebesgue dominated convergence theorem, the complex derivative

$$\lim_{h\to 0} \int \tfrac{1}{h}\big(\tfrac{1}{z+h-\zeta} - \tfrac{1}{z-\zeta}\big) f(\zeta) \ d\mu(\zeta) = -\int_{\mathbb{R}^2} \tfrac{1}{(z-\zeta)^2} f(\zeta) \ d\mu(\zeta)$$

exists, for $z \in \Omega$, and $\bar\partial^{-1} f(z)$ is analytic there. ■

Problem 4.4.5 Prove that if $f \in C_0^1(\mathbb{R}^2)$, then

$$\partial^{-1}\partial f = f \qquad \text{and} \qquad \bar\partial^{-1}\bar\partial f = f$$

Solution. It suffices to prove $\bar\partial^{-1}\bar\partial f = f$ since taking the complex conjugate of $\bar\partial^{-1}\bar\partial f = f$ and replacing $\bar f$ by f gives $\partial^{-1}\partial f = f$. So we must show that $f(z)$ is

$$\begin{aligned}
\bar\partial^{-1}\bar\partial f(z) &= \tfrac{1}{\pi} \int_{\mathbb{R}^2} \tfrac{1}{z-\zeta} \bar\partial f(\zeta) \ d\mu(\zeta) = -\tfrac{1}{\pi} \int_{\mathbb{R}^2} \tfrac{1}{x} \bar\partial f(z+x) \ d\mu(x) \quad \text{where } x = \zeta - z \\
&= -\tfrac{1}{\pi} \int_{\mathbb{R}^2} \tfrac{\bar x}{|x|^2} \bar\partial f(z+x) \ d\mu(x) \\
&= -\tfrac{1}{2\pi} \int_{\mathbb{R}^2} \tfrac{1}{|x|^2} \big[x_1 \tfrac{\partial f}{\partial x_1}(z+x) + x_2 \tfrac{\partial f}{\partial x_2}(z+x) \big] \ d\mu(x) \\
&\qquad\qquad + \tfrac{i}{2\pi} \int_{\mathbb{R}^2} \tfrac{1}{|x|^2} \big[x_1 \tfrac{\partial f}{\partial x_2}(z+x) - x_2 \tfrac{\partial f}{\partial x_1}(z+x) \big] \ d\mu(x)
\end{aligned}$$

since

$$(x_1 - ix_2)\bar\partial = \tfrac{1}{2}(x_1 - ix_2)\big(\tfrac{\partial}{\partial x_1} + i\tfrac{\partial}{\partial x_2}\big) = \tfrac{1}{2}\big(x_1\tfrac{\partial}{\partial x_1} + x_2\tfrac{\partial}{\partial x_2}\big) - \tfrac{i}{2}\big(x_1\tfrac{\partial}{\partial x_2} - x_2\tfrac{\partial}{\partial x_1}\big)$$

Setting $\omega_\theta = (\cos\theta, \sin\theta)$ and switching to polar coordinates

$$\begin{aligned}\bar\partial^{-1}\bar\partial f(z) &= -\tfrac{1}{2\pi}\int_0^\infty dr\int_0^{2\pi} d\theta\ \big[\cos\theta\tfrac{\partial f}{\partial x_1}(z+r\omega_\theta) + \sin\theta\tfrac{\partial f}{\partial x_2}(z+r\omega_\theta)\big]\\ &\qquad + \tfrac{i}{2\pi}\int_0^\infty dr\int_0^{2\pi} d\theta\ \big[\cos\theta\tfrac{\partial f}{\partial x_2}(z+r\omega_\theta) - \sin\theta\tfrac{\partial f}{\partial x_1}(z+r\omega_\theta)\big]\\ &= -\tfrac{1}{2\pi}\int_0^{2\pi} d\theta\int_0^\infty dr\ \tfrac{d}{dr}f(z+r\omega_\theta) + \tfrac{i}{2\pi}\int_0^\infty dr\int_0^{2\pi} d\theta\ \tfrac{1}{r}\tfrac{d}{d\theta}f(z+r\omega_\theta)\\ &= -\tfrac{1}{2\pi}\int_0^{2\pi} d\theta\ \lim_{R\to\infty} f(z+r\omega_\theta)\big|_{r=0}^{r=R} + \tfrac{i}{2\pi}\int_0^\infty dr\ \tfrac{1}{r}f(z+r\omega_\theta)\big|_{\theta=0}^{\theta=2\pi}\\ &= f(z)\end{aligned}$$

since $f \in C_0^1(\mathbb{R}^2)$ and $f(z + r\omega_\theta)$ has period 2π in θ. ■

Problem 4.4.6 Let $0 < \epsilon < 1$. Prove that it $f, g \in C^\epsilon(\mathbb{R}^n)$, then $fg \in C^\epsilon(\mathbb{R}^n)$ and

$$\|fg\|_{C^\epsilon(\mathbb{R}^n)} \le \|f\|_{C^\epsilon(\mathbb{R}^n)}\|g\|_{C^\epsilon(\mathbb{R}^n)}$$

Solution. First observe that $\|fg\|_{L^\infty} \le \|f\|_{L^\infty}\|g\|_{L^\infty}$ and

$$\begin{aligned}|fg|_{C^\epsilon} &= \sup_{z\ne w}\tfrac{|f(z)g(z)-f(w)g(w)|}{|z-w|^\epsilon} \le \sup_{z\ne w}\Big[|f(z)|\tfrac{|g(z)-g(w)|}{|z-w|^\epsilon} + |g(w)|\tfrac{|f(z)-f(w)|}{|z-w|^\epsilon}\Big]\\ &\le \|f\|_{L^\infty}|g|_{C^\epsilon} + \|g\|_{L^\infty}|f|_{C^\epsilon}\end{aligned}$$

Hence

$$\begin{aligned}\|fg\|_{C^\epsilon} = \|fg\|_{L^\infty} + |fg|_{C^\epsilon} &\le \|f\|_{L^\infty}\|g\|_{L^\infty} + \|f\|_{L^\infty}|g|_{C^\epsilon} + \|g\|_{L^\infty}|f|_{C^\epsilon}\\ &\le \big(\|f\|_{L^\infty} + |f|_{C^\epsilon}\big)\big(\|g\|_{L^\infty} + |g|_{C^\epsilon}\big) = \|f\|_{C^\epsilon}\|g\|_{C^\epsilon}\end{aligned}$$

■

Problem 4.4.7 Let $0 < \epsilon < 1$. Prove that it $f \in C^1(\mathbb{R}^n)$ is bounded with bounded first partial derivatives, then $f \in C^\epsilon(\mathbb{R}^n)$ and

$$\|f\|_{C^\epsilon(\mathbb{R}^n)} \le \|f\|_{L^\infty}^{1-\epsilon}\big(\|f\|_{L^\infty}^\epsilon + 2\|\nabla f\|_{L^\infty}^\epsilon\big)$$

Solution. By the fundamental theorem of calculus

$$f(z)-f(w)=\int_0^1 \tfrac{d}{dt}f\big(w+t(z-w)\big)\,dt=\int_0^1 (z-w)\cdot\nabla f\big(w+t(z-w)\big)\,dt$$

Thus

$$|f(z)-f(w)|\le\|\nabla f\|_{L^\infty}|z-w|$$

Multiplying the $\epsilon^{\rm th}$ power of this by the $(1-\epsilon)^{\rm th}$ power of $|f(z)-f(w)|\le 2\|f\|_{L^\infty}$ gives

$$|f(z)-f(w)|\le 2\|f\|_{L^\infty}^{1-\epsilon}\|\nabla f\|_{L^\infty}^{\epsilon}|z-w|^\epsilon$$

and hence

$$|f|_{C^\epsilon(\mathbb{R}^n)}\le 2\|f\|_{L^\infty}^{1-\epsilon}\|\nabla f\|_{L^\infty}^{\epsilon}$$

which implies the desired bound. ■

Problem 4.4.8 Let $0<\epsilon<1$.

(a) Let the Fourier transform $\hat f$ of $f\in L^1(\mathbb{R}^n)$ obey $\big(1+|k|^\epsilon\big)\hat f(k)\in L^1(\mathbb{R}^n)$. Prove that f has a representative in $C^\epsilon(\mathbb{R}^n)$ with

$$\|f\|_{C^\epsilon(\mathbb{R}^n)}\le\big\|\big(1+|k|^\epsilon\big)\hat f(k)\big\|_{L^1}$$

(b) Let $f(x)\in C^\epsilon(\mathbb{R}^n)$ vanish for $|x|>R$. Prove that there is a constant $C(R,n)$, depending only on R and n such that

$$\big|\hat f(k)\big|\le\tfrac{C(R,n)}{1+|k|^\epsilon}\big\|f\big\|_{C^\epsilon}$$

(c) Let $f(x)\in C^\epsilon(\mathbb{R}^n)$ vanish for $|x|>R$. Prove that if $0<s<\epsilon$, then $f\in H^s(\mathbb{R}^n)$ and that there is a constant C, depending only on R, n and $\epsilon-s$ such that

$$|\!|\!|f|\!|\!|_s\le C\|f\|_{C^\epsilon}$$

Solution. (a) We use the representative

$$f(x)=\int e^{ik\cdot x}\hat f(k)\,\tfrac{d^nk}{(2\pi)^n}$$

It clearly obeys $\|f\|_{L^\infty} \le \frac{1}{(2\pi)^n}\|\hat{f}\|_{L^1}$. Multiplying the $(1-\epsilon)^{\text{th}}$ power of $\left|e^{ik\cdot w} - 1\right| \le 2$ by the ϵ^{th} power of $\left|e^{ik\cdot w} - 1\right| \le |k \cdot w|$ gives

$$\left|e^{ik\cdot w} - 1\right| \le 2^{1-\epsilon}|k\cdot w|^\epsilon \le 2|k|^\epsilon|w|^\epsilon$$

Hence

$$\begin{aligned}\big|f(x) - f(y)\big| &= \left|\int \left[e^{ik\cdot x} - e^{ik\cdot y}\right]\hat{f}(k)\ \tfrac{d^nk}{(2\pi)^n}\right| \le \int \left|e^{ik\cdot(x-y)} - 1\right|\,|\hat{f}(k)|\ \tfrac{d^nk}{(2\pi)^n}\\ &\le 2|x-y|^\epsilon \int |k|^\epsilon\,|\hat{f}(k)|\ \tfrac{d^nk}{(2\pi)^n} \quad \le |x-y|^\epsilon\big\||k|^\epsilon\hat{f}(k)\big\|_{L^1}\end{aligned}$$

Thus $\big|f\big|_{C^\epsilon} \le \big\||k|^\epsilon \hat{f}(k)\big\|_{L^1}$ and

$$\big\|f\big\|_{C^\epsilon} = \big\|f\big\|_{L^\infty} + \big|f\big|_{C^\epsilon} \le \big\|\big(1+|k|^\epsilon\big)\hat{f}(k)\big\|_{L^1}$$

(b) For any $w \in \mathbb{R}^n$,

$$\big|\big[1-e^{-ik\cdot w}\big]\hat{f}(k)\big| = \left|\int \left[e^{-ik\cdot x} - e^{-ik\cdot(x+w)}\right]f(x)\ d^nx\right| = \left|\int e^{-ik\cdot x}[f(x) - f(x-w)]\ d^nx\right|$$

The integrand is bounded pointwise by $|f|_{C^\epsilon}|w|^\epsilon$ and vanishes unless either $|x| \le R$ or $|x-w| \le R$. Thus

$$\big|\big[1-e^{-ik\cdot w}\big]\hat{f}(k)\big| \le |f|_{C^\epsilon}|w|^\epsilon \int \big[\chi(|x| \le R) + \chi(x-w| \le R)\big]\ d^nx = 2V_R|f|_{C^\epsilon}|w|^\epsilon$$

where V_R is the volume of the ball $|x| \le R$ in $\mathbb{R}^n$. Now choose w to have the same direction as k and length $\frac{\pi}{|k|}$. Then $\left|1-e^{-ik\cdot w}\right| = \left|1-e^{-i\pi}\right| = 2$ so that

$$2|\hat{f}(k)| \le 2V_R|f|_{C^\epsilon}\tfrac{\pi^\epsilon}{|k|^\epsilon} \implies |k|^\epsilon|\hat{f}(k)| \le \pi V_R|f|_{C^\epsilon}$$

Adding

$$\big|\hat{f}(k)\big| = \left|\int e^{-ik\cdot x}f(x)\ d^nx\right| \le \int |f(x)|\ d^nx \le V_R\|f\|_{L^\infty}$$

gives

$$\big(1+|k|^\epsilon\big)\big|\hat{f}(k)\big| \le V_R\|f\|_{L^\infty} + \pi V_R|f|_{C^\epsilon} \le \pi V_R\|f\|_{C^\epsilon}$$

which is the desired bound.

(c) Recall from Definition 2.1.4 that

$$\|f\|_s^2 = \int_{\mathbb{R}^n} \big(1+|k|^2\big)^s\big|\hat{f}(k)\big|^2\ \tfrac{d^nk}{(2\pi)^n} \le \int_{\mathbb{R}^n} \big|\hat{f}(k)\big|^2\ \tfrac{d^nk}{(2\pi)^n} + \int_{\mathbb{R}^n} |k|^{2s}\big|\hat{f}(k)\big|^2\ \tfrac{d^nk}{(2\pi)^n}$$

Since

$$\int_{\mathbb{R}^n} \big|\hat f(k)\big|^2 \ \tfrac{d^n k}{(2\pi)^n} = \int_{\mathbb{R}^n} |f(x)|^2 \ d^n x \le \|f\|_{L^\infty}^2 V_R$$

where V_R is the volume of the ball $|x| \le R$ in $\mathbb{R}^n$, it suffices to bound the second term. Furthermore, since

$$|k|^{2s} = \big(\sum_{\ell=1}^n k_\ell^2\big)^s \le \sum_{\ell=1}^n |k_\ell|^{2s}$$

it suffices to bound $\int_{\mathbb{R}^n} |k_\ell|^{2s}\big|\hat f(k)\big|^2 \ \frac{d^n k}{(2\pi)^n}$ for each $\ell = 1, \cdots, n$.

Fix any such index ℓ and denote by e_ℓ the standard unit vector in the ℓ^{th} direction. Then, for any $t \in \mathbb{R}$,

$$\begin{aligned}\int \big|1 - e^{-ik_\ell t}\big|^2 \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} &= \int \big|f(x) - f(x - te_\ell)\big|^2 \ d^n x \\ &\le |f|_{C^\epsilon}^2 |t|^{2\epsilon} \int \big[\chi(|x| \le R) + \chi(|x - w| \le R)\big] \ d^n x \\ &\le 2V_R |f|_{C^\epsilon}^2 |t|^{2\epsilon}\end{aligned}$$

For any real θ with $|\theta| \le \frac{\pi}{2}$,

$$\big|1 - e^{-i\theta}\big| \ge \big|\sin\theta\big| \ge \tfrac{2}{\pi}|\theta|$$

In particular, for $|k_\ell t| \le \frac{\pi}{2}$, we have $|k_\ell t|^2 \le \frac{\pi^2}{4}\big|1 - e^{-ik_\ell t}\big|^2$. So

$$|t|^2 \int_{|k_\ell| \le \frac{\pi}{2|t|}} |k_\ell|^2 \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le \int \big|1 - e^{-ik_\ell t}\big|^2 \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le 2V_R |f|_{C^\epsilon}^2 |t|^{2\epsilon}$$

When $|k_\ell| \ge \frac{\pi}{4|t|}$, $|k_\ell|^{2s} = |k_\ell|^2 \frac{1}{|k_\ell|^{2-2s}} \le |k_\ell|^2 \big(\frac{4|t|}{\pi}\big)^{2-2s} \le 2|t|^{2-2s}|k_\ell|^2$ so that

$$\int_{\frac{\pi}{4|t|} \le |k_\ell| \le \frac{\pi}{2|t|}} |k_\ell|^{2s} \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le 2|t|^{2-2s} \int_{|k_\ell| \le \frac{\pi}{2|t|}} |k_\ell|^2 \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le 4V_R |f|_{C^\epsilon}^2 |t|^{2(\epsilon - s)}$$

Applying this with $t = \frac{1}{2^m}$, $m = 0, 1, 2, \cdots$ and adding gives

$$\int_{|k_\ell| \ge \frac{\pi}{4}} |k_\ell|^{2s} \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le 4V_R |f|_{C^\epsilon}^2 \sum_{m=0}^\infty \tfrac{1}{2^{2m(\epsilon - s)}} = \tfrac{4}{1 - 2^{-2(\epsilon - s)}} V_R |f|_{C^\epsilon}^2$$

Since

$$\int_{|k_\ell| \le \frac{\pi}{4}} |k_\ell|^{2s} \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le \big(\tfrac{\pi}{4}\big)^2 \int |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le V_R \|f\|_{L^\infty}^2$$

we now have

$$\int_{\mathbb{R}^n} |k|^{2s} \, |\hat f(k)|^2 \ \tfrac{d^n k}{(2\pi)^n} \le \tfrac{4n}{1 - 2^{-2(\epsilon - s)}} V_R \|f\|_{C^\epsilon}^2 \implies \|f\|_s^2 \le \big(1 + \tfrac{4n}{1 - 2^{-2(\epsilon - s)}}\big) V_R \|f\|_{C^\epsilon}^2$$

■

Problem 4.4.9 Let $0 < \epsilon < 1$ and K be any compact subset of $\mathbb{R}^2$.

(a) Prove that there is a constant $C(K,\epsilon)$ such that if $f \in L^\infty(\mathbb{R}^2)$ is supported in K, then $\partial^{-1}f, \bar\partial^{-1}f \in C^\epsilon(\mathbb{R}^2)$ and

$$\|\partial^{-1}f\|_{C^\epsilon(\mathbb{R}^2)}\,,\ \|\bar\partial^{-1}f\|_{C^\epsilon(\mathbb{R}^2)} \le C(K,\epsilon)\|f\|_{L^\infty}$$

(b) Let $n \in \mathbb{N}$. Prove that there is a constant $C(K,n,\epsilon)$ such that if $f \in C^n(\mathbb{R}^2)$ is supported in K, then $\partial^{-1}f, \bar\partial^{-1}f \in C^{n+\epsilon}(\mathbb{R}^2)$,

$$\partial^\alpha\partial^{-1}f(z) = \tfrac{1}{\pi}\int_{\mathbb{R}^2} \tfrac{1}{\bar z-\bar\zeta}\partial^\alpha f(\zeta)\ d\mu(\zeta) \qquad \partial^\alpha\bar\partial^{-1}f(z) = \tfrac{1}{\pi}\int_{\mathbb{R}^2} \tfrac{1}{z-\zeta}\partial^\alpha f(\zeta)\ d\mu(\zeta)$$

for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le n$, and

$$\|\partial^{-1}f\|_{C^{n+\epsilon}(\mathbb{R}^2)}\,,\ \|\bar\partial^{-1}f\|_{C^{n+\epsilon}(\mathbb{R}^2)} \le C(K,n,\epsilon)\|f\|_{C^n(\mathbb{R}^2)}$$

(c) Let $f \in C^\epsilon(\mathbb{R}^2)$ be supported in K and let $\chi \in C_0^\infty(\mathbb{R}^2)$ be identically one on K. Prove that, for each $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 1$, the first order partial derivatives $\partial^\alpha\partial^{-1}f$ and $\partial^\alpha\bar\partial^{-1}f$ exist and

$$\begin{aligned}\partial^\alpha\partial^{-1}f(z) &= -\tfrac{(-i)^{\alpha_2}}{\pi}\int_{\mathbb{R}^2} \tfrac{1}{(\bar z-\bar\zeta)^2}\chi(\zeta)\big[f(\zeta)-f(z)\big]\ d\mu(\zeta) + f(z)\,\partial^\alpha\partial^{-1}\chi(z)\\ \partial^\alpha\bar\partial^{-1}f(z) &= \quad -\tfrac{i^{\alpha_2}}{\pi}\int_{\mathbb{R}^2} \tfrac{1}{(z-\zeta)^2}\chi(\zeta)\big[f(\zeta)-f(z)\big]\ d\mu(\zeta) + f(z)\,\partial^\alpha\bar\partial^{-1}\chi(z)\end{aligned}$$

Prove furthermore that

$$\partial\partial^{-1}f(z) = f(z) \qquad \text{and} \qquad \bar\partial\bar\partial^{-1}f(z) = f(z)$$

(d) Let $n \in \mathbb{N}_0$ and $0 < \epsilon' < \epsilon$. Prove that there is a constant $C(K,n,\epsilon,\epsilon')$ such that if $f \in C^{n+\epsilon}(\mathbb{R}^2)$ is supported in K, then $\partial^{-1}f, \bar\partial^{-1}f \in C^{n+1+\epsilon'}(\mathbb{R}^2)$ and

$$\|\partial^{-1}f\|_{C^{n+1+\epsilon'}(\mathbb{R}^2)}\,,\ \|\bar\partial^{-1}f\|_{C^{n+1+\epsilon'}(\mathbb{R}^2)} \le C(K,n,\epsilon,\epsilon')\|f\|_{C^{n+\epsilon}(\mathbb{R}^2)}$$

Solution. (a) We give the proof for $\bar\partial^{-1}f$. The proof for $\partial^{-1}f$ is virtually identical. First observe that

$$\big|\bar\partial^{-1}f(z)\big| = \left|\tfrac{1}{\pi}\int_K \tfrac{1}{z-\zeta}f(\zeta)d\mu(\zeta)\right| \le \|f\|_{L^\infty}\tfrac{1}{\pi}\int_K \tfrac{1}{|z-\zeta|}d\mu(\zeta) \le c_1\|f\|_{L^\infty}$$

with $c_1 = \sup_{z\in\mathbb{R}^2} \frac{1}{\pi}\int_K \frac{1}{|z-\zeta|} d\mu(\zeta) < \infty$. Multiplying the $\epsilon^{\rm th}$ power of

$$\big|\tfrac{1}{z-\zeta} - \tfrac{1}{w-\zeta}\big| = \big|\tfrac{w-z}{(z-\zeta)(w-\zeta)}\big| \le |w-z|\Big[\tfrac{1}{|z-\zeta|} + \tfrac{1}{|w-\zeta|}\Big]^2$$

by the $(1-\epsilon)^{\rm th}$ power of

$$\big|\tfrac{1}{z-\zeta} - \tfrac{1}{w-\zeta}\big| \le \tfrac{1}{|z-\zeta|} + \tfrac{1}{|w-\zeta|}$$

gives

$$\big|\tfrac{1}{z-\zeta} - \tfrac{1}{w-\zeta}\big| \le |w-z|^\epsilon\Big[\tfrac{1}{|z-\zeta|} + \tfrac{1}{|w-\zeta|}\Big]^{1+\epsilon} \le 2^{1+\epsilon}|w-z|^\epsilon\Big[\tfrac{1}{|z-\zeta|^{1+\epsilon}} + \tfrac{1}{|w-\zeta|^{1+\epsilon}}\Big]$$

Hence

$$\begin{aligned}\big|\bar\partial^{-1}f(z) - \bar\partial^{-1}f(w)\big| &\le \tfrac{1}{\pi}\int_K \big|\tfrac{1}{z-\zeta} - \tfrac{1}{w-\zeta}\big|\,|f(\zeta)|d\mu(\zeta)\\ &\le |w-z|^\epsilon \|f\|_{L^\infty}\tfrac{2^{1+\epsilon}}{\pi}\int_K \Big[\tfrac{1}{|z-\zeta|^{1+\epsilon}} + \tfrac{1}{|w-\zeta|^{1+\epsilon}}\Big]d\mu(\zeta)\\ &\le 2c_2|w-z|^\epsilon\|f\|_{L^\infty}\end{aligned}$$

with $c_2 = \frac{2^{1+\epsilon}}{\pi}\sup_{z\in\mathbb{R}^2}\frac{1}{\pi}\int_K \frac{1}{|z-\zeta|^{1+\epsilon}}d\mu(\zeta) < \infty$ since $0 < 1+\epsilon < 2$. This gives the desired bound with $C(K,\epsilon) = c_1 + 2c_2$.

(b) We prove that $\partial^\alpha\partial^{-1}f(z) = \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{1}{\bar z-\bar\zeta}\partial^\alpha f(\zeta)\ d\mu(\zeta)$. The proof of the corresponding formula for $\partial^\alpha\bar\partial^{-1}f$ is similar. Once these formulae are proven, the bounds follow from those of part (a). By induction, it suffices to consider $|\alpha| = 1$. For any $h\in\mathbb{R}^2$,

$$\begin{aligned}\partial^{-1}f(z+h) - \partial^{-1}f(z) &= \tfrac{1}{\pi}\int_{\mathbb{R}^2}\tfrac{1}{\bar z+\bar h-\bar\zeta}f(\zeta)\ d\mu(\zeta) - \tfrac{1}{\pi}\int_{\mathbb{R}^2}\tfrac{1}{\bar z-\bar\zeta}f(\zeta)\ d\mu(\zeta)\\ &= \tfrac{1}{\pi}\int_{\mathbb{R}^2}\tfrac{1}{\bar z-\bar\zeta}f(\zeta+h)\ d\mu(\zeta) - \tfrac{1}{\pi}\int_{\mathbb{R}^2}\tfrac{1}{\bar z-\bar\zeta}f(\zeta)\ d\mu(\zeta)\\ &= \tfrac{1}{\pi}\int_{\mathbb{R}^2}\tfrac{1}{\bar z-\bar\zeta}\big[f(\zeta+h) - f(\zeta)\big]\ d\mu(\zeta)\end{aligned}$$

The existence of the limit defining $\partial^\alpha\partial^{-1}f$, as well as the desired formula, now follow by the Lebesgue dominated convergence theorem.

(c) We prove the formula for $\bar\partial^{-1}f$, evaluated at any $w\in\mathbb{R}^2$. For any $z\in\mathbb{R}^2$,

$$\bar\partial^{-1}f(z) = \tfrac{1}{\pi}F(z,w) + f(w)\bar\partial^{-1}\chi(z) \text{ with } F(z,w) = \int_{\mathbb{R}^2}\tfrac{1}{z-\zeta}\chi(\zeta)\big[f(\zeta) - f(w)\big]\ d\mu(\zeta)$$

By part (b), $\bar\partial^{-1}\chi$ is C^∞, so it suffices to prove that

$$\lim_{h\to 0}\tfrac{1}{h}\big[F(h+w,w) - F(w,w)\big] = -\int_{\mathbb{R}^2}\tfrac{1}{(w-\zeta)^2}\chi(\zeta)\big[f(\zeta) - f(w)\big]\ d\mu(\zeta)$$

In the limit, h tends to zero in $\mathbb{C}$. To get the first partial derivative, just restrict h to the real axis. To get the second partial derivative, set $h = iy$ with y real. Then you have to multiply by i to convert $\frac{1}{h}$ to $\frac{1}{y}$. So we have to prove that

$$\begin{aligned}\int_{\mathbb{R}^2} \Big\{\tfrac{1}{h}\big[\tfrac{1}{h+w-\zeta} - \tfrac{1}{w-\zeta}\big] + \tfrac{1}{(w-\zeta)^2}\Big\}\chi(\zeta)\big[f(\zeta) - f(w)\big]\ d\mu(\zeta) \\ = \int_{\mathbb{R}^2} \Big\{ - \tfrac{1}{(h+w-\zeta)(w-\zeta)} + \tfrac{1}{(w-\zeta)^2}\Big\}\chi(\zeta)\big[f(\zeta) - f(w)\big]\ d\mu(\zeta) \\ = \int_{\mathbb{R}^2} \tfrac{h}{(h+w-\zeta)(w-\zeta)^2}\chi(\zeta)\big[f(\zeta) - f(w)\big]\ d\mu(\zeta)\end{aligned}$$

converges to zero as h tends to zero. Since $f \in C^\epsilon(\mathbb{R}^2)$, this integral is bounded in magnitude by

$$\begin{aligned}|f|_{C^\epsilon} \int_{\mathbb{R}^2} \tfrac{|h|}{|h+w-\zeta|\,|w-\zeta|^{2-\epsilon}}\ d\mu(\zeta) &= |f|_{C^\epsilon} \int_{\mathbb{R}^2} \tfrac{|h|}{|h-\zeta'|\,|\zeta'|^{2-\epsilon}}\ d\mu(\zeta') \quad \text{where } \zeta = \zeta' + w \\ &= |f|_{C^\epsilon}|h|^\epsilon \int_{\mathbb{R}^2} \tfrac{1}{|\hat{h}-\xi|\,|\xi|^{2-\epsilon}}\ d\mu(\xi) \text{ where } \zeta' = |h|\xi,\ \hat{h} = \tfrac{h}{|h|}\end{aligned}$$

The integral $\int_{\mathbb{R}^2} \frac{1}{|\hat{h}-\xi|\,|\xi|^{2-\epsilon}}\ d\mu(\xi)$ is convergent (there are two singularities separated by distance one, each of degree strictly less than the dimension of $\mathbb{R}^2$, and the integrand decays like $\frac{1}{|\xi|^{3-\epsilon}}$ at infinity) and, by rotation invariance, is independent of the unit vector $\hat{h}$. So

$$\lim_{h\to 0}\left| \int_{\mathbb{R}^2} \Big\{\tfrac{1}{h}\big[\tfrac{1}{h+w-\zeta} - \tfrac{1}{w-\zeta}\big] + \tfrac{1}{(w-\zeta)^2}\Big\}\chi(\zeta)\big[f(\zeta) - f(w)\big]\ d\mu(\zeta)\right| \le \lim_{h\to 0} C|f|_{C^\epsilon}|h|^\epsilon = 0$$

We now prove $\partial\partial^{-1}f(z) = f(z)$. The proof that $\bar\partial\bar\partial^{-1}f(z) = f(z)$ is similar. The coefficient of $\int_{\mathbb{R}^2} \frac{1}{(\bar z-\bar\zeta)^2}\chi(\zeta)\big[f(\zeta) - f(z)\big]\ d\mu(\zeta)$ in $\partial\partial^{-1}f(z) = \frac{1}{2}\big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\big)\partial^{-1}f(x+iy)$ is $-\frac{1}{2}\frac{1-i(-i)}{\pi} = 0$, so it suffices to prove that $\partial\partial^{-1}\chi(z) = \chi(z)$. But by part (b), $\partial\partial^{-1}\chi(z) = \partial^{-1}\partial\chi(z)$ and, by Problem 4.4.5, $\partial^{-1}\partial\chi(z) = \chi(z)$.

(d) It suffices to consider $n = 0$ by the formulae of part (b). Again, we only consider $\bar\partial^{-1}f$. We show that, for each $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 1$,

$$\big|\partial^\alpha\bar\partial^{-1}f(w) - \partial^\alpha\bar\partial^{-1}f(z)\big| \le C(K,\epsilon,\epsilon')\|f\|_{C^\epsilon(\mathbb{R}^2)}|w-z|^{\epsilon'}$$

The required bounds on $\|\bar\partial^{-1}f\|_{L^\infty}$ and $\|\partial^\alpha\bar\partial^{-1}f\|_{L^\infty}$ are much easier. Observe that

$$\begin{aligned}\partial^\alpha\bar\partial^{-1}f(w) - \partial^\alpha\bar\partial^{-1}f(z) = -\tfrac{i^{\alpha_2}}{\pi}\int_{\mathbb{R}^2} \big[\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\big]\chi(\zeta)\ d\mu(\zeta) \\ + f(w)\,\partial^\alpha\bar\partial^{-1}\chi(w) - f(z)\,\partial^\alpha\bar\partial^{-1}\chi(z)\end{aligned}$$

Since $\bar\partial^{-1}\chi$ is C^∞ and $f \in C^\epsilon$, it is easy to bound

$$\begin{aligned}
&\big|f(w)\,\partial^\alpha\bar\partial^{-1}\chi(w) - f(z)\,\partial^\alpha\bar\partial^{-1}\chi(z)\big| \\
&\qquad\qquad \le \big|f(w)-f(z)\big|\,\big|\partial^\alpha\bar\partial^{-1}\chi(w)\big| + \big|f(z)\big|\,\big|\partial^\alpha\bar\partial^{-1}\chi(w) - \partial^\alpha\bar\partial^{-1}\chi(z)\big| \\
&\qquad\qquad \le C(K)|f|_{C^\epsilon}|w-z|^\epsilon + C(K)|w-z|\,\|f\|_{L^\infty}
\end{aligned}$$

so we concentrate on the first part.

$$\begin{aligned}
\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2} &= \big[f(\zeta)-f(w)\big]\Big[\tfrac{1}{(w-\zeta)^2} - \tfrac{1}{(z-\zeta)^2}\Big] + \tfrac{f(z)-f(w)}{(z-\zeta)^2} \\
&= \big[f(\zeta)-f(w)\big]\tfrac{z^2-w^2-2z\zeta+2w\zeta}{(w-\zeta)^2(z-\zeta)^2} + \tfrac{f(z)-f(w)}{(z-\zeta)^2} \\
&= (z-w)\big[f(\zeta)-f(w)\big]\tfrac{z+w-2\zeta}{(w-\zeta)^2(z-\zeta)^2} + \tfrac{f(z)-f(w)}{(z-\zeta)^2} \\
&= (z-w)\big[f(\zeta)-f(w)\big]\Big[\tfrac{1}{(w-\zeta)^2(z-\zeta)} + \tfrac{1}{(w-\zeta)(z-\zeta)^2}\Big] + \tfrac{f(z)-f(w)}{(z-\zeta)^2}
\end{aligned}$$

This is bounded by

$$\begin{aligned}
\Big|\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\Big| &\le |z-w|\,|f|_{C^\epsilon}\Big[\tfrac{1}{|w-\zeta|^{2-\epsilon}|z-\zeta|} + \tfrac{1}{|w-\zeta|^{1-\epsilon}|z-\zeta|^2}\Big] + |f|_{C^\epsilon}\tfrac{|z-w|^\epsilon}{|z-\zeta|^2} \\
&\le 2|z-w|\,|f|_{C^\epsilon}\max\Big\{\tfrac{1}{|w-\zeta|},\tfrac{1}{|z-\zeta|}\Big\}^{3-\epsilon} + |f|_{C^\epsilon}\tfrac{|z-w|^\epsilon}{|z-\zeta|^2} \\
&\equiv \qquad\qquad I \qquad\qquad\qquad\qquad + \qquad II
\end{aligned}$$

and also by

$$\Big|\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\Big| \le |f|_{C^\epsilon}\big[\tfrac{1}{|w-\zeta|^{2-\epsilon}} + \tfrac{1}{|z-\zeta|^{2-\epsilon}}\big] \le 2|f|_{C^\epsilon}\max\Big\{\tfrac{1}{|w-\zeta|},\tfrac{1}{|z-\zeta|}\Big\}^{2-\epsilon} \equiv III$$

We split the domain of integration for

$$\int_{\mathbb{R}^2}\Big[\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\Big]\chi(\zeta)\;d\mu(\zeta)$$

into two subdomains. In the first subdomain $I \ge II$ and then we bound

$$\Big|\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\Big| \le (2I)^{\epsilon'}III^{1-\epsilon'} \le 4|z-w|^{\epsilon'}\,|f|_{C^\epsilon}\max\Big\{\tfrac{1}{|w-\zeta|},\tfrac{1}{|z-\zeta|}\Big\}^{2+\epsilon'-\epsilon}$$

In the second subdomain $II \ge I$ and then we bound

$$\Big|\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\Big| \le (2II)^{\epsilon'/\epsilon}III^{1-\epsilon'/veps} \le 2|z-w|^{\epsilon'}\,|f|_{C^\epsilon}\max\Big\{\tfrac{1}{|w-\zeta|},\tfrac{1}{|z-\zeta|}\Big\}^{2+\epsilon'-\epsilon}$$

Since $2+\epsilon'-\epsilon < 2$

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}\Big[\tfrac{f(\zeta)-f(w)}{(w-\zeta)^2} - \tfrac{f(\zeta)-f(z)}{(z-\zeta)^2}\Big]\chi(\zeta)\;d\mu(\zeta)\Big| \\
&\qquad \le 4|z-w|^{\epsilon'}\,|f|_{C^\epsilon}\int_{\mathbb{R}^2}\max\Big\{\tfrac{1}{|w-\zeta|},\tfrac{1}{|z-\zeta|}\Big\}^{2+\epsilon'-\epsilon}\chi(\zeta)\;d\mu(\zeta) \\
&\qquad \le 4|z-w|^{\epsilon'}\,|f|_{C^\epsilon}\int_{\mathbb{R}^2}\Big[\tfrac{1}{|w-\zeta|^{2+\epsilon'-\epsilon}} + \tfrac{1}{|z-\zeta|^{2+\epsilon'-\epsilon}}\Big]\chi(\zeta)\;d\mu(\zeta) \\
&\qquad \le C|z-w|^{\epsilon'}\,|f|_{C^\epsilon}
\end{aligned}$$

as desired. ■

Problem 4.4.10 Let $1 \le p < q \le \infty$ and $\delta, \delta' \in \mathbb{R}$ with $\delta' < \delta - \frac{n}{p}\frac{q-p}{q}$ (when $q = \infty$, $\delta' < \delta - \frac{n}{p}$). Prove that if $f \in L^q_\delta(\mathbb{R}^n)$, then $f \in L^p_{\delta'}(\mathbb{R}^n)$ and $\|f\|_{L^p_{\delta'}} \le C\|f\|_{L^q_\delta}$ for some constant C that depends only on δ, δ', p, q and n.

Solution. First suppose that $q < \infty$. Let $r = \frac{q}{p} > 1$ and $r' = \frac{q}{q-p}$. For any $s > \frac{n}{2r'}$,

$$\begin{aligned}
\|f\|^p_{L^p_{\delta'}} &= \int \big(1+|x|^2\big)^{p\delta'/2}|f(x)|^p \, d^n x \\
&= \int \big(1+|x|^2\big)^{-s}\big(1+|x|^2\big)^{s+p\delta'/2}|f(x)|^p \, d^n x \\
&\le \big\|\big(1+|x|^2\big)^{-s}\big\|_{L^{r'}} \big\|\big(1+|x|^2\big)^{s+p\delta'/2}|f(x)|^p\big\|_{L^r} \\
&= \big\|\big(1+|x|^2\big)^{-s}\big\|_{L^{r'}} \big\|\big(1+|x|^2\big)^{s/p+\delta'/2}f(x)\big\|^p_{L^q} \\
&= \big\|\big(1+|x|^2\big)^{-s}\big\|_{L^{r'}} \|f\|^p_{L^q_\delta}
\end{aligned}$$

with $\delta = \delta' + \frac{2s}{p} > \delta' + n\frac{q-p}{qp}$. The first factor is finite since $s > \frac{n}{2r'}$. The same argument works when $q = \infty$. Just use $r' = 1$ and skip the middle inequality. ■

Problem 4.4.11 Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and let ∂^α refer to the α^{th} weak derivative. Let $f, u, v \in L^1_\delta(\mathbb{R}^n)$ for some $\delta \in \mathbb{R}$.

(a) Prove that if $\partial^\alpha f = u$ and $\partial^\alpha f = v$, then $u = v$.

(b) Prove that if f is continuously differentiable and the α^{th} classical derivative equals u, then $\partial^\alpha f = u$.

Solution. (a) If $\partial^\alpha f = u$ and $\partial^\alpha f = v$, then, by definition, $\langle \varphi, u\rangle = \langle \varphi, v\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. That is, $\langle \varphi, u - v\rangle = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Now let E be any bounded measurable subset of $\mathbb{R}^n$. Its characteristic function $\chi_E \in L^1(\mathbb{R}^n)$. So there is a sequence of C_0^∞ functions that converge in $L^1(\mathbb{R}^n)$ to χ_E. Since χ_E is bounded in magnitude by one and is of bounded support, we can always choose the approximating functions to be bounded in magnitude by two and supported in some bounded set E'. But every convergent sequence in $L^1(\mathbb{R}^n)$ has an almost everywhere pointwise convergent subsequence. So there is a sequence of functions in C_0^∞ that is bounded in magnitude by two, supported in E' and converges pointwise almost everywhere to χ_E. By the Lebesgue dominated convergence theorem $\int_E (u-v) \, d^n x = 0$ for all bounded measurable sets. But this forces $u = v$ almost everywhere.

(b) By integration by parts u is also a weak α^{th} derivative of f. Now just apply part (a). ■

Problem 4.4.12 Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and let ∂^α refer to the α^{th} weak derivative. Let $\delta \in \mathbb{R}$ and $f, u \in L^1_\delta(\mathbb{R}^n)$. Suppose that $\{f_j\}_{j\in\mathbb{N}}$ is a sequence in $L^1_\delta(\mathbb{R}^n)$ such that f_j converges to f in $L^1_\delta(\mathbb{R}^n)$ and $\partial^\alpha f_j$ converges to u in $L^1_\delta(\mathbb{R}^n)$. Prove that $\partial^\alpha f = u$.

Solution. By hypothesis

$$-\langle \partial^\alpha \varphi, f_j \rangle = \langle \varphi, \partial^\alpha f_j \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. But if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi, \partial^\alpha \varphi \in L^\infty_{-\delta}(\mathbb{R}^n)$. As

$$\big|\, \langle \psi, g \rangle \,\big| \le \|\psi\|_{L^\infty_{-\delta}} \|g\|_{L^1_\delta}$$

for all $\psi \in L^\infty_{-\delta}(\mathbb{R}^n)$ and $g \in L^1_\delta(\mathbb{R}^n)$, we have

$$\lim_{j\to\infty} \langle \partial^\alpha \varphi, f_j \rangle = \langle \partial^\alpha \varphi, f \rangle \qquad \text{and} \qquad \lim_{j\to\infty} \langle \varphi, \partial^\alpha f_j \rangle = \langle \varphi, u \rangle$$

so that $-\langle \partial^\alpha \varphi, f \rangle = \langle \varphi, u \rangle$ as desired. ■

Problem 4.4.13 Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and let ∂^α refer to the α^{th} weak derivative. Let $f \in L^1_\delta(\mathbb{R}^n)$ for some $\delta \in \mathbb{R}$.

(a) Let ψ be once continuously differentiable with polynomially bounded derivatives. Prove that $\partial^\alpha(\psi f) = \psi \partial^\alpha f + (\partial^\alpha \psi) f$.

(b) Let $\psi \in C_0^\infty(\mathbb{R}^n)$. Prove that $\partial^\alpha(\psi * f) = \psi * (\partial^\alpha f)$.

(c) Let $\psi : \mathbb{R} \to \mathbb{R}$ be once continuously differentiable. Suppose that f is continuous. Suppose further that there are monotone increasing functions $\Psi, F : [0,\infty)$ such that $|\psi(t)|, |\psi'(t)| \le \Psi(|t|)$, $|f(x)| \le F(|x|)$ and $\Psi \circ F$ is polynomially bounded. Prove that $\partial^\alpha(\psi \circ f) = (\psi' \circ f)\partial^\alpha f$.

Solution. (a) First suppose that ψ is C^∞ with polynomially bounded derivatives. Then, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $\bar\psi\varphi \in \mathcal{S}(\mathbb{R}^n)$ and

$$\begin{aligned}
-\langle \partial^\alpha \varphi, \psi f \rangle &= -\big\langle \bar\psi \partial^\alpha \varphi, f \big\rangle = -\big\langle \partial^\alpha(\bar\psi\varphi) - \varphi \partial^\alpha \bar\psi, f \big\rangle = -\big\langle \partial^\alpha(\bar\psi\varphi), f \big\rangle + \big\langle \varphi \partial^\alpha \bar\psi, f \big\rangle \\
&= \big\langle \bar\psi\varphi, \partial^\alpha f \big\rangle + \langle \varphi, f \partial^\alpha \psi \rangle \\
&= \langle \varphi, \psi \partial^\alpha f + f \partial^\alpha \psi \rangle
\end{aligned}$$

Now consider general ψ. By convolving with C_0^∞ mollifiers, we can construct a sequence ψ_n

- consisting of C^∞ functions with polynomially bounded derivatives

- with ψ_n, $\partial^\alpha\psi_n$ bounded by a common polynomial for all n and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$
- with ψ_n converging pointwise to ψ and $\partial^\alpha\psi_n$ converging pointwise to $\partial^\alpha\psi$ all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$.

Then

$$-\langle \partial^\alpha\varphi, \psi_n f\rangle = \langle \varphi, \psi_n\partial^\alpha f + f\partial^\alpha\psi_n\rangle$$

for all n and, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \langle \partial^\alpha\varphi, \psi_n f\rangle = \langle \partial^\alpha\varphi, \psi f\rangle \qquad \lim_{n\to\infty} \langle \varphi, \psi_n\partial^\alpha f + f\partial^\alpha\psi_n\rangle = \langle \varphi, \psi\partial^\alpha f + f\partial^\alpha\psi\rangle$$

giving

$$-\langle \partial^\alpha\varphi, \psi f\rangle = \langle \varphi, \psi\partial^\alpha f + f\partial^\alpha\psi\rangle$$

as desired.

(b) Let $\tilde\psi(x) = \overline{\psi(-x)}$. Since $\tilde\psi * \partial^\alpha\varphi$, $\tilde\psi * \varphi \in \mathcal{S}$ and $\psi * f$, $\tilde\psi * \partial^\alpha f \in L^1_\delta$,

$$-\langle \partial^\alpha\varphi, \psi * f\rangle = -\left\langle \tilde\psi * \partial^\alpha\varphi, f\right\rangle = -\left\langle \partial^\alpha(\tilde\psi * \varphi), f\right\rangle = \left\langle \tilde\psi * \varphi, \partial^\alpha f\right\rangle = \left\langle \varphi, \tilde\psi * \partial^\alpha f\right\rangle$$

(c) Let $\zeta \in C_0^\infty$ be nonnegative and obey $\int \zeta(x)\, d^n x = 1$. Set $\zeta_m(x) = m^n\zeta(mx)$. Then $f_m = \zeta_m \circ f$ is C^∞, so the classical derivative $\partial^\alpha(\psi \circ f_m) = (\psi' \circ f_m)\partial^\alpha f_m$. Furthermore $|f_m(x)| \le F\big(C+|x|\big)$ so that $|\psi\circ f_m(x)|$ and $|\psi'\circ f_m(x)| \le \Psi\circ F\big(C+|x|\big)$ are polynomially bounded. So

$$-\langle \partial^\alpha\varphi, \psi\circ f_m\rangle = \langle \varphi, (\psi'\circ f_m)\partial^\alpha f_m\rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and all $m \in \mathbb{N}$. Now $\psi\circ f_m$ converges pointwise to $\psi\circ f$ and is uniformly bounded by $\Psi\circ F\big(C+|x|\big)$. So, by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} -\langle \partial^\alpha\varphi, \psi\circ f_m\rangle = -\langle \partial^\alpha\varphi, \psi\circ f\rangle$$

Write

$$\langle \varphi, (\psi'\circ f_m)\partial^\alpha f_m\rangle = \langle \varphi, (\psi'\circ f_m)\zeta_m\circ\partial^\alpha f\rangle = \left\langle \tilde\zeta_m\circ\big(\,(\overline{\psi'\circ f_m})\,\varphi\big), \partial^\alpha f\right\rangle$$

where $\tilde\zeta_m(x) = \overline{\zeta_m(-x)}$. As f_m converges uniformly on compact sets to f, $\tilde\zeta_m\circ\big(\,(\overline{\psi'\circ f_m})\,\varphi\big)$ converges pointwise to $(\overline{\psi'\circ f})\,\varphi$. Again by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} \langle \varphi, (\psi'\circ f_m)\partial^\alpha f_m\rangle = \big\langle (\overline{\psi'\circ f})\,\varphi, \partial^\alpha f\big\rangle = \langle \varphi, (\psi'\circ f)\partial^\alpha f\rangle$$

as desired. ■

Problem 4.4.14 The purpose of this problem is to start providing some intuition concerning the behaviour of $\mathfrak{d}^{-1}f(z)$. Define

$$D(z,\zeta)=\tfrac{1}{z-\zeta}+\tfrac{\chi(\zeta)}{\zeta}\qquad S_\sigma(\zeta)=\begin{cases}\frac{1}{|\zeta|^\sigma} & \text{if } |\zeta|\le 1\\ 0 & \text{otherwise}\end{cases}\qquad L_\lambda(\zeta)=\begin{cases}\frac{1}{|\zeta|^\lambda} & \text{if } |\zeta|\ge 2\\ 0 & \text{otherwise}\end{cases}$$

Observe that $S_\sigma\in L^p(\mathbb{R}^2)$ if and only if $\sigma<\frac{2}{p}$ and that $L_\lambda\in L^p(\mathbb{R}^2)$ if and only if $\lambda>\frac{2}{p}$. Assume that $0<\sigma<2$ and that $\lambda>0$. Prove that there are constants C_σ and C_λ such that, if $|z|\le 1$, then

$$\int_{\mathbb{R}^2}|D(z,\zeta)|S_\sigma(\zeta)\;d\mu(\zeta)\le C_\sigma\left\{\begin{array}{ll}\frac{1}{|z|^{\sigma-1}} & \text{if } 1<\sigma<2\\ \ln\frac{1}{|z|} & \text{if } \sigma=1\\ 1 & \text{if } 0<\sigma<1\end{array}\right\}$$

$$\int_{\mathbb{R}^2}|D(z,\zeta)|L_\lambda(\zeta)\;d\mu(\zeta)\le C_\lambda|z|$$

and if $|z|\ge 2$, then

$$\int_{\mathbb{R}^2}|D(z,\zeta)|S_\sigma(\zeta)\;d\mu(\zeta)\le C_\sigma\tfrac{1}{|z|}$$

$$\int_{\mathbb{R}^2}|D(z,\zeta)|L_\lambda(\zeta)\;d\mu(\zeta)\le C_\lambda\left\{\begin{array}{ll}|z|^{1-\lambda} & \text{if } 0<\lambda<1\\ \ln|z| & \text{if } \lambda=1\\ 1 & \text{if } \lambda>1\end{array}\right\}$$

Solution. Observe that

$$\int_{\mathbb{R}^2}|D(z,\zeta)|S_\sigma(\zeta)\;d\mu(\zeta)=\int_{|\zeta|\le 1}\tfrac{1}{|z-\zeta|\,|\zeta|^\sigma}\;d\mu(\zeta)$$

$$\int_{\mathbb{R}^2}|D(z,\zeta)|L_\lambda(\zeta)\;d\mu(\zeta)=\int_{|\zeta|\ge 2}\tfrac{|z|}{|z-\zeta|\,|\zeta|^{1+\lambda}}\;d\mu(\zeta)$$

Hence, if $|z|\le 1$,

$$\int_{\mathbb{R}^2}|D(z,\zeta)|L_\lambda(\zeta)\;d\mu(\zeta)\le|z|\int_{|\zeta|\ge 2}\tfrac{1}{\frac{1}{2}|\zeta|\,|\zeta|^{1+\lambda}}\;d\mu(\zeta)\le C_\lambda|z|$$

and, if $|z|\ge 2$,

$$\int_{\mathbb{R}^2}|D(z,\zeta)|S_\sigma(\zeta)\;d\mu(\zeta)\le\tfrac{2}{|z|}\int_{|\zeta|\le 1}\tfrac{1}{|\zeta|^\sigma}\;d\mu(\zeta)\le C_\sigma\tfrac{1}{|z|}$$

For the remaining cases, we make the change of variables $\zeta = z\xi$ giving

$$\int_{\mathbb{R}^2} |D(z,\zeta)| S_\sigma(\zeta)\ d\mu(\zeta) = |z|^{1-\sigma} \int_{|\xi| \le \frac{1}{|z|}} \tfrac{1}{|1-\xi|\,|\xi|^\sigma}\ d\mu(\xi) \tag{S4.7}$$

$$\int_{\mathbb{R}^2} |D(z,\zeta)| L_\lambda(\zeta)\ d\mu(\zeta) = |z|^{1-\lambda} \int_{|\xi| \ge \frac{2}{|z|}} \tfrac{1}{|1-\xi|\,|\xi|^{1+\lambda}}\ d\mu(\xi) \tag{S4.8}$$

Since $\sigma < 2$, $\frac{1}{|1-\xi|\,|\xi|^\sigma}$ is locally integrable so that, for $|z| \le 1$,

$$\begin{aligned}
\int_{|\xi| \le \frac{1}{|z|}} \tfrac{1}{|1-\xi|\,|\xi|^\sigma}\ d\mu(\xi) &\le C_\sigma + \int_{\max\{2,\frac{1}{|z|}\} \le |\xi| \le \frac{1}{|z|}} \tfrac{1}{|1-\xi|\,|\xi|^\sigma}\ d\mu(\xi) \\
&\le C_\sigma + 2\int_{1 \le |\xi| \le \frac{1}{|z|}} \tfrac{1}{|\xi|^{1+\sigma}}\ d\mu(\xi) \\
&= C_\sigma + 2\int_0^{2\pi} d\theta \int_1^{\frac{1}{|z|}} dr\ r^{-\sigma} \\
&\le C_\sigma + 4\pi \begin{cases} \frac{1}{(1-\sigma)|z|^{1-\sigma}} & \text{if } \sigma < 1 \\ \ln \frac{1}{|z|} & \text{if } \sigma = 1 \\ \frac{1}{\sigma-1} & \text{if } \sigma > 1 \end{cases}
\end{aligned}$$

Substituting this into (S4.7) gives the desired bound on $\int_{\mathbb{R}^2} |D(z,\zeta)| S_\sigma(\zeta)\ d\mu(\zeta)$ when $|z| \le 1$. Since $\lambda > 0$, $\frac{1}{|1-\xi|\,|\xi|^{1+\lambda}}$ is integrable both at infinity and at $\xi = 1$ so that, for $|z| \ge 2$,

$$\begin{aligned}
\int_{|\xi| \ge \frac{2}{|z|}} \tfrac{1}{|1-\xi|\,|\xi|^{1+\lambda}}\ d\mu(\xi) &\le C_\lambda + \int_{\frac{2}{|z|} \le |\xi| \le \min\{\frac{1}{2},\frac{2}{|z|}\}} \tfrac{1}{|1-\xi|\,|\xi|^{1+\lambda}}\ d\mu(\xi) \\
&\le C_\lambda + 2\int_{\frac{2}{|z|} \le |\xi| \le 1} \tfrac{1}{|\xi|^{1+\lambda}}\ d\mu(\xi) \\
&= C_\lambda + 2\int_0^{2\pi} d\theta \int_{\frac{2}{|z|}}^{1} dr\ r^{-\lambda} \\
&\le C_\lambda + 4\pi \begin{cases} \frac{1}{1-\lambda} & \text{if } \lambda < 1 \\ \ln \frac{|z|}{2} & \text{if } \lambda = 1 \\ \frac{|z|^{\lambda-1}}{(\lambda-1)2^{\lambda-1}} & \text{if } \lambda > 1 \end{cases}
\end{aligned}$$

Substituting this into (S4.8) gives the desired bound on $\int_{\mathbb{R}^2} |D(z,\zeta)| L_\lambda(\zeta)\ d\mu(\zeta)$ when $|z| \ge 2$.

■

Problem 4.4.15 Let $\langle X,\mu\rangle$ and $\langle Y,\nu\rangle$ be measure spaces and let $k(x,y)=k_1(x,y)k_2(x,y)$ be a measurable function on $X\times Y$. Set

$$L=\sup_{x\in X}\Big\{\int_Y |k_1(x,y)|^2\ d\nu(y)\Big\}^{1/2}$$
$$R=\sup_{y\in Y}\Big\{\int_X |k_2(x,y)|^2\ d\mu(x)\Big\}^{1/2}$$

Prove that, if $L<\infty$ and $R<\infty$, then the map

$$(Kf)(x)=\int_Y k(x,y)f(y)\ d\nu(y)$$

is a bounded linear operator from $L^2(Y,\nu)$ to $L^2(X,\mu)$ with operator norm $\|K\|\le LR$.

Solution. Applying Cauchy Schwarz in $L^2(X\times Y,\mu\otimes\nu)$ gives that, for all $f\in L^2(Y,\nu)$ and $g\in L^2(X,\mu)$,

$$\begin{aligned}\big|\,\langle g,Kf\rangle\,\big| &= \Big|\int_{X\times Y}\big[\bar g(x)k_1(x,y)\big]\ \big[k_2(x,y)f(y)\big]\ d\mu(x)d\nu(y)\Big|\\ &\le \Big\{\int_{X\times Y}|g(x)k_1(x,y)|^2\ d\mu(x)d\nu(y)\Big\}^{1/2}\Big\{\int_{X\times Y}|k_2(x,y)f(y)|^2\ d\mu(x)d\nu(y)\Big\}^{1/2}\end{aligned}$$

The desired bound now follows from

$$\begin{aligned}\int_{X\times Y}|g(x)k_1(x,y)|^2\ d\mu(x)d\nu(y) &= \int_X |g(x)|^2\Big\{\int_Y |k_1(x,y)|^2\ d\nu(y)\Big\}\ d\mu(x)\\ &\le L^2\int_X |g(x)|^2\ d\mu(x)\\ \int_{X\times Y}|k_2(x,y)f(y)|^2\ d\mu(x)d\nu(y) &= \int_Y |f(y)|^2\Big\{\int_X |k_2(x,y)|^2\ d\mu(x)\Big\}\ d\nu(y)\\ &\le R^2\int_X |f(y)|^2\ d\nu(y)\end{aligned}$$

■

Problem 4.4.16 Let

$$D(z,\zeta)=\tfrac{1}{z-\zeta}+\tfrac{\chi(\zeta)}{\zeta}$$

and set

$$L_{\alpha_1,\beta_1}=\sup_{z\in\mathbb{R}^2}\Big\{\int_{\mathbb{R}^2}\big(1+|z|^2\big)^{\alpha_1\delta}|D(z,\zeta)|^{2\beta_1}\ d\mu(\zeta)\Big\}^{1/2}$$
$$R_{\alpha_2,\beta_2}=\sup_{\zeta\in\mathbb{R}^2}\Big\{\int_{\mathbb{R}^2}\big(1+|z|^2\big)^{\alpha_2\delta}|D(z,\zeta)|^{2\beta_2}\ d\mu(z)\Big\}^{1/2}$$

Prove that L_{α_1,β_1} and R_{α_2,β_2} are finite if $\frac{1}{2} < \beta_1 < 1$, $\beta_2 < 1$, $\alpha_1(-\delta) \ge 1 - \beta_1$ and $\alpha_2(-\delta) > 1$.

Solution. For any $a,\ b,\ \alpha > 0$, $(a+b)^\alpha \le \big(2\max\{a,b\}\big)^\alpha \le 2^\alpha\big(a^\alpha + b^\alpha\big)$. Combining

$$\sup_z \int_{|\zeta|\le 2} |D(z,\zeta)|^{2\beta_1}\ d\mu(\zeta) \le 2^{2\beta_1} \sup_z \int_{|\zeta|\le 2} \tfrac{1}{|z-\zeta|^{2\beta_1}}\ d\mu(\zeta) + 2^{2\beta_1} \int_{|\zeta|\le 2} \tfrac{\chi(\zeta)^{2\beta_1}}{|\zeta|^{2\beta_1}}\ d\mu(\zeta) < \infty$$

for all $2\beta_1 < 2$ and

$$\begin{aligned}
\sup_z \big(1+|z|^2\big)^{\alpha_1\delta} \int_{|\zeta|\ge 2} |D(z,\zeta)|^{2\beta_1}\ d\mu(\zeta) &= \sup_z \big(1+|z|^2\big)^{\alpha_1\delta} \int_{|\zeta|\ge 2} \tfrac{|z|^{2\beta_1}}{|z-\zeta|^{2\beta_1}|\zeta|^{2\beta_1}}\ d\mu(\zeta) \\
&= \sup_z \big(1+|z|^2\big)^{\alpha_1\delta} \int_{|\xi|\ge \frac{2}{|z|}} \tfrac{|z|^{2-2\beta_1}}{|1-\xi|^{2\beta_1}|\xi|^{2\beta_1}}\ d\mu(\xi) \\
&\le \sup_z \big(1+|z|^2\big)^{1+\alpha_1\delta-\beta_1} \int_{\mathbb{R}^2} \tfrac{1}{|1-\xi|^{2\beta_1}|\xi|^{2\beta_1}}\ d\mu(\xi) \\
&< \infty
\end{aligned}$$

(where we substituted $\zeta = z\xi$) for all $1 < 2\beta_1 < 2,\ 1+\alpha_1\delta-\beta_1 \le 0$ gives that $L_{\alpha_1,\beta_1} < \infty$ provided $\frac{1}{2} < \beta_1 < 1$ and $\alpha_1(-\delta) \ge 1-\beta_1$. Similarly,

$$\begin{aligned}
R^2_{\alpha_2,\beta_2} &= \sup_\zeta \int \big(1+|z|^2\big)^{\alpha_2\delta} |D(z,\zeta)|^{2\beta_2}\ d\mu(z) \\
&\le 2^{2\beta_2} \sup_\zeta \int \big(1+|z|^2\big)^{\alpha_2\delta} \tfrac{1}{|z-\zeta|^{2\beta_2}}\ d\mu(z) + 2^{2\beta_2} \sup_\zeta \int \big(1+|z|^2\big)^{\alpha_2\delta} \tfrac{\chi(\zeta)^{2\beta_2}}{|\zeta|^{2\beta_2}}\ d\mu(z) \\
&\le 2^{2\beta_2} \sup_\zeta \int_{|z-\zeta|\ge 1} \big(1+|z|^2\big)^{\alpha_2\delta}\ d\mu(z) + 2^{2\beta_2} \sup_\zeta \int_{|z-\zeta|\le 1} |z-\zeta|^{-2\beta_2}\ d\mu(z) \\
&\qquad + 2^{2\beta_2} \int \big(1+|z|^2\big)^{\alpha_2\delta}\ d\mu(z) \\
&< \infty
\end{aligned}$$

provided $-2\alpha_2\delta > 2$ and $2\beta_2 < 2$. ■

Problem 4.4.17 Let $2 < p < \infty$, $\varepsilon < 1 - \frac{2}{p}$ and let K be any compact subset of $\mathbb{R}^2$. Prove that there is a constant $C(K,\epsilon,p)$ such that if $f \in L^p(\mathbb{R}^2)$ is supported in K, then $\partial^{-1}f, \bar\partial^{-1}f \in C^\epsilon(\mathbb{R}^2)$ and

$$\|\partial^{-1}f\|_{C^\epsilon(\mathbb{R}^2)}\,,\ \|\bar\partial^{-1}f\|_{C^\epsilon(\mathbb{R}^2)} \le C(K,\epsilon)\|f\|_{L^p(\mathbb{R}^2)}$$

Solution. Since $\partial^{-1}f = \overline{\bar{\partial}^{-1}\bar{f}}$, it suffices to consider $\bar{\partial}f$. The required bound on the L^∞ part of the $\|\bar{\partial}^{-1}f\|_{C^\epsilon}$ is provided by Hölder's inequality.

$$\left|\bar{\partial}^{-1}f(z)\right| = \tfrac{1}{\pi}\left|\int_K \tfrac{1}{z-\zeta}f(\zeta)d\mu(\zeta)\right| \le \left[\int_K \tfrac{1}{|z-\zeta|^{p'}}d\mu(\zeta)\right]^{1/p'} \|f\|_{L^p}$$

The integral converges since $p' < 2$.

Since $f \in L^p$ is of compact support, it is also in L^2 with $\|f\|_{L^2} \le C(K)\|f\|_{L^p}$ and $\bar{\partial}^{-1}f$ differs from $\bar{\mathfrak{d}}^{-1}f(z)$ by the finite constant (with respect to z) $\frac{1}{\pi}\int_K \frac{\chi(\zeta)}{\zeta}f(\zeta)\,d\mu(\zeta)$. Hence, by Lemma 4.4.4.b,

$$\begin{aligned}\left|\bar{\partial}^{-1}f(z+h) - \bar{\partial}^{-1}f(z)\right| &= \left|\bar{\mathfrak{d}}^{-1}f(z+h) - \bar{\mathfrak{d}}^{-1}f(z)\right| \\ &\le C|h|^\varepsilon\left[\|f\|_{L^2(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)}\right] \\ &\le C|h|^\varepsilon\|f\|_{L^p(\mathbb{R}^2)}\end{aligned}$$

for all $z \in \mathbb{C}$ and $|h| \le 1$, as we wish. ■

Problem 4.4.18 Let

$$D = \begin{bmatrix}\bar{\partial} & 0 \\ 0 & \partial\end{bmatrix}$$

and let E_k be the map on 2×2 matrix valued functions $A = A(z)$ defined by

$$E_k\begin{bmatrix}a_1 & a_2 \\ a_3 & a_4\end{bmatrix} = \begin{bmatrix}a_1 & e^{-iz\bar{k}-i\bar{z}k}a_2 \\ e^{izk+i\bar{z}\bar{k}}a_3 & a_4\end{bmatrix}$$

Prove that $D_k = E_k^{-1}DE_k$.

Proof: Since

$$E_k^{-1}\begin{bmatrix}a_1 & a_2 \\ a_3 & a_4\end{bmatrix} = \begin{bmatrix}a_1 & e^{iz\bar{k}+i\bar{z}k}a_2 \\ e^{-izk-i\bar{z}\bar{k}}a_3 & a_4\end{bmatrix}$$

we have

$$\begin{aligned}E_k^{-1}DE_k\begin{bmatrix}a_1 & a_2 \\ a_3 & a_4\end{bmatrix} &= E_k^{-1}\begin{bmatrix}\bar{\partial} & 0 \\ 0 & \partial\end{bmatrix}\begin{bmatrix}a_1 & e^{-iz\bar{k}-i\bar{z}k}a_2 \\ e^{izk+i\bar{z}\bar{k}}a_3 & a_4\end{bmatrix} \\ &= E_k^{-1}\begin{bmatrix}\bar{\partial}a_1 & e^{-iz\bar{k}-i\bar{z}k}(\bar{\partial}a_2 - ika_2) \\ e^{izk+i\bar{z}\bar{k}}(\partial a_3 + ika_3) & \partial a_4\end{bmatrix} \\ &= \begin{bmatrix}\bar{\partial}a_1 & \bar{\partial}a_2 - ika_2 \\ \partial a_3 + ika_3 & \partial a_4\end{bmatrix}\end{aligned}$$

as desired. ■

Problem 4.4.19 Define the operator

$$\tilde{\mathcal{J}}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} a_4 & a_3 \\ a_2 & a_1 \end{bmatrix} \qquad \text{or equivalently} \qquad \tilde{\mathcal{J}}A = \tilde{J}A\tilde{J} \qquad \text{with} \qquad \tilde{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Prove that

$$E_k\tilde{\mathcal{J}}A = \tilde{\mathcal{J}}\overline{E_{\bar k}\bar A} \quad E_k^{-1}\tilde{\mathcal{J}}A = \tilde{\mathcal{J}}\overline{E_{\bar k}^{-1}\bar A} \quad D\tilde{\mathcal{J}}A = \tilde{\mathcal{J}}\overline{D\bar A} \quad \tilde{\mathcal{J}}Q = \bar Q \quad D_k\tilde{\mathcal{J}}A = \tilde{\mathcal{J}}\overline{D_{\bar k}\bar A}$$

Solution. For the first, we just explicitly compute the right and left hand sides.

$$\begin{aligned} \tilde{\mathcal{J}}\overline{E_{\bar k}\bar A} &= \tilde{\mathcal{J}}\overline{\begin{bmatrix} \bar a_1 & e^{-izk-i\bar z\bar k}\bar a_2 \\ e^{iz\bar k+i\bar z k}\bar a_3 & \bar a_4 \end{bmatrix}} = \tilde{\mathcal{J}}\begin{bmatrix} a_1 & e^{i\bar z\bar k+izk}a_2 \\ e^{-i\bar z k-iz\bar k}a_3 & a_4 \end{bmatrix} \\ &= \begin{bmatrix} a_4 & e^{-i\bar z k-iz\bar k}a_3 \\ e^{i\bar z\bar k+izk}a_2 & a_1 \end{bmatrix} = E_k\begin{bmatrix} a_4 & a_3 \\ a_2 & a_1 \end{bmatrix} = E_k\tilde{\mathcal{J}}A \end{aligned}$$

Also for the third.

$$\tilde{\mathcal{J}}\overline{D\bar A} = \tilde{\mathcal{J}}\overline{\begin{bmatrix} \bar\partial\bar a_1 & \bar\partial\bar a_2 \\ \partial\bar a_3 & \partial\bar a_4 \end{bmatrix}} = \tilde{\mathcal{J}}\begin{bmatrix} \partial a_1 & \partial a_2 \\ \bar\partial a_3 & \bar\partial a_4 \end{bmatrix} = \begin{bmatrix} \bar\partial a_4 & \bar\partial a_3 \\ \partial a_2 & \partial a_1 \end{bmatrix} = D\begin{bmatrix} a_4 & a_3 \\ a_2 & a_1 \end{bmatrix} = D\tilde{\mathcal{J}}A$$

And the fourth.

$$\tilde{\mathcal{J}}Q = \tilde{\mathcal{J}}\begin{bmatrix} 0 & q \\ \bar q & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar q \\ q & 0 \end{bmatrix} = \bar Q$$

The second follows from the first, since $E_k^{-1} = E_{-k}$. Finally, for the fifth, we just combine the first three.

$$D_k\tilde{\mathcal{J}}A = E_k^{-1}DE_k\tilde{\mathcal{J}}A = E_k^{-1}D\tilde{\mathcal{J}}\overline{E_{\bar k}\bar A} = E_k^{-1}\tilde{\mathcal{J}}\overline{DE_{\bar k}\bar A} = \tilde{\mathcal{J}}\overline{E_{\bar k}^{-1}DE_{\bar k}\bar A} = \tilde{\mathcal{J}}\overline{D_{\bar k}\bar A}$$

■

Problem 4.4.20 Prove that

$$D^{-1}\tilde{\mathcal{J}}A = \tilde{\mathcal{J}}\overline{D^{-1}\bar A} \qquad D_k^{-1}\tilde{\mathcal{J}}A = \tilde{\mathcal{J}}\overline{D_{\bar k}^{-1}\bar A}$$

Proof: For the first, we just explicitly compute the right and left hand sides.

$$\begin{aligned}
\tilde{\mathcal{J}}\overline{D^{-1}\bar{A}} &= \frac{1}{\pi}\tilde{\mathcal{J}}\overline{\int_{\mathbb{R}^2}\begin{bmatrix} z-\zeta & 0 \\ 0 & \bar{z}-\bar{\zeta}\end{bmatrix}^{-1}\overline{A(\zeta)}\;d\mu(\zeta)} \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2}\tilde{\mathcal{J}}\begin{bmatrix} \bar{z}-\bar{\zeta} & 0 \\ 0 & z-\zeta\end{bmatrix}^{-1} A(\zeta)\;d\mu(\zeta) \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2}\tilde{J}\begin{bmatrix} \bar{z}-\bar{\zeta} & 0 \\ 0 & z-\zeta\end{bmatrix}^{-1} A(\zeta)\tilde{J}\;d\mu(\zeta) \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2}\tilde{J}\begin{bmatrix} \bar{z}-\bar{\zeta} & 0 \\ 0 & z-\zeta\end{bmatrix}^{-1} \tilde{J}\tilde{J}A(\zeta)\tilde{J}\;d\mu(\zeta) \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2}\begin{bmatrix} z-\zeta & 0 \\ 0 & \bar{z}-\bar{\zeta}\end{bmatrix}^{-1} \tilde{\mathcal{J}}A(\zeta)\;d\mu(\zeta) = D^{-1}\tilde{\mathcal{J}}A
\end{aligned}$$

For the second, we just combine the first with the corresponding properties of E_k and E_k^{-1} from Problem 4.4.19.

$$\begin{aligned}
D_k^{-1}\tilde{\mathcal{J}}A &= E_k^{-1}D^{-1}E_k\tilde{\mathcal{J}}A = E_k^{-1}D^{-1}\tilde{\mathcal{J}}\overline{E_{\bar{k}}\bar{A}} = E_k^{-1}\tilde{\mathcal{J}}\overline{D^{-1}E_{\bar{k}}\bar{A}} = \tilde{\mathcal{J}}\overline{E_{\bar{k}}^{-1}D^{-1}E_{\bar{k}}\bar{A}} \\
&= \tilde{\mathcal{J}}\overline{D_{\bar{k}}^{-1}\bar{A}}
\end{aligned}$$

■

Problem 4.4.21 Let $\mathfrak{M}_n$ denote the space of all $n \times n$ matrices with complex entries. Define the norm

$$\left|\big[a_{i,j}\big]_{1\le i,j\le n}\right| = \sqrt{\sum_{i,j=1}^{n}|a_{i,j}|^2}$$

on $\mathfrak{M}_n$. Prove that $|AB| \le |A|\,|B|$ for all $A, B \in \mathfrak{M}_n$.

Solution. If $A = \big[a_{i,j}\big]_{1\le i,j\le n}$ and $B = \big[b_{i,j}\big]_{1\le i,j\le n}$, then

$$\begin{aligned}
|AB|^2 &= \sum_{i,j=1}^{n}\Big|\sum_{k=1}^{n} a_{i,k}b_{k,j}\Big|^2 \\
&\le \sum_{i,j=1}^{n}\Big[\sum_{k=1}^{n}|a_{i,k}|^2\Big]\Big[\sum_{\ell=1}^{n}|b_{\ell,j}|^2\Big] \quad \text{by Cauchy–Schwarz} \\
&= |A|^2\,|B|^2
\end{aligned}$$

■

Problem 4.4.22 Prove that, for any p and r with $p < 2 < r$,

$$\|D^{-1}f\|_{L^\infty} \le C(p,r)(\|f\|_{L^p} + \|f\|_{L^r}).$$

where D^{-1} is the operator of (4.4.9).

Proof: It suffices to add

$$\left|\frac{1}{\pi}\int_{|z-\zeta|\le 1}\begin{bmatrix} z-\zeta & 0 \\ 0 & \bar z - \bar\zeta \end{bmatrix}^{-1} f(\zeta)\, d\mu(\zeta)\right| \le \frac{1}{\pi}\left[\int_{|z-\zeta|\le 1} \tfrac{1}{|z-\zeta|^{r'}}\, d\mu(\zeta)\right]^{1/r'} \quad \|f\|_{L^r}$$

(by Hölder with $r' = \left(1-\frac{1}{r}\right)^{-1} < 2$) and

$$\left|\frac{1}{\pi}\int_{|z-\zeta|\ge 1}\begin{bmatrix} z-\zeta & 0 \\ 0 & \bar z - \bar\zeta \end{bmatrix}^{-1} f(\zeta)\, d\mu(\zeta)\right| \le \frac{1}{\pi}\left[\int_{|z-\zeta|\ge 1} \tfrac{1}{|z-\zeta|^{p'}}\, d\mu(\zeta)\right]^{1/p'} \quad \|f\|_{L^p}$$

(by Hölder with $p' = \left(1-\frac{1}{p}\right)^{-1} > 2$). ■

Problem 4.4.23 Suppose that $D_k h - Qh = 0$ or that $h - D_k^{-1}Qh = 0$ where Q was defined in (4.4.5), D_k was defined following (4.4.8) and D_k^{-1} was defined following (4.4.10). Prove that each of $u_\pm(z) = h_{11}(z) \pm e^{-izk-i\bar z\bar k}\overline{h_{21}(z)}$ and $v_\pm(z) = e^{-iz\bar k - i\bar z k}h_{12}(z) \pm \overline{h_{22}(z)}$ satisfies an equation of the form $\bar\partial u = r\bar u$ where $|r| = |q|$.

Solution. We have, in the two cases,

$$E_k^{-1}DE_kh = Qh \qquad \text{or} \qquad h = D_k^{-1}Qh = E_k^{-1}D^{-1}E_kQh$$

Applying E_k and, in the second case D, to both sides gives

$$DE_kh = E_kQh$$

Write

$$e_+ = e^{izk+i\bar z\bar k} \qquad \text{and} \qquad e_- = e^{-iz\bar k - i\bar z k}$$

Then

$$\begin{aligned} DE_kh &= \begin{bmatrix} \bar\partial & 0 \\ 0 & \partial \end{bmatrix}\begin{bmatrix} h_{11} & e_-h_{12} \\ e_+h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \bar\partial h_{11} & \bar\partial(e_-h_{12}) \\ \partial(e_+h_{21}) & \partial h_{22} \end{bmatrix} \\ E_kQh &= E_k\begin{bmatrix} qh_{21} & qh_{22} \\ \bar q h_{11} & \bar q h_{12} \end{bmatrix} = \begin{bmatrix} qh_{21} & e_-qh_{22} \\ e_+\bar q h_{11} & \bar q h_{12} \end{bmatrix} \end{aligned}$$

Taking the complex conjugate of the lower row of

$$\begin{bmatrix} \bar\partial h_{11} & \bar\partial(e_- h_{12}) \\ \partial(e_+ h_{21}) & \partial h_{22} \end{bmatrix} = \begin{bmatrix} q h_{21} & e_- q h_{22} \\ e_+ \bar q h_{11} & \bar q h_{12} \end{bmatrix}$$

and using that $\overline{\partial f} = \bar\partial \bar f$ gives

$$\begin{bmatrix} \bar\partial h_{11} & \bar\partial(e_- h_{12}) \\ \bar\partial(\bar e_+ \bar h_{21}) & \bar\partial \, \bar h_{22} \end{bmatrix} = \begin{bmatrix} q h_{21} & e_- q h_{22} \\ \bar e_+ q \bar h_{11} & q \bar h_{12} \end{bmatrix}$$

Thus

$$\begin{aligned} \bar\partial(h_{11} \pm \bar e_+ \bar h_{21}) &= q(h_{21} \pm \bar e_+ \bar h_{11}) = \pm \bar e_+ q(\,\overline{h_{11} \pm \bar e_+ h_{21}}\,) \\ \bar\partial(e_- h_{12} \pm \bar h_{22}) &= q(e_- h_{22} \pm \bar h_{12}) = \pm e_- q(\,\overline{e_- h_{12} \pm \bar h_{22}}\,) \end{aligned}$$

Since e_+ and e_- both have modulus one, these are the desired differential equations. ■

Problem 4.4.24 Let, for each sufficiently small $h \in \mathbb{C}$, A_h be a bounded linear operator on the Banach space $\mathcal{B}$. Suppose that

- $\mathbb{1} - A_0$ has a bounded inverse on $\mathcal{B}$
- $\lim_{h\to 0} \|A_h - A_0\| = 0$
- for each $f \in \mathcal{B}$, the map $h \mapsto A_h f$ is differentiable at $h = 0$ in $\mathcal{B}$.

Prove that $\mathbb{1} - A_h$ has a bounded inverse on $\mathcal{B}$ for all sufficiently small h and that for each $f \in \mathcal{B}$, the map $h \mapsto (\mathbb{1} - A_h)^{-1} f$ is differentiable at $h = 0$ in $\mathcal{B}$, with the derivative being $-(\mathbb{1} - A_0)^{-1} A_0' (\mathbb{1} - A_0)^{-1} f$.

Solution. If h is small enough that $\|A_h - A_0\| < \big\|(\mathbb{1} - A_0)^{-1}\big\|$, the geometric series

$$B_h = \sum_{n=0}^{\infty} (\mathbb{1} - A_0)^{-1} \Big((A_0 - A_h)(\mathbb{1} - A_0)^{-1} \Big)^n = \sum_{n=0}^{\infty} \Big((\mathbb{1} - A_0)^{-1} (A_0 - A_h) \Big)^n (\mathbb{1} - A_0)^{-1}$$

converges in norm and obeys

$$\begin{aligned} (\mathbb{1} - A_h) B_h &= (\mathbb{1} - A_0) B_h + (A_h - A_0) B_h \\ &= \sum_{n=0}^{\infty} \Big((A_0 - A_h)(\mathbb{1} - A_0)^{-1} \Big)^n - \sum_{n=0}^{\infty} \Big((A_0 - A_h)(\mathbb{1} - A_0)^{-1} \Big)^{n+1} = \mathbb{1} \end{aligned}$$

Similarly $B_h(\mathbb{1} - A_h) = \mathbb{1}$ so that B_h is a bounded inverse for $\mathbb{1} - A_h$.

We now prove that the map $h \mapsto (\mathbb{1} - A_h)^{-1} f$ is differentiable at $h = 0$ in $\mathcal{B}$. By hypothesis, the map $h \mapsto A_h (\mathbb{1} - A_0)^{-1} f$ is differentiable at $h = 0$ in $\mathcal{B}$. This means that there there is a $V \in \mathcal{B}$ such that $(A_h - A_0)(\mathbb{1} - A_0)^{-1} f = hV + o(h)$. The $o(h)$ signifies that the norm in $\mathcal{B}$ divided by h goes to zero as h goes to zero. Thus

$$
\begin{aligned}
(\mathbb{1} - A_h)^{-1} f - (\mathbb{1} - A_0)^{-1} f &= \sum_{n=1}^{\infty} \Big((\mathbb{1} - A_0)^{-1} (A_0 - A_h) \Big)^n (\mathbb{1} - A_0)^{-1} f \\
&= \sum_{n=0}^{\infty} \Big((\mathbb{1} - A_0)^{-1} (A_0 - A_h) \Big)^n (\mathbb{1} - A_0)^{-1} (A_0 - A_h)(\mathbb{1} - A_0)^{-1} f \\
&= -h \sum_{n=0}^{\infty} \Big((\mathbb{1} - A_0)^{-1} (A_0 - A_h) \Big)^n (\mathbb{1} - A_0)^{-1} V + o(h)
\end{aligned}
$$

Since $\|A_h - A_0\|$ converges to zero as $h \to 0$

$$
h \sum_{n=1}^{\infty} \Big((\mathbb{1} - A_0)^{-1} (A_0 - A_h) \Big)^n (\mathbb{1} - A_0)^{-1} V = o(h)
$$

so that

$$
(\mathbb{1} - A_h)^{-1} f - (\mathbb{1} - A_0)^{-1} f = -h (\mathbb{1} - A_0)^{-1} V + o(h)
$$

which is what we had to show. ■

Problem 4.4.25 Let $r > 2$ and $A(z) \in L^1(\mathbb{R}^2, \mathfrak{M}_2) \cap L^r(\mathbb{R}^2, \mathfrak{M}_2)$. Denote by A the multiplication operator

$$
f(z) \in L^\infty(\mathbb{R}^2, \mathfrak{M}_2) \mapsto A(z) f(z) \in L^1(\mathbb{R}^2, \mathfrak{M}_2) \cap L^r(\mathbb{R}^2, \mathfrak{M}_2)
$$

Prove that if $S \in \mathfrak{M}_2$ is purely off–diagonal, then

$$
(D_k^{-1} A E_k^{-1} S)(z) = (D_{\bar{k}}^{-1} A)(z)(E_k^{-1} S)(z)
$$

Solution. Both sides depend linearly on A, so it suffices to consider separately the cases that A is purely diagonal and that A is purely off–diagonal.

The diagonal case: If $A(\zeta)$ is a diagonal matrix for all $\zeta \in \mathbb{R}^2$, then S, $(E_k^{-1} S)(\zeta)$ and $\big(A E_k^{-1} S\big)(\zeta)$ are all off-diagonal for all $\zeta \in \mathbb{R}^2$ and

$$
\big(E_k A E_k^{-1} S\big)(\zeta) = \Lambda_k^{-1}(\zeta) A(\zeta) \Lambda_k(\zeta) S = A(\zeta) S
$$

since all diagonal matrices commute with each other. Thus the left hand side

$$
(D_k^{-1} A E_k^{-1} S)(z) = \Lambda_k(z)(D^{-1} A S)(z) = \Lambda_k(z)(D^{-1} A)(z) S
$$

since S is constant. Since A is diagonal and S is off–diagonal, the right hand side

$$(D_{\bar{k}}^{-1}A)(z)(E_k^{-1}S)(z) = (D^{-1}A)(z)\ \Lambda_k(z)S$$

The right and left hand sides agree, because the diagonal matrices $(D^{-1}A)(z)$ and $\Lambda_k(z)$ commute.

The off–diagonal case: If $A(\zeta)$ is an off-diagonal matrix for all $\zeta \in \mathbb{R}^2$, then $\big(AE_k^{-1}S\big)(\zeta) = A(\zeta)\Lambda_k(\zeta)S$ is a diagonal matrix for all $\zeta \in \mathbb{R}^2$ and

$$\big(E_k AE_k^{-1}S\big)(\zeta) = A(\zeta)\Lambda_k(\zeta)S = \Lambda_{\bar{k}}^{-1}(\zeta)A(\zeta)S = (E_{\bar{k}}A)(\zeta)S$$

Thus the left hand side

$$(D_k^{-1}AE_k^{-1}S)(z) = (E_k^{-1}D^{-1}E_kAE_k^{-1}S)(z) = (D^{-1}E_kAE_k^{-1}S)(z) = (D^{-1}E_{\bar{k}}A)(z)\ S$$

since S is constant. Since A and S are both off–diagonal, the right hand side

$$(D_{\bar{k}}^{-1}A)(z)(E_k^{-1}S)(z) = (E_{\bar{k}}^{-1}D^{-1}E_{\bar{k}}A)(z)\ \Lambda_k(z)S = (D^{-1}E_{\bar{k}}A)(z)\Lambda_k^{-1}(z)\ \Lambda_k(z)S$$

The right and left hand sides again agree. ■

Problem 4.4.26 Let $\tilde{\mathcal{J}}$ be the row/column exchange operator of Problem 4.4.19. Prove that $\tilde{\mathcal{J}}m(z,k) = \overline{m(z,\bar{k})}$.

Solution. By definition, suppressing the z's, $m(k) - D_k^{-1}Qm(k) = 1$. So, by Problems 4.4.19 and 4.4.20,

$$1 = \tilde{\mathcal{J}}1 = \tilde{\mathcal{J}}m(k) - \tilde{\mathcal{J}}D_k^{-1}Qm(k) = \tilde{\mathcal{J}}m(k) - \overline{D_{\bar{k}}^{-1}\tilde{\mathcal{J}}\,\overline{Qm(k)}} = \tilde{\mathcal{J}}m(k) - \overline{D_{\bar{k}}^{-1}(\tilde{\mathcal{J}}\,\bar{Q})(\tilde{\mathcal{J}}\overline{m(k)})}$$

Taking the complex conjugate of both sides and using the facts that $\tilde{\mathcal{J}}Q = \bar{Q}$ and $\tilde{\mathcal{J}}\bar{A} = \overline{\tilde{\mathcal{J}}A}$,

$$1 = \overline{\tilde{\mathcal{J}}m(k)} - D_{\bar{k}}^{-1}Q\overline{\tilde{\mathcal{J}}m(k)}$$

Thus $\overline{\tilde{\mathcal{J}}m(k)}$ obeys the defining equation of $m(\bar{k})$. By the uniqueness provision of Proposition 4.4.6

$$\overline{\tilde{\mathcal{J}}m(k)} = m(\bar{k})$$

Taking the complex conjugate of both sides gives the desired result. ■

Problem 4.4.27 Let $f \in L^1(\mathbb{R}^n)$. Prove that

$$\lim_{r\to 0+} \sup_{c\in\mathbb{R}^n} \int_{B_r(c)} |f(x)|\; d^n x = 0$$

where $B_r(c)$ is the ball of radius r centred on c.

Solution. It suffices to prove that, for every $\varepsilon > 0$,

$$\limsup_{r\to 0+} \sup_{c\in\mathbb{R}^n} \int_{B_r(c)} |f(x)|\; d^n x < \varepsilon \tag{S4.9}$$

Fix any $\varepsilon > 0$. Since $f \in L^1(\mathbb{R}^n)$,

$$\lim_{R\to\infty} \int_{|x|\ge R} |f(x)|\; d^n x = 0$$

In particular, there is an $R > 0$ such that

$$\int_{|x|\ge R} |f(x)|\; d^n x < \varepsilon$$

So it suffices to consider c with $|c| \le R+1$. If (S4.9) fails, there is a sequence $\{c_i\}_{i\in\mathbb{N}} \subset \overline{B_{R+1}(0)}$ and a sequence $\{r_i\}_{i\in\mathbb{N}} \subset (0,1)$ such that $\lim\limits_{i\to\infty} r_i = 0$ and $\int_{B_{r_i}(c_i)} |f(x)|\; d^n x \ge \varepsilon$. Since $\overline{B_{R+1}(0)}$ is compact, we may assume that the c_i's converge to some $c \in B_{R+r}(0)$. But then

$$\lim_{i\to\infty} \chi_{B_{r_i}(c_i)}(x) = 0 \qquad \text{for all } x \ne c$$

and, by the Lebesgue dominated convergence theorem

$$\lim_{i\to\infty} \int_{B_{r_i}(c_i)} |f(x)|\; d^n x = 0$$

This provides a contradiction to the assumption that (S4.9) fails. ■

Problem 4.4.28 Let $u \in H^1(\Omega)$ where Ω is a convex, bounded, open subset of $\mathbb{R}^2$ with smooth boundary. Let S_1 and S_2 be two measurable subsets of Ω. Prove that

$$\big|(u)_{S_1} - (u)_{S_2}\big| \le \sqrt{\pi}\,(\operatorname{diam}\Omega)^2 \Big(\tfrac{1}{|S_1|} + \tfrac{1}{|S_2|}\Big) \|\nabla u\|_{L^2(\Omega)}$$

Solution. Think of $(u)_{S_1} - (u)_{S_2}$ as a constant function, defined on Ω. Then

$$\begin{aligned}\big|(u)_{S_1} - (u)_{S_2}\big| &= \tfrac{1}{|\Omega|^{1/2}} \big\|(u)_{S_1} - (u)_{S_2}\big\|_{L^2(\Omega)} = \tfrac{1}{|\Omega|^{1/2}} \big\|(u)_{S_1} - u + u - (u)_{S_2}\big\|_{L^2(\Omega)} \\ &\le \tfrac{1}{|\Omega|^{1/2}} \big[\big\|u - (u)_{S_1}\big\|_{L^2(\Omega)} + \big\|u - (u)_{S_2}\big\|_{L^2(\Omega)}\big]\end{aligned}$$

By Poincaré's inequality, Proposition 2.3.10,

$$\big|(u)_{S_1} - (u)_{S_2}\big| \le \sqrt{\pi}\, \|\nabla u\|_{L^2(\Omega)} \big[\tfrac{(\operatorname{diam}\Omega)^2}{|S_1|} + \tfrac{(\operatorname{diam}\Omega)^2}{|S_2|}\big]$$

as desired. ■

Problem 4.4.29 Let $\delta \in \mathbb{R}$. Let $\chi \in C_0^\infty(\mathbb{R}^2)$, $u \in L^2_{loc} \cap L^1_\delta$ and $g \in L^1_\delta$. Assume further that u has a weak derivative $\bar\partial u \in L^1_\delta$ and that g is continuous and has a weak derivative $\bar\partial g \in L^2_{loc}$. Prove that

$$\bar\partial\big(\chi u e^{-g}\big) = u e^{-g}\bar\partial\chi + \chi e^{-g}\bar\partial u - \chi u e^{-g}\bar\partial g$$

Solution. Let $\tilde\chi \in C_0^\infty(\mathbb{R}^2)$ be identically one on the support of χ. By replacing u by $\tilde\chi u$ and g by $\tilde\chi g$, we may assume that $u \in L^2 \cap L^1$, $\bar\partial u \in L^1$ and that g is bounded and continuous with $\bar\partial g \in L^2$.

Let $\zeta \in C_0^\infty(\mathbb{R}^2)$ be nonnegative and obey $\int \zeta(x)\ d^nx = 1$. Set $\zeta_m(x) = m^n\zeta(mx)$. Then $g_m = \zeta_m \circ g$ is C^∞ and furthermore

- g_m converges pointwise to g and is uniformly bounded by $\sup_x |g(x)|$ and
- $\bar\partial g_m = \zeta_m * \bar\partial g$ converges in L^2 to $\bar\partial g$. This is easily seen by applying the Lebesgue dominated convergence theorem to the L^2 norm of the Fourier transform, $\hat g(k) - \hat\zeta(k/m)\hat g(k)$, of $g(x) - g_m(x)$.

By the classical product and chain rules

$$\bar\partial\big(\chi e^{-g_m}\big) = e^{-g_m}\bar\partial\chi - \chi e^{-g_m}\bar\partial g_m$$

and, by part (a) of Problem 4.4.13,

$$\bar\partial\big(\chi u e^{-g_m}\big) = u e^{-g_m}\bar\partial\chi + \chi e^{-g_m}\bar\partial u - \chi u e^{-g_m}\bar\partial g_m$$

To complete the proof, we apply Problem 4.4.12. To verify the hypotheses of Problem 4.4.12, namely that $\chi u e^{-g_m}$ and $\bar\partial\big(\chi u e^{-g_m}\big)$ converge in $L^1_{\delta'}$ for some $\delta' \in \mathbb{R}$, we observe that, by the Lebesgue dominated convergence theorem

- $\chi u e^{-g_m}$ converges in L^1 to $\chi u e^{-g}$ as $m \to \infty$,
- $u e^{-g_m}\bar\partial\chi$ converges in L^1 to $u e^{-g}\bar\partial\chi$,
- $\chi e^{-g_m}\bar\partial u$ converges in L^1 to $\chi e^{-g}\bar\partial u$ and
- since $u \in L^2$, $\chi u e^{-g_m}$ converges in L^2 to $\chi u e^{-g}$. Since $\bar\partial g_m$ converges in L^2 to ∂g, the product $\chi u e^{-g_m}\bar\partial g_m$ converges in L^1 to $\chi u e^{-g}\bar\partial g$. ■

Problem 4.4.30 Let $\varphi(x)$ be a smooth nonnegative function that vanishes for $|x| > 1$ and that is normalized by $\int \varphi(x)\, d\mu(x) = 1$. Let $f \in C^\epsilon(\mathbb{R}^2)$ and set, for $0 < t \le 1$, $f_t = \varphi_t * f$ where $\varphi_t(x) = t^{-2}\varphi\big(\frac{x}{t}\big)$. Prove that

$$\begin{aligned} \|f - f_t\|_{L^\infty} &\le |f|_{C^\epsilon} t^\epsilon \\ |f_t|_{C^\epsilon} &\le |f|_{C^\epsilon} \\ \big\|\tfrac{\partial^\alpha f_t}{\partial x^\alpha}\big\|_{L^\infty} &\le C_\alpha |f|_{C^\epsilon} t^{\epsilon-|\alpha|} \qquad \text{if } |\alpha| \ge 1 \end{aligned}$$

Solution. Since $\int \varphi_t(x)\, d\mu(x) = 1$,

$$\begin{aligned}\big|f(x) - f_t(x)\big| &= \bigg|\int t^{-2}\varphi\big(\tfrac{x-y}{t}\big)\ \big[f(x) - f(y)\big]\ d\mu(y)\bigg| \\ &\le \int t^{-2}\varphi\big(\tfrac{x-y}{t}\big)\ |f|_{C^\epsilon}|x-y|^\epsilon\ d\mu(y)\end{aligned}$$

Making the change of variables $y = x - ty'$,

$$\begin{aligned}\big|f(x) - f_t(x)\big| &\le t^\epsilon |f|_{C^\epsilon} \int \varphi(y')\,|y'|^\epsilon\ d\mu(y') \\ &\le t^\epsilon |f|_{C^\epsilon} \int \varphi(y')\ \ d\mu(y') \\ &= t^\epsilon |f|_{C^\epsilon}\end{aligned}$$

To prove the second bound, note that

$$\begin{aligned}\big|f_t(x+z) - f_t(x)\big| &= \bigg|\int \big[t^{-2}\varphi\big(\tfrac{x+z-y}{t}\big) - t^{-2}\varphi\big(\tfrac{x-y}{t}\big)\big] f(y)\ d\mu(y)\bigg| \\ &= \bigg|\int t^{-2}\varphi\big(\tfrac{x-y}{t}\big)\ \big[f(y+z) - f(y)\big]\ d\mu(y)\bigg| \\ &\le \int t^{-2}\varphi\big(\tfrac{x-y}{t}\big)\ |f|_{C^\epsilon}|z|^\epsilon\ d\mu(y) \\ &= |f|_{C^\epsilon}\,|z|^\epsilon\end{aligned}$$

For every $\alpha \in \mathbb{N}_0^2$ and $t > 0$,

$$\big|\tfrac{\partial^\alpha}{\partial x^\alpha}\varphi_t(x-y)f(y)\big| = \big|t^{-2-|\alpha|}\varphi^{(\alpha)}\big(\tfrac{x-y}{t}\big)f(y)\big| \le t^{-2-|\alpha|}\|f\|_{L^\infty}\|\varphi^{(\alpha)}\|_{L^\infty}\chi\big(|x-y| < 2\big)$$

Consequently, $f_t(x)$ is C^∞ in x and

$$f_t^{(\alpha)}(x) = \int t^{-2-|\alpha|}\varphi^{(\alpha)}\big(\tfrac{x-y}{t}\big)f(y)\ d\mu(y)$$

If $|\alpha| \ge 1$, $t^{-2-|\alpha|}\varphi^{(\alpha)}\big(\tfrac{x-y}{t}\big) = (-1)^{|\alpha|}\tfrac{\partial^\alpha}{\partial y^\alpha}\varphi_t(x-y)$ is a perfect derivative and

$$\int t^{-2-|\alpha|}\varphi^{(\alpha)}\big(\tfrac{x-y}{t}\big)f(x)\ d\mu(y) = 0$$

so that

$$\begin{aligned}\big|f_t^{(\alpha)}(x)\big| &= \bigg|\int t^{-2-|\alpha|}\varphi^{(\alpha)}\big(\tfrac{x-y}{t}\big)[f(y) - f(x)]\ d\mu(y)\bigg| \\ &\le \int t^{-2-|\alpha|}\big|\varphi^{(\alpha)}\big(\tfrac{x-y}{t}\big)\big|\ |f|_{C^\epsilon}|y-x|^\epsilon\ d\mu(y) \\ &= |f|_{C^\epsilon}t^{\epsilon-|\alpha|}\int \big|\varphi^{(\alpha)}(y')\big|\ |y'|^\epsilon\ d\mu(y) \quad \text{with } y = x - ty' \\ &\le |f|_{C^\epsilon}t^{\epsilon-|\alpha|}\int \big|\varphi^{(\alpha)}(y')\big|\ \ d\mu(y)\end{aligned}$$

which is the desired bound with $C_\alpha = \int \big|\varphi^{(\alpha)}(y')\big|\ \ d\mu(y)$. ■

Problem 4.4.31 Let $0 < \epsilon \le \frac{1}{2}$ and suppose that Q is in C^{ϵ}, compactly supported and $Q^d = 0$. Prove that, for each $k \in \mathbb{C}$, $m(\,\cdot\,, k) \in C^{1+\epsilon'}$ for all $\epsilon' < \epsilon$. Let $n \in \mathbb{N}$. Prove that if, in addition, $Q \in C^n$, then for each $k \in \mathbb{C}$, $m(\,\cdot\,, k) \in C^{n+\epsilon'}$ for all $\epsilon' < 1$.

Solution. Set, for each $m \in \mathbb{N}_0$ and $0 < \eta \le 1$

$$D_{m,\eta} = \left\{ \ f : \mathbb{R}^2 \to \mathfrak{M}_2 \ \middle|\ f \in C^{n+\epsilon} \text{ for all } 0 \le \epsilon < \eta \ \right\}$$

Then, for each fixed $k \in \mathbb{R}^2$ (the bounds will not be uniform in k, but that's allowed in this problem)

$$\text{(S4.10)} \qquad \begin{aligned} f \in D_{m,\eta} &\implies Qf \in D_{m,\eta} && \text{if } Q \in C^{m+\eta} \\ &\implies E_k Q f \in D_{m,\eta} && \text{since } \Lambda_k(z) \in C^{\infty} \\ &\implies D^{-1} E_k Q f \in D_{m+1,\eta} && \text{by part (d) of Problem 4.4.9} \\ &\implies D_k^{-1} Q f \in D_{m+1,\eta} && \text{since } \Lambda_k(z)^{-1} \in C^{\infty} \end{aligned}$$

Since $m = 1 + D_k^{-1} Q m$ and $m \in C^{\epsilon}$, by Corollary 4.4.13, the first claim by applying (S4.10) with $m = 0$, $\eta = \epsilon$. The second claim follows by applying (S4.10) with $\eta = 1$ and $m = 0, 1, \cdots, n-1$. ■