Appendix S1: Problem Solutions for §1

Problem 1.1.1 Find the Dirichlet to Neumann map when $\Omega = \left\{ x \in \mathbb{R}^2 \bigm| |x| < 1 \right\}$ and the conductivity $\gamma(x) \equiv 1$.

Solution. In polar coordinates

$$\nabla \cdot \nabla u(r,\theta) = \tfrac{\partial^2 u}{\partial r^2}(r,\theta) + \tfrac{1}{r}\tfrac{\partial u}{\partial r}(r,\theta) + \tfrac{1}{r^2}\tfrac{\partial^2 u}{\partial \theta^2}(r,\theta)$$

We look for a solution to the boundary value problem

$$\begin{array}{lll} \text{(a)} & \tfrac{\partial^2 u}{\partial r^2}(r,\theta) + \tfrac{1}{r}\tfrac{\partial u}{\partial r}(r,\theta) + \tfrac{1}{r^2}\tfrac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0 & \text{for all } \ r<1,\ 0 \le \theta \le 2\pi \\ \text{(b)} & u(1,\theta) = f(\theta) & \text{for all } 0 \le \theta \le 2\pi \\ \text{(c)} & u(r,\theta) \text{ is continuous at } r=0 & \end{array}$$

of the form $u(r,\theta) = \sum\limits_{n=-\infty}^{\infty} a_n(r) e^{in\theta}$. Substituting this into the differential equation (a) gives

$$\sum_{n=-\infty}^{\infty} e^{in\theta}\left[a_n''(r) + \tfrac{1}{r}a_n'(r) - \tfrac{n^2}{r^2}a_n(r)\right] = 0$$

This is satisfied for all θ if and only if

$$a_n''(r) + \tfrac{1}{r}a_n'(r) - \tfrac{n^2}{r^2}a_n(r) = 0$$

for all n. The guess $a_n(r) = r^\alpha$ solves this O.D.E. if and only if $\alpha(\alpha-1)+\alpha-n^2=0$. For $n \ne 0$, the general solution to the second order O.D.E. is $a_n(r) = C_n r^{|n|} + D_n r^{-|n|}$. For $n=0$, the general solution to the second order O.D.E. is $a_0(r) = C_0 + D_0 \ln r$. We reject $r^{-|n|}$ and $\ln r$ to achieve continuity at $r=0$. Thus

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} C_n r^{|n|} e^{in\theta}$$

satisfies the differential equation (a) and the continuity requirement (c) for all choices of the C_n's that decay sufficiently quickly as $|n| \to \infty$. To satisfy the boundary condition (b) we need

$$u(1,\theta) = \sum_{n=-\infty}^{\infty} C_n e^{in\theta} = f(\theta) \iff C_n = \tfrac{1}{2\pi}\int_0^{2\pi} f(\varphi) e^{-in\varphi}\, d\varphi$$

Hence

$$\begin{aligned}u(r,\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \tfrac{1}{2\pi}\int_0^{2\pi} f(\varphi)e^{-in\varphi}\, d\varphi \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi + \sum_{n=1}^{\infty} r^{|n|}\tfrac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n(\theta-\varphi)\, d\varphi\end{aligned}$$

The associated current on $r=1$ is

$$k(\theta) = \tfrac{\partial u}{\partial r}(1,\theta) = \boxed{\sum_{n=1}^{\infty} \tfrac{|n|}{\pi}\int_0^{2\pi} f(\varphi)\cos n(\theta-\varphi)\, d\varphi}$$

Problem 1.1.2 Let $\Omega = (-\infty, 0)\times S^1$. Functions on Ω can be identified with those functions $u(x,\theta)$ that are defined for $x<0$ and all $\theta\in\mathbb{R}$ and that are periodic of period 2π in θ. The gradient operator for Ω is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial\theta}\right)$. Find the Dirichlet to Neumann map when the conductivity $\gamma(x,\theta)\equiv 1$. Assume that potentials $u(x,\theta)$ must remain bounded in the limit $x\to-\infty$.

Solution. We look for a solution to the boundary value problem

$$\begin{array}{lll}\text{(a)} & \frac{\partial^2 u}{\partial x^2}(x,\theta) + \frac{\partial^2 u}{\partial\theta^2}(x,\theta) = 0 & \text{for all } \ x<0,\ 0\le\theta\le 2\pi \\ \text{(b)} & u(0,\theta) = f(\theta) & \text{for all } 0\le\theta\le 2\pi \\ \text{(c)} & u(x,\theta) \text{ remains bounded as } x\to-\infty & \\ \text{(d)} & u(x,\theta+2\pi) = u(x,\theta) & \text{for all } \ x<0,\ \theta\in\mathbb{R}\end{array}$$

By (d), u must be of the form $u(x,\theta) = \sum\limits_{n=-\infty}^{\infty} a_n(x)e^{in\theta}$. Substituting this into the differential equation (a) gives

$$\sum_{n=-\infty}^{\infty} e^{in\theta}\left[a_n''(x) - n^2 a_n(x)\right] = 0$$

This is satisfied for all θ if and only if

$$a_n''(x) - n^2 a_n(x) = 0$$

for all n and all $x<0$. The guess $a_n(x) = e^{\alpha x}$ solves this O.D.E. if and only if $\alpha^2 - n^2 = 0$. For $n\ne 0$, the general solution to the second order O.D.E. is $a_n(x) = C_n e^{|n|x} + D_n e^{-|n|x}$. For $n=0$, the general solution to the second order O.D.E. is $a_0(x) = C_0 + D_0 x$. We reject $e^{-|n|x}$ and x to achieve boundedness at $x=-\infty$. Thus

$$u(x,\theta) = \sum_{n=-\infty}^{\infty} C_n e^{|n|x} e^{in\theta}$$

satisfies the differential equation (a), boundedness requirement (c) and the periodicity requirement (d) for all choices of the C_n's that decay sufficiently quickly as $|n| \to \infty$. To satisfy the boundary condition (b) we need

$$u(0,\theta) = \sum_{n=-\infty}^{\infty} C_n e^{in\theta} = f(\theta) \iff C_n = \tfrac{1}{2\pi}\int_0^{2\pi} f(\varphi)e^{-in\varphi}\, d\varphi$$

Hence

$$\begin{aligned} u(x,\theta) &= \sum_{n=-\infty}^{\infty} e^{|n|x} e^{in\theta} \tfrac{1}{2\pi}\int_0^{2\pi} f(\varphi)e^{-in\varphi}\, d\varphi \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi + \sum_{n=1}^{\infty} e^{|n|x} \tfrac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n(\theta-\varphi)\, d\varphi \end{aligned}$$

The associated current on $x = 0$ is

$$k(\theta) = \tfrac{\partial u}{\partial x}(0,\theta) = \boxed{\sum_{n=1}^{\infty} \tfrac{|n|}{\pi}\int_0^{2\pi} f(\varphi)\cos n(\theta-\varphi)\, d\varphi}$$

Problem 1.1.3 Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the divergence theorem is applicable to Ω. Let $\gamma(x)$ be a real–valued C^∞ function on Ω all of whose derivatives are bounded. Suppose that the complex numbers λ, μ and the, not identically zero, functions $\varphi,\ \psi \in C^2(\overline{\Omega})$ obey

$$\begin{aligned} &\nabla\cdot[\gamma(x)\nabla\varphi(x)] = \lambda\varphi(x) && \text{for all } \ x\in\Omega \\ &\varphi(x) = 0 && \text{for all } x\in\partial\Omega \\ &\nabla\cdot[\gamma(x)\nabla\psi(x)] = \mu\psi(x) && \text{for all } \ x\in\Omega \\ &\psi(x) = 0 && \text{for all } x\in\partial\Omega \end{aligned}$$

We say that φ and ψ are eigenfunctions for the differential operator $u \mapsto \nabla\cdot[\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. The numbers λ and μ are the corresponding eigenvalues.

(a) Prove that $\lambda, \mu \in \mathbb{R}$.

(b) Prove that if $\lambda \neq \mu$ then φ and ψ are orthogonal in $L^2(\Omega)$. In other words, prove that $\int_\Omega \varphi(x)\overline{\psi(x)}\, d^n x = 0$.

(c) Suppose that $\gamma(x) > 0$ for all $x \in \Omega$. Prove that $\lambda, \mu < 0$.

(d) Let $\mathcal{H}$ be the closure of the subspace of $L^2(\Omega)$ spanned by the eigenfunctions for the differential operator $u \mapsto \nabla\cdot[\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Prove that there is an orthonormal basis for $\mathcal{H}$ consisting of real–valued eigenfunctions.

Solution. (a) We prove that λ is real. The proof that μ is real is identical. Since λ is the eigenvalue for the eigenfunction φ

$$\begin{aligned}\lambda\int_\Omega \varphi(x)\overline{\varphi(x)}\ d^nx &= \int_\Omega \lambda\varphi(x)\overline{\varphi(x)}\ d^nx = \int_\Omega \nabla\cdot[\gamma(x)\nabla\varphi(x)]\overline{\varphi(x)}\ d^nx \\ &= -\int_\Omega [\gamma(x)\nabla\varphi(x)]\cdot\overline{\nabla\varphi(x)}\ d^nx\end{aligned}$$

by the divergence theorem applied to $\overline{\varphi(x)}\gamma(x)\nabla\varphi(x)$. The boundary term vanished because $\overline{\varphi(x)}$ vanishes on $\partial\Omega$. Applying the divergence theorem a second time

$$\begin{aligned}\lambda\int_\Omega \varphi(x)\overline{\varphi(x)}\ d^nx &= \int_\Omega \varphi(x)\overline{\nabla\cdot[\gamma(x)\nabla\varphi(x)]}\ d^nx = \int_\Omega \varphi(x)\overline{\lambda\varphi(x)}\ d^nx \\ &= \overline{\lambda}\int_\Omega \varphi(x)\overline{\varphi(x)}\ d^nx\end{aligned}$$

Since $\int_\Omega |\varphi(x)|^2 \neq 0$, $\lambda = \overline{\lambda}$.

(b)

$$\begin{aligned}\lambda\int_\Omega \varphi(x)\overline{\psi(x)}\ d^nx = \int_\Omega \lambda\varphi(x)\overline{\psi(x)}\ d^nx \qquad &= \int_\Omega \nabla\cdot[\gamma(x)\nabla\varphi(x)]\overline{\psi(x)}\ d^nx \\ = -\int_\Omega [\gamma(x)\nabla\varphi(x)]\cdot\overline{\nabla\psi(x)}\ d^nx &= \int_\Omega \varphi(x)\overline{\nabla\cdot[\gamma(x)\nabla\psi(x)]}\ d^nx \\ = \int_\Omega \varphi(x)\overline{\mu\psi(x)}\ d^nx \qquad &= \mu\int_\Omega \varphi(x)\overline{\psi(x)}\ d^nx\end{aligned}$$

Since $\mu \neq \lambda$, $\int_\Omega \varphi(x)\overline{\psi(x)}\ d^nx = 0$.

(c) We prove that $\lambda < 0$. The proof that $\mu < 0$ is identical. We have already seen in the solution to part (a) that

$$\lambda\int_\Omega |\varphi(x)|^2\ d^nx = -\int_\Omega [\gamma(x)\nabla\varphi(x)]\cdot\overline{\nabla\varphi(x)}\ d^nx = -\int_\Omega \gamma(x)|\nabla\varphi(x)|^2\ d^nx$$

Since $\gamma > 0$ and φ cannot be constant (since it is not identically zero and vanishes on $\partial\Omega$), the right hand side is strictly negative. Thus λ must also be strictly negative.

(d) Eigenspaces for different eigenvalues are orthogonal, so it suffices to consider a single eigenspace. If φ is an eigenfunction, then both $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ obey the differential equation and the boundary conditions. So the set of real valued eigenfunctions spans the eigenspace. Since $L^2(\Omega)$ is separable (it is a subspace of $L^2(\mathbb{R}^n)$, which is separable by [RS, Chapter 2, Problem 11] and [RS, Theorem II.10]), the eigenspace is separable too and we may use the Gram-Schmidt process to produce an orthonormal basis.

■

Problem 1.1.4 Let Ω and γ be as in Problem 1.1.3. Assume that we already know

- an orthonormal basis for $L^2(\Omega)$ consisting of C^2 eigenfunctions for the differential operator $u \mapsto \nabla\cdot[\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Call the eigenfunctions and corresponding eigenvalues $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.
- a linear map $E : C^\infty(\partial\Omega)\to C^\infty(\overline{\Omega})$ such that $(Ef)(x) = f(x)$ for all $x\in\partial\Omega$).

Find the Dirichlet to Neumann map for conductivity γ.

Solution. Let $f\in C^\infty(\partial\Omega)$. Let u obey

$$\nabla\cdot\gamma\nabla u = 0 \text{ in } \Omega \qquad u = f \text{ on } \partial\Omega$$

Then $v = u - Ef$ obeys

$$\nabla\cdot\gamma\nabla v = -\nabla\cdot\gamma\nabla Ef \text{ in } \Omega \qquad v = 0 \text{ on } \partial\Omega$$

Let $v=\sum_{\ell=1}^\infty \alpha_\ell\varphi_\ell$ be the expansion of v in terms of the specified orthonormal basis. As v satisfies the differential equation above,

$$\sum_{\ell=1}^\infty \alpha_\ell\lambda_\ell\varphi_\ell = -\nabla\cdot\gamma\nabla Ef \implies \alpha_\ell = -\tfrac{1}{\lambda_\ell}\int_\Omega \overline{\varphi_\ell}\,\nabla\cdot\gamma\nabla Ef\ d^nx$$

and

$$\boxed{\Lambda_\gamma(f) = \tfrac{\partial}{\partial\nu}Ef\big|_{\partial\Omega} + \sum_{\ell=1}^\infty \alpha_\ell\tfrac{\partial\varphi_\ell}{\partial\nu}\big|_{\partial\Omega}}$$

with the α_ℓ's given above.

Problem 1.1.5 Apply the method of Problem 1.1.4 to find the Dirichlet to Neumann map for $\left\{\, x\in\mathbb{R}^2 \;\middle|\; |x|<1 \,\right\}$ with conductivity $\gamma\equiv 1$. You may assume that a suitable orthonormal basis exists.

Solution. We first find the eigenfunctions. We know that all of the eigenvalues are strictly negative, so write $\lambda = -\mu^2$ with $\mu > 0$. Recall that in polar coordinates

$$\Delta = \tfrac{\partial^2}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2}{\partial\theta^2}$$

Writing the eigenfunction in polar coordinates and expanding in a Fourier series, $\varphi = \sum_{n\in\mathbb{Z}} a_n(r)e^{in\theta}$, with respect to the angular variable

$$\begin{aligned}\Delta\varphi = \lambda\varphi &\iff a_n''(r) + \tfrac{1}{r}a_n'(r) - \tfrac{n^2}{r^2}a_n(r) = \lambda a_n(r)\\ &\iff r^2a_n''(r) + ra_n'(r) + (\mu^2r^2 - n^2)a_n(r) = 0\end{aligned}$$

Aside from a scaling by a factor μ, this is Bessel's equation of order n. The general solution that is bounded at the origin is a constant times the Bessel function $J_{|n|}(\mu r)$. If $J_{|n|}(\mu r)$ is to vanish at $r=1$, μ must be a zero of $J_{|n|}(\mu)$. Call these zeros $\mu_{m,|n|}$, $m\in\mathbb{N}$. Thus the eigenvalues are $\big\{\ -\mu^2_{m,|n|}\ \big|\ m\in\mathbb{N},\ n\in\mathbb{Z}\ \big\}$ and the corresponding normalized eigenfunctions are $\frac{1}{L_{m,|n|}}J_{|n|}\big(\mu_{m,|n|}r\big)e^{in\theta}$ where the normalization constants are determined by

$$L^2_{m,|n|}=2\pi\int_0^1 J^2_{|n|}\big(\mu_{m,|n|}r\big)\ rdr$$

Next we find a suitable extension operator. This is easy. Just define

$$(Ef)(x)=\chi(|x|)f\big(\tfrac{x}{|x|}\big)$$

where χ is C^∞ function that takes values in $[0,1]$, is zero for all x is a neighbourhood of 0 and is one for x is a neighbourhood of 1. In a neighbourhood of $r=1$, Ef is a function of θ only. So $\frac{\partial}{\partial\nu}Ef\big|_{\partial\Omega}=\frac{\partial}{\partial r}Ef\big|_{\partial\Omega}=0$ and the answer from Problem 1.1.4, applied to this specific problem, becomes

$$\Lambda_\gamma(f)=\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\alpha_{m,n}\tfrac{\partial}{\partial r}\tfrac{1}{L_{m,|n|}}J_{|n|}(\mu_{m,|n|}r)e^{in\theta}\big|_{r=1}$$

where

$$\alpha_{m,n}=\tfrac{1}{\mu^2_{m,|n|}L_{m,|n|}}\int_0^{2\pi}d\theta\int_0^1 dr\ rJ_{|n|}\big(\mu_{m,|n|}r\big)e^{-in\theta}\,\Delta(Ef)$$

Substituting

$$\begin{aligned}&\tfrac{d}{dr}J_n(\mu r)=\tfrac{\mu}{2}\big[J_{n-1}(\mu r)-J_{n+1}(\mu r)\big]\text{ if }n\in\mathbb{N}\\&\tfrac{d}{dr}J_0(\mu r)=-\mu J_1(\mu r)\end{aligned}$$

(see, for example, [Hi, §4.9]) gives

$$\begin{aligned}\Lambda_\gamma(f)=&\sum_{m,n=1}^\infty\tfrac{\mu_{m,n}}{2L_{m,n}}\big[J_{n-1}(\mu_{m,n})-J_{n+1}(\mu_{m,n})\big]\big[\alpha_{m,n}e^{in\theta}+\alpha_{m,-n}e^{-in\theta}\big]\\&-\sum_{m=1}^\infty\alpha_{m,0}\tfrac{\mu_{m,0}}{L_{m,0}}J_1(\mu_{m,0})\end{aligned}$$

Problem 1.1.6 Let Ω be an open subset of $\mathbb{R}^n$. Let $\gamma\in C^1(\Omega)$ be bounded away from zero. Find $q,\beta\in C^1(\Omega)$ such that

$$\nabla\cdot[\gamma\nabla u]=0\iff(-\Delta+q)v=0\text{ for }v=\beta u$$

Solution. Let $\alpha = \frac{1}{\beta}$. By the usual vector identities

$$\begin{aligned}
\nabla\cdot\gamma\nabla\alpha v &= \nabla\cdot\gamma\big[\alpha\nabla v + v\nabla\alpha\big]\\
&= \nabla\gamma\cdot\big[\alpha\nabla v + v\nabla\alpha\big] + \gamma\nabla\cdot\big[\alpha\nabla v + v\nabla\alpha\big]\\
&= \alpha\nabla\gamma\cdot\nabla v + v\nabla\gamma\cdot\nabla\alpha + \alpha\gamma\Delta v + 2\gamma\nabla\alpha\cdot\nabla v + (\gamma\Delta\alpha)v\\
&= (\alpha\gamma)\Delta v + (2\gamma\nabla\alpha + \alpha\nabla\gamma)\cdot\nabla v + (\gamma\Delta\alpha + \nabla\gamma\cdot\nabla\alpha)v
\end{aligned}$$

Choosing $\alpha = \gamma^{-1/2}$, the coefficient of ∇v becomes

$$2\gamma\nabla\alpha + \alpha\nabla\gamma = 2\gamma\big(-\tfrac{1}{2}\gamma^{-3/2}\big)\nabla\gamma + \gamma^{-1/2}\nabla\gamma = 0$$

the coefficient of Δv becomes $\alpha\gamma = \gamma^{1/2}$ and the coefficient of v becomes

$$\begin{aligned}
\gamma\nabla\cdot\nabla\alpha + \nabla\gamma\cdot\nabla\alpha &= \gamma\nabla\cdot\big(-\tfrac{1}{2}\gamma^{-3/2}\big)\nabla\gamma + \nabla\gamma\cdot\big(-\tfrac{1}{2}\gamma^{-3/2}\big)\nabla\gamma\\
&= -\tfrac{1}{2}\gamma^{-1/2}\Delta\gamma + \tfrac{3}{4}\gamma^{-3/2}\big|\nabla\gamma\big|^2 - \tfrac{1}{2}\gamma^{-3/2}\big|\nabla\gamma\big|^2\\
&= -\tfrac{1}{2}\gamma^{-1/2}\Delta\gamma + \tfrac{1}{4}\gamma^{-3/2}\big|\nabla\gamma\big|^2
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla\cdot\gamma\nabla u = 0 &\iff \gamma^{1/2}\Delta v + \big[-\tfrac{1}{2}\gamma^{-1/2}\Delta\gamma + \tfrac{1}{4}\gamma^{-3/2}\big|\nabla\gamma\big|^2\big]v = 0\\
&\iff (-\Delta + q)v = 0
\end{aligned}$$

where $\boxed{q = \frac{1}{2}\gamma^{-1}\Delta\gamma - \frac{1}{4}\gamma^{-2}|\nabla\gamma|^2}$ and $v = \beta u$ with $\boxed{\beta = \sqrt{\gamma}}$.

Problem 1.1.7 Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\Lambda_\gamma(f)$ denote the Dirichlet to Neumann map for the conductivity $\gamma(x)$. Let $\beta(x)$ be a C^∞ function on Ω all of whose derivatives are bounded. Compute $\frac{d}{dt}\Lambda_{\gamma+t\beta}(f)\big|_{t=0}$. Assume that we already know

- a complete orthonormal basis for $L^2(\Omega)$ consisting of C^2 eigenfunctions for the differential operator $u \mapsto \nabla\cdot[\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Call the eigenfunctions and corresponding eigenvalues $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.
- the solution to the boundary value problem $\nabla\cdot[\gamma\nabla u] = 0$ in Ω, $u = f$ on $\partial\Omega$. Call the solution $u_0(x)$.

Solution. Let $u_t(x,\theta)$ denote the solution to

$$\begin{aligned}
&\text{(a)}\quad \nabla\cdot\big(\gamma(x) + t\beta(x)\big)\nabla u_t(x) = 0 && \text{for all } x\in\Omega\\
&\text{(b)}\quad u_t(x) = f(x) && \text{for all } x\in\partial\Omega
\end{aligned}$$

Apply $\frac{d}{dt}\,\big|_{t=0}$ to these conditions. In terms of $\frac{d}{dt}u_t(x)\big|_{t=0} = \dot u(x)$, we have

$$\begin{aligned}
&\text{(a}'\text{)}\quad \nabla\cdot\gamma(x)\nabla\dot u(x) = -\nabla\cdot\beta(x)\nabla u_0(x) && \text{for all } x\in\Omega\\
&\text{(b}'\text{)}\quad \dot u(x) = 0 && \text{for all } x\in\partial\Omega
\end{aligned}$$

We look for a solution of the form $\dot{u}(x) = \sum\limits_{\ell=1}^{\infty} a_\ell \varphi_\ell(x)$. Substituting this into the differential equation (a$'$) gives

$$\sum_{\ell=1}^{\infty} \lambda_\ell a_\ell \varphi_\ell(x) = F(x)$$

where $F(x) = -\nabla \cdot \beta(x) \nabla u_0(x)$. This is satisfied if and only if

$$\lambda_\ell a_\ell = \frac{\int_\Omega F(x) \overline{\varphi_\ell(x)}\ d^n x}{\int_\Omega |\varphi_\ell(x)|^2\ d^n x}$$

for all $\ell \in \mathbb{N}$. Hence, by the divergence theorem,

$$a_\ell = \tfrac{1}{\lambda_\ell} \frac{\int_\Omega \beta(x) \nabla u_0(x) \cdot \overline{\nabla \varphi_\ell(x)}\ d^n x}{\int_\Omega |\varphi_\ell(x)|^2\ d^n x}$$

We are after

$$\begin{aligned} \tfrac{d}{dt} \Lambda_{\gamma+t\beta}(f)\big|_{t=0} &= \tfrac{d}{dt} \big(\gamma(x) + t\beta(x)\big) \tfrac{\partial u_t}{\partial \nu}(x) \Big|_{\substack{t=0 \\ x \in \partial\Omega}} = \big[\beta(x) \tfrac{\partial u_0}{\partial \nu}(x) + \gamma(x) \tfrac{\partial \dot{u}}{\partial \nu}(x)\big]_{x \in \partial\Omega} \\ &= \boxed{\tfrac{\beta}{\gamma} \Lambda_\gamma(f) + \textstyle\sum_{\ell=1}^{\infty} a_\ell \gamma(x) \tfrac{\partial \varphi_\ell}{\partial \nu}(x) \big|_{x \in \partial\Omega}} \end{aligned}$$

Problem 1.2.1 The purpose of this problem is to prove that there is a unique continuous, rotation invariant, mass one linear functional on $C_{\mathbb{R}}(S^{n-1})$. The argument is similar to that used to prove the existence of a Haar measure on a compact topological group. For each $f \in C_{\mathbb{R}}(S^{n-1})$ set

$$\mathcal{F}(f) = \Big\{ \sum_{i=1}^{m} a_i L_{\alpha_i} f \ \Big|\ m \in \mathbb{N}, \alpha_i \in SO(n), a_i > 0, \sum_{i=1}^{m} a_i = 1 \Big\} \subset C_{\mathbb{R}}(S^{n-1})$$

where, for $f \in C_{\mathbb{R}}(S^{n-1})$ and $\alpha \in SO(n)$,

$$(L_\alpha f)(x) = f\big(\alpha^{-1} x\big)$$

Denote by $\overline{\mathcal{F}(f)}$ the closure of $\mathcal{F}(f)$ in $C_{\mathbb{R}}(S^{n-1})$.

(a) Use Arzelà–Ascoli to show that $\overline{\mathcal{F}(f)}$ is a compact subset of $C_{\mathbb{R}}(S^{n-1})$.

(b) Show that $\overline{\mathcal{F}(f)}$ contains a constant function.

(c) Show that the constant function in $\overline{\mathcal{F}(f)}$ is $c(f) = \frac{1}{\Omega_n} \int_{S^{n-1}} f(x)\ d\sigma(x)$ where $d\sigma(x)$ is the surface measure on S^{n-1} and $\Omega_n = \int_{s^{n-1}} 1\ d\sigma(x)$ is the volume of S^{n-1}.

(d) Let $\mathcal{L} : C_{\mathbb{R}}(S^{n-1}) \to \mathbb{R}$ be continuous and linear and obey $\mathcal{L}(1) = 1$ and

$$\mathcal{L}(L_\alpha f) = \mathcal{L}(f)$$

for all $f \in C_{\mathbb{R}}(S^{n-1})$ and $\alpha \in SO(n)$. Prove that $\mathcal{L}(f) = c(f)$ for all $f \in C_{\mathbb{R}}(S^{n-1})$.

Solution. (a) Let $\|f\|_\infty = \sup_{x\in S^{n-1}} |f(x)|$. Note first that

$$\Big\| \sum_{i=1}^m a_i L_{\alpha_i} f \Big\|_\infty \le \sum_{i=1}^m a_i \|L_{\alpha_i} f\|_\infty = \sum_{i=1}^m a_i \|f\|_\infty = \|f\|_\infty \tag{S1.1}$$

In other words, $\overline{\mathcal{F}(f)}$ is a bounded subset of $C_{\mathbb{R}}(S^{n-1})$ that is contained in the closed ball of radius $\|f\|_\infty$ centered on the origin. We next claim that $\mathcal{F}(f)$ is an equicontinuous subset of $C_{\mathbb{R}}(S^{n-1})$. Pick any $\varepsilon > 0$. Since f is continuous on the compact set S^{n-1}, f is uniformly continuous so that there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Then

$$\begin{aligned}\Big| \sum_{i=1}^m a_i (L_{\alpha_i} f)(x) - \sum_{i=1}^m a_i (L_{\alpha_i} f)(y) \Big| &\le \sum_{i=1}^m a_i \big| (L_{\alpha_i} f)(x) - (L_{\alpha_i} f)(y) \big| \\ &\le \sum_{i=1}^m a_i \big| f(\alpha_i^{-1} x) - f(\alpha_i^{-1} y) \big| \le \sum_{i=1}^m a_i \varepsilon = \varepsilon\end{aligned}$$

whenever $|x-y| < \delta$ since $|\alpha_i^{-1}x - \alpha_i^{-1}y| = |x-y| < \delta$ for all $1 \le i \le m$. Hence the closure $\overline{\mathcal{F}(f)}$ is also equicontinuous and is, by Arzelà–Ascoli, a compact subset of $C_{\mathbb{R}}(S^{n-1})$.

(b) Set

$$M(f) = \max_{x\in S^{n-1}} f(x) \qquad m(f) = \min_{x\in S^{n-1}} f(x) \qquad v(f) = M(f) - m(f)$$

The function $v(f)$ is continuous on $C_{\mathbb{R}}(S^{n-1})$ and in particular on the compact subset $\overline{\mathcal{F}(f)}$. Therefore $v(f)$ attains its minimum value at a point $f_* \in \overline{\mathcal{F}(f)}$. Either $v(f_*) = 0$, and f_* is a constant function, or $v(f_*) \ne 0$ and $M(f_*) > m(f_*)$. We now show that the latter possibility cannot happen. Suppose that $M(f_*) > m(f_*)$. Then

$$\mathcal{B} = \Big\{ \, x \in S^{n-1} \, \Big| \, f_*(x) > \tfrac{M(f_*)+m(f_*)}{2} \, \Big\}$$

is a nonempty subset of S^{n-1}. For each $\alpha \in SO(n)$, $\alpha\mathcal{B} = \big\{ \, \alpha x \, \big| \, x \in \mathcal{B} \, \big\}$ is also open and $\{\alpha\mathcal{B}\}_{\alpha\in SO(n)}$ is an open cover of S^{n-1}. Since S^{n-1} is compact, $S^{n-1} \subset \alpha_1\mathcal{B} \cup \cdots \cup \alpha_m\mathcal{B}$, for some $m \in \mathbb{N}$ and $\alpha_1, \cdots, \alpha_m \in SO(n)$. Set

$$\tilde{f}_*(x) = \tfrac{1}{m} \sum_{i=1}^m L_{\alpha_i} f_*(x)$$

Then

- $\tilde{f}_* \in \overline{\mathcal{F}(f)}$ since $\frac{1}{m}\sum_{i=1}^m L_{\alpha_i} g \in \mathcal{F}(f)$ for all $g \in \mathcal{F}(f)$.
- $M(\tilde{f}_*) \le M(f_*)$
- $m(\tilde{f}_*) \ge m(f_*) + \frac{1}{2m} v(f_*)$. To see this, observe that, for any $x \in S^{n-1}$, there is an $1 \le \ell \le m$ such that $x \in \alpha_\ell \mathcal{B}$. Then $\alpha_\ell^{-1} x \in \mathcal{B}$ and, by the definition of $\mathcal{B}$, $f_*(\alpha_\ell^{-1}x) > \frac{M(f_*)+m(f_*)}{2}$. Hence

$$\begin{aligned}\tilde{f}_*(x) &= \tfrac{1}{m} f_*(\alpha_\ell^{-1}x) + \tfrac{1}{m}\sum_{i\ne\ell} f_*(\alpha_i^{-1}x) > \tfrac{1}{m}\tfrac{M(f_*)+m(f_*)}{2} + \tfrac{1}{m}\sum_{i\ne m} m(f_*) \\ &= \tfrac{1}{2m} M(f_*) + \big(1-\tfrac{1}{2m}\big) m(f_*) \quad = m(f_*) + \tfrac{1}{2m} v(f_*)\end{aligned}$$

  This is true for all $x \in S^{n-1}$, so $m(\tilde{f}_*) \ge m(f_*) + \frac{1}{2m} v(f_*)$.

Hence $v(\tilde{f}_*) = M(\tilde{f}_*) - m(\tilde{f}_*) \le M(f_*) - m(f_*) - \frac{1}{2n} v(f_*) < M(f_*) - m(f_*) = v(f_*)$ which is a contradiction. Consequently $v(f_*) = 0$ and f_* is a constant function.

(c) Let ℓ be any constant function in $\overline{\mathcal{F}(f)}$ and let $\varepsilon > 0$. Choose $\alpha_i \in SO(n), a_i > 0$, $1 \le i \le m$ such that $\sum\limits_{i=1}^m a_i = 1$ and

$$\big\|\ell - \sum_{i=1}^m a_i L_{\alpha_i} f\big\|_\infty \le \varepsilon$$

Then

$$\begin{aligned}\left|\tfrac{1}{\Omega_n}\int_{S^{n-1}} \Big[\ell - \sum_{i=1}^m a_i L_{\alpha_i} f(x)\Big]\, d\sigma(x)\right| &\le \tfrac{1}{\Omega_n}\int_{S^{n-1}} \Big|\ell - \sum_{i=1}^m a_i L_{\alpha_i} f(x)\Big|\, d\sigma(x) \\ &\le \tfrac{1}{\Omega_n}\int_{S^{n-1}} \varepsilon\, d\sigma(x) = \varepsilon\end{aligned}$$

As

$$\begin{aligned}\tfrac{1}{\Omega_n}\int_{S^{n-1}} \Big[\ell - \sum_{i=1}^m a_i L_{\alpha_i} f(x)\Big]\, d\sigma(x) &= \tfrac{1}{\Omega_n}\int_{S^{n-1}} \ell\, d\sigma(x) - \sum_{i=1}^m \tfrac{a_i}{\Omega_n}\int_{S^{n-1}} L_{\alpha_i} f(x)\, d\sigma(x) \\ &= \ell - \sum_{i=1}^m \tfrac{a_i}{\Omega_n}\int_{S^{n-1}} f(y)\, d\sigma(y) \text{ where } y = \alpha_i^{-1}x \\ &= \ell - \sum_{i=1}^m a_i c(f) \\ &= \ell - c(f)\end{aligned}$$

we have $\big|\ell - c(f)\big| \le \varepsilon$ for all $\varepsilon > 0$. Hence $\ell = c(f)$.

(d) By linearity and rotation invariance, if $\sum_{i=1}^m a_i L_{\alpha_i} f \in \mathcal{F}(f)$, then

$$\mathcal{L}\big(\sum_{i=1}^m a_i L_{\alpha_i} f\big) = \sum_{i=1}^m a_i \mathcal{L}\big(L_{\alpha_i} f\big) = \sum_{i=1}^m a_i \mathcal{L}(f) = \mathcal{L}(f)$$

By continuity $\mathcal{L}(g) = \mathcal{L}(f)$ for all $g \in \overline{\mathcal{F}(f)}$. In particular, since $c(f) \in \overline{\mathcal{F}(f)}$,

$$c(f) = c(f)\mathcal{L}(1) = \mathcal{L}\big(c(f)\big) = \mathcal{L}(f)$$

■

Problem 1.2.2

(a) Use residues to show that $\int \frac{e^{ip_1}}{p_1^2+q^2}\ dp_1 = \pi\frac{e^{-|q|}}{|q|}$ for all $0 \neq q \in \mathbb{R}$.

(b) Let $n = 3$. Show that $\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\ d^np = 2\pi^2$.

(c) Let $n = 2$. Use $\int \frac{dp_3}{p_3^2+q^2} = \frac{\pi}{|q|}$ to show that

$$\int \frac{e^{ip_1}}{\sqrt{p_1^2+p_2^2}}\ d^2p = \frac{1}{\pi}\int \frac{e^{ip_1}}{p_1^2+p_2^2+p_3^2}\ d^3p = 2\pi$$

(d) Let $n \geq 3$. Show that

$$\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\ d^np = \Omega_{n-2}\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}\frac{\sqrt{\pi}}{2}\int \frac{e^{ip_1}}{\sqrt{p_1^2+p_2^2}}\ dp_1dp_2 = 2\pi^{n/2}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}$$

Solution. (a) It suffices to consider $q > 0$. Then

$$\int \frac{e^{ip_1}}{p_1^2+q^2}\ dp_1 = \int \frac{e^{ip_1}}{(p_1+iq)(p_1-iq)}\ dp_1 = 2\pi i\frac{e^{ip_1}}{p_1+iq}\bigg|_{p_i=iq} = \pi\frac{e^{-q}}{q}$$

(b) By part (a) $\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\ d^np = \int d^2q \int dp_1\ \frac{e^{ip_1}}{p_1^2+q^2} = \pi\int \frac{e^{-|q|}}{|q|}\ d^2q$. Switching to polar coordinates

$$\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\ d^np = \pi\int_0^{2\pi} d\theta \int_0^\infty dr\ r\frac{e^{-r}}{r} = 2\pi^2\int_0^\infty e^{-r}\ dr = -2\pi^2 e^{-r}\Big|_0^\infty = 2\pi^2$$

(c) is obvious from part (b).

(d)
$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\ d^np &= \int dp_1dp_2\ e^{ip_1}\int d^{n-2}q\frac{1}{(p_1^2+p_2^2+q^2)^{\frac{n-1}{2}}}\\
&= \int dp_1dp_2\ e^{ip_1}\Omega_{n-2}\int_0^\infty dr\ r^{n-3}\frac{1}{(p_1^2+p_2^2+r^2)^{\frac{n-1}{2}}}\\
&= \int dp_1dp_2\ e^{ip_1}\Omega_{n-2}\Big(\sqrt{p_1^2+p_2^2}\Big)^{n-2-(n-1)}\int_0^\infty ds\ s^{n-3}\frac{1}{(1+s^2)^{\frac{n-1}{2}}}\\
&\qquad\qquad \text{where } r = s\sqrt{p_1^2+p_2^2}\\
&= \Omega_{n-2}\int dp_1dp_2\ \frac{e^{ip_1}}{\sqrt{p_1^2+p_2^2}}\int_0^\infty ds\ s^{n-3}\frac{1}{(1+s^2)^{\frac{n-1}{2}}}\\
&= 2\pi\Omega_{n-2}\int_0^\infty ds\ \frac{s^{n-3}}{(1+s^2)^{\frac{n-1}{2}}}\\
&= 2\pi\Omega_{n-2}\int_0^{\pi/2} d\theta\ \sec^2\theta\ \frac{\tan^{n-3}\theta}{\sec^{n-1}\theta} \qquad \text{where } s = \tan\theta\\
&= 2\pi\Omega_{n-2}\int_0^{\pi/2} d\theta\ \sin^{n-3}\theta
\end{aligned}$$

To compute the remaining integral we derive, using integration by parts, the standard recursion relation

$$\begin{aligned}
\int_0^{\pi/2} \sin^m \theta \; d\theta &= \int_0^{\pi/2} \sin^{m-1} \theta \; d\big(-\cos\theta\big) \\
&= -\sin^{m-1}\theta\cos\theta\big|_0^{\pi/2} + (m-1)\int_0^{\pi/2} \cos^2\theta \sin^{m-2}\theta \; d\theta \\
&= (m-1)\int_0^{\pi/2} \sin^{m-2}\theta \; d\theta - (m-1)\int_0^{\pi/2} \sin^m\theta \; d\theta \\
\implies \int_0^{\pi/2} \sin^m \theta \; d\theta &= \tfrac{m-1}{m}\int_0^{\pi/2} \sin^{m-2}\theta \; d\theta \qquad \text{for all } m \ge 2
\end{aligned}$$

Iterating

$$\begin{aligned}
\int_0^{\pi/2} \sin^{n-3} \theta \; d\theta &= \tfrac{n-4}{n-3}\tfrac{n-6}{n-5}\cdots \begin{cases} \frac{1}{2}\frac{\pi}{2} & n \text{ odd} \\ \frac{2}{3} & n \text{ even} \end{cases} \\
&= \tfrac{\frac{n}{2}-2}{\frac{n}{2}-\frac{3}{2}}\tfrac{\frac{n}{2}-3}{\frac{n}{2}-\frac{5}{2}}\cdots \begin{cases} \frac{1/2}{1}\sqrt{\pi}\frac{\sqrt{\pi}}{2} & n \text{ odd} \\ \frac{1}{3/2}\frac{1}{1/2}\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} & n \text{ even} \end{cases} \\
&= \tfrac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-\frac{1}{2})}\tfrac{\sqrt{\pi}}{2}
\end{aligned}$$

since $\Gamma(x) = (x-1)\Gamma(x-1)$ and $\Gamma(1/2) = \sqrt{\pi}$. All together

$$\begin{aligned}
\int_{\mathbb{R}^n} \tfrac{e^{ip_1}}{|p|^{n-1}} \; d^np &= 2\pi\Omega_{n-2}\int_0^{\pi/2} d\theta \;\; \sin^{n-3}\theta = 2\pi\Omega_{n-2}\tfrac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-\frac{1}{2})}\tfrac{\sqrt{\pi}}{2} \\
&= 2\pi\tfrac{2\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2})}\tfrac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-\frac{1}{2})}\tfrac{\sqrt{\pi}}{2} = 2\pi^{n/2}\tfrac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}
\end{aligned}$$

as desired. ■

Problem 1.3.1 Set $\Phi(x) = -\frac{e^{ik|x|}}{4\pi|x|}$.

(a) Prove that $\Delta\Phi(x) + k^2\Phi(x) = 0$ for all $x \neq 0$.

(b) Let S_ε be the sphere of radius ε centered on the origin and let $d\sigma_\varepsilon$ be the surface measure on S_ε. Prove that, for any continuous function $\psi(x)$,

$$\lim_{\varepsilon\to 0+} \iint_{S_\varepsilon} \tfrac{\psi(x)}{|x|^p} \; d\sigma_\varepsilon = \begin{cases} 4\pi\psi(0) & \text{if } p = 2 \\ 0 & \text{if } p < 2 \end{cases}$$

(c) Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$. Prove that

$$\iiint \Phi(x)\big[\Delta\psi(x) + k^2\psi(x)\big] \; d^3x = \psi(0)$$

Solution. (a) Require $x \neq 0$ throughout this argument. Then $\nabla |x| = \frac{x}{|x|}$. Hence, setting $F(r) = -\frac{e^{ikr}}{4\pi r}$,

$$\nabla\Phi(x) = F'(|x|)\tfrac{x}{|x|} = \Big[-ik\tfrac{e^{ik|x|}}{4\pi|x|} + \tfrac{e^{ik|x|}}{4\pi|x|^2}\Big]\tfrac{x}{|x|} = \Big[-ik\tfrac{e^{ik|x|}}{4\pi|x|^2} + \tfrac{e^{ik|x|}}{4\pi|x|^3}\Big]x$$

Now using

$$\nabla\cdot\big[G(|x|)x\big] = x\cdot\nabla G(|x|) + G(|x|)\nabla\cdot x = G'(|x|)|x| + 3G(|x|)$$

gives

$$\Delta\Phi(x) = \Big[k^2\tfrac{e^{ik|x|}}{4\pi|x|^2} + 3ik\tfrac{e^{ik|x|}}{4\pi|x|^3} - 3\tfrac{e^{ik|x|}}{4\pi|x|^4}\Big]|x| + 3\Big[-ik\tfrac{e^{ik|x|}}{4\pi|x|^2} + \tfrac{e^{ik|x|}}{4\pi|x|^3}\Big] = k^2\tfrac{e^{ik|x|}}{4\pi|x|}$$

as desired.

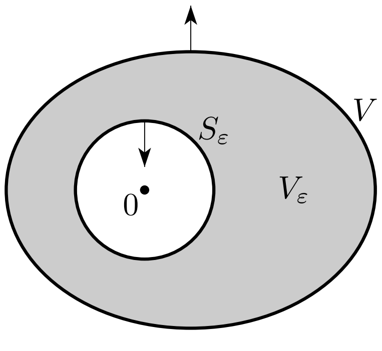
(b) For any continuous function $\psi(x)$, $M(\varepsilon) = \sup_{|x|=\varepsilon}\big|\psi(x) - \psi(0)\big|$ converges to zero as $\varepsilon \to 0$. As

$$D(\varepsilon) = \tfrac{1}{4\pi\varepsilon^2}\iint_{S_\varepsilon}\big[\psi(x) - \psi(0)\big]\,d\sigma_\varepsilon$$

is bounded in magnitude by $M(\varepsilon)$, $D(\varepsilon)$ also converges to zero as $\varepsilon \to 0$ and

$$\begin{aligned}\lim_{\varepsilon\to0+}\iint_{S_\varepsilon}\tfrac{\psi(x)}{\|x\|^p}\,d\sigma_\varepsilon &= \lim_{\varepsilon\to0+}\tfrac{1}{\varepsilon^p}\iint_{S_\varepsilon}\psi(x)\,d\sigma_\varepsilon = \lim_{\varepsilon\to0+}\tfrac{4\pi\varepsilon^2}{\varepsilon^p}\big[\psi(0) + D(\varepsilon)\big]\\ &= \begin{cases}4\pi\psi(0) & \text{if } p = 2\\ 0 & \text{if } p < 2\end{cases}\end{aligned}$$

(c) Let V be a bounded open set in $\mathbb{R}^3$ that contains the support of ψ and let V_ε be the set of points in V with $|x| > \varepsilon$. Note that the boundary ∂V_ε of V_ε consists of two parts — the boundary ∂V of V and the sphere S_ε — and that the unit outward normal to ∂V_ε on S_ε is $-\frac{x}{|x|}$. By the divergence theorem



$$\text{(S1.2)}\qquad \iiint_{V_\varepsilon}\nabla\cdot\big(\Phi\nabla\psi - \psi\nabla\Phi\big)d^3x = \iint_{S_\varepsilon}\big(\Phi\nabla\psi - \psi\nabla\Phi\big)\cdot\big(-\tfrac{x}{|x|}\big)d\sigma_\varepsilon$$

The ∂V boundary term has vanished because the support of ψ is contained in V. Subbing in $\nabla\Phi = \Big[-ik\frac{e^{ik|x|}}{4\pi|x|^2} + \frac{e^{ik|x|}}{4\pi|x|^3}\Big]x$, from part (a), and applying part (b)

(S1.3)

$$\begin{aligned}\lim_{\varepsilon\to0+}\iint_{S_\varepsilon}\big(\Phi\nabla\psi - \psi\nabla\Phi\big)\cdot\big(-\tfrac{x}{|x|}\big)\ d\sigma_\varepsilon &= \lim_{\varepsilon\to0+}\iint_{S_\varepsilon}\big(x\cdot\nabla\psi + \psi - ik|x|\psi\big)\tfrac{e^{ik|x|}}{4\pi|x|^2}d\sigma_\varepsilon \\ &= \Big[\big(x\cdot\nabla\psi + \psi - ik|x|\psi\big)e^{ik|x|}\Big]_{x=0} \\ &= \psi(0)\end{aligned}$$

For any vector field F and any complex–valued function f, $\nabla\cdot\big(fF\big) = \nabla f\cdot F + f\nabla\cdot F$. Applying this twice, we see that the integrand of the left hand side of (S1.2) is

$$\begin{aligned}\nabla\cdot\big(\Phi\nabla\psi - \psi\nabla\Phi\big) &= \nabla\Phi\cdot\nabla\psi + \Phi\Delta\psi - \nabla\psi\cdot\nabla\Phi - \psi\Delta\Phi \\ &= \Phi(\Delta + k^2)\psi\end{aligned} \tag{S1.4}$$

since $\Delta\Phi = -k^2\Phi$ on V_ε. So applying $\lim\limits_{\varepsilon\to0+}$ to (S1.2) and using (S1.4) and (S1.3) gives

$$\iiint_V \Phi(\Delta + k^2)\psi\ d^3x = \psi(0)$$

as desired. ■

Problem 1.3.2 Prove that $\Phi(x-y)$ has the asymptotic behaviour

$$\Phi(x-y) = -\tfrac{e^{ik|x|}}{4\pi|x|}e^{-ik\hat{x}\cdot y} + O\big(\tfrac{1}{|x|^2}\big)$$

for $|y|$ bounded and $|x|$ large.

Solution. Observe that

$$\begin{aligned}|x-y| = \sqrt{x^2 - 2x\cdot y + y^2} &= |x|\sqrt{1 - 2\hat{x}\cdot y\tfrac{1}{|x|} + \tfrac{y^2}{x^2}} \\ &= |x|\Big[1 + \tfrac{1}{2}\big(-2\hat{x}\cdot y\tfrac{1}{|x|} + \tfrac{y^2}{x^2}\big) + O\big(\tfrac{1}{|x|^2}\big)\Big] \\ &= |x| - \hat{x}\cdot y + O\big(\tfrac{1}{|x|}\big)\end{aligned}$$

so that

$$e^{ik|x-y|} = e^{ik|x|}e^{-ik\hat{x}\cdot y}e^{O(1/|x|)} = e^{ik|x|}e^{-ik\hat{x}\cdot y} + O\big(\tfrac{1}{|x|}\big)$$

and

$$\tfrac{1}{|x-y|} = \tfrac{1}{|x|}\tfrac{1}{1+O(1/|x|)} = \tfrac{1}{|x|}\Big(1 + O\big(\tfrac{1}{|x|}\big)\Big) = \tfrac{1}{|x|} + O\big(\tfrac{1}{|x|^2}\big)$$

Multiplying

$$\Phi(x-y) = -\tfrac{e^{ik|x-y|}}{4\pi|x-y|} = -\tfrac{1}{4\pi}\Big[e^{ik|x|}e^{-ik\hat{x}\cdot y} + O\big(\tfrac{1}{|x|}\big)\Big]\Big[\tfrac{1}{|x|} + O\big(\tfrac{1}{|x|^2}\big)\Big] = -\tfrac{e^{ik|x|}}{4\pi|x|}e^{-ik\hat{x}\cdot y} + O\big(\tfrac{1}{|x|^2}\big)$$

as desired. ■

Problem 1.3.3 Let $f \in C_0^\infty(\mathbb{R}^3)$. Prove that

$$F(x) = \int \Phi(x-y) f(y)\ d^3y$$

obeys $\Delta F + k^2 F = f$ and the radiation condition.

Solution. Making the change of variables $y \to x - y$

$$F(x) = \int \Phi(y) f(x-y)\ d^3y$$

Since f is C_0^∞ and Φ is L^1,

$$(\Delta + k^2)F(x) = \int \Phi(y)(\Delta_x + k^2) f(x-y)\ d^3y = \int \Phi(y)(\Delta_y + k^2) f(x-y)\ d^3y = f(x)$$

by part (c) of Problem 1.3.1.

For x separated from 0 and the support of f

$$\big(\tfrac{\partial}{\partial r} - ik\big)F(x) = \int \big(\tfrac{x}{|x|} \cdot \nabla_x - ik\big)\Phi(x-y) f(y)\ d^3y$$

We have

$$\begin{aligned}\big(\tfrac{x}{|x|} \cdot \nabla_x - ik\big)\Phi(x-y) &= -\big(\tfrac{x}{|x|} \cdot \nabla_x - ik\big)\tfrac{e^{ik|x-y|}}{4\pi|x-y|} \\ &= -\big(ik\tfrac{x}{|x|} \cdot \tfrac{x-y}{|x-y|} - \tfrac{x}{|x|} \cdot \tfrac{x-y}{|x-y|^2} - ik\big)\tfrac{e^{ik|x-y|}}{4\pi|x-y|}\end{aligned}$$

As in the solution to Problem 1.3.2,

$$\tfrac{1}{|x-y|} = \tfrac{1}{|x|} + O\big(\tfrac{1}{|x|^2}\big)$$

so that

$$\big(\tfrac{x}{|x|} \cdot \nabla_x - ik\big)\Phi(x-y) = -\big(ik + O\big(\tfrac{1}{|x|}\big) - ik\big)\tfrac{e^{ik|x-y|}}{4\pi|x-y|} = O\big(\tfrac{1}{|x|^2}\big)$$

and the radiation condition follows. ■