§1. Introduction

§1.1. Dirichlet to Neumann Problems

Consider a wire $0 \le x \le \ell$ with voltage $u(x)$ at x. By Ohm's law, the voltage difference between the points at x and $x+h$ is the current, I, flowing through the wire times the resistance between x and $x+h$. If the resistance density $\rho(x)$, called the resistivity, is continuous, then

$$u(x+h) - u(x) = -I\rho(x')h$$

for some x' between x and $x+h$. Dividing across by h and taking the limit $h \to 0$,

$$u'(x) = -I\rho(x)$$

We have been assuming that charge is not allowed to accumulate anywhere inside the wire so that I is a constant. We may eliminate it from the equation just by dividing $\rho(x)$ across and differentiating. In terms of the conductivity, $\gamma(x) = \frac{1}{\rho(x)}$,

$$\gamma(x)u'(x) = -I \implies \big(\gamma(x)u'(x)\big)' = 0 \tag{1.1.1}$$

Now suppose that we may only measure the voltages and currents at the ends of the wire. That is, we may only measure $u(0), u(\ell)$, $\gamma(0)u'(0)$ and $\gamma(\ell)u'(\ell)$. By (1.1.1), $\gamma(x)u'(x)$ is a constant and so takes the value $\gamma(0)u'(0)$ everywhere. Thus

$$u'(x) = \gamma(0)u'(0)\tfrac{1}{\gamma(x)} \implies u(\ell) - u(0) = \gamma(0)u'(0)\int_0^\ell \tfrac{dx}{\gamma(x)}$$

The only property of the wire that you can determine by measurements at the ends of the wire is the total resistance $\int_0^\ell \frac{dx}{\gamma(x)}$.

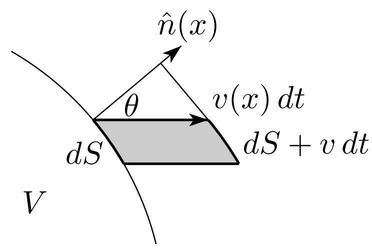
Replacing the wire by a two or higher dimensional body changes the picture completely. In $\mathbb{R}^n$, $n \ge 2$, the current $i(x)$ is a vector and Ohm's Law is

$$i(x) = -\gamma(x)\nabla u(x) \tag{1.1.2}$$

Assuming that charge is still not allowed to accumulate anywhere in the body, the net rate of charge flow across the boundary ∂V of any region V must vanish, so that

$$\int_{\partial V} i(x)\cdot\hat{n}(x)\ dS = 0$$

where $\hat{n}(x)$ is the outward unit normal to ∂V at x. To derive this condition, concentrate on the charge that, at time t, is on an infinitesimal piece, dS, of the surface of V. If this charge has velocity $v(x)$, then at the end of an infinitesimal time interval dt it has moved to a surface element that is the translate by $v(x)dt$ of dS. In the figure, this surface element is denoted $dS + v\,dt$. The charge that has left V through dS during this time interval now



fills a tube whose ends are dS and $dS + v\,dt$. The tube has cross–sectional area dS and height $|v(x)|dt\cos\theta = \hat{n}\cdot v\,dt$. Hence the tube has volume $v(x)\cdot\hat{n}(x)\,dt\,dS$. If the charge density at x is $\kappa(x)$, the tube contains charge $\kappa(x)v(x)\cdot\hat{n}(x)\,dSdt = i(x)\cdot\hat{n}(x)\ dSdt$. The total charge that leaves V during the time interval dt is $dt\int_{\partial V} i(x)\cdot\hat{n}(x)\ dS$.

As we are not allowing charge to accumulate anywhere, $0 = \int_{\partial V} i(x)\cdot\hat{n}(x)\ dS = \int_V \nabla\cdot i(x)\ dV$, by the divergence theorem. This is true for all regions V. So, assuming that $\nabla\cdot i(x)$ is continuous,

$$\nabla\cdot i(x) = 0 \implies \nabla\cdot\big(\gamma(x)\nabla u(x)\big) = 0$$

Suppose now that we have a conductor filling a region Ω and that we apply a voltage f on the boundary $\partial\Omega$ of Ω and measure the current that then flows out of the region. By measuring the rate at which charge is leaving various parts of $\partial\Omega$, we are measuring the current flux through $\partial\Omega$, which determines $\gamma(x)\nabla u(x)\cdot\hat{n}(x) = \gamma(x)\frac{\partial u}{\partial\nu}(x)$ on $\partial\Omega$, where $\frac{\partial u}{\partial\nu}$ is the normal derivative of u. For a given γ and f, the boundary value problem

$$\nabla\cdot\big(\gamma(x)\nabla u(x)\big) = 0 \text{ in } \Omega \qquad u = f \text{ on } \partial\Omega$$

determines u on Ω (see Theorem 3.3.5) and hence $k(x) = \gamma(x)\frac{\partial u}{\partial\nu}(x) \restriction \partial\Omega$. Let $\Lambda_\gamma(f)$ be the k that results from a given γ and f. The map $f \mapsto \Lambda_\gamma(f)$, which clearly depends linearly on f, is called the Dirichlet to Neumann Map. Pretend for a minute that the boundary $\partial\Omega$ contains only a finite number, m, of points and call the value of f at the j^{th} boundary point f_j and the value of $\Lambda_\gamma(f)$ at the i^{th} boundary point $\Lambda_\gamma(f)_i$. Then the map $f \mapsto \Lambda_\gamma(f)$ is a linear map from $f \in \mathbb{R}^m$ to $\Lambda_\gamma(f) \in \mathbb{R}^m$ and hence is of the form

$$\Lambda_\gamma(f)_i = \sum_{j=1}^{m} \lambda_{i,j} f_j$$

where $\lambda_{i,j}$ is the current that results at i^{th} boundary point when a unit voltage is applied at the j^{th} boundary point. The analogous formula for the true, continuous, boundary $\partial\Omega$ is

$$\Lambda_\gamma(f) = \int_{\partial\Omega} \lambda_\gamma(x,y) f(y)\, d\sigma(y)$$

where $d\sigma$ is the surface measure on $\partial\Omega$ and $\lambda_\gamma(x,y)$ the current density that results at x when a unit voltage is applied at y. If we measure the current k that results from all applied surface voltages f, we know $\lambda_\gamma(x,y)$ for all $x,y \in \partial\Omega$. This is a function of $2(n-1)$ variables. The conductivity $\gamma(x)$ is a function of n variables. So for $n=1$, $\gamma(x)$ is a function of more variables than $\lambda_\gamma(x,y)$. We have already seen that, for $n=1$, $\lambda_\gamma(x,y)$ cannot possibly determine $\gamma(x)$. For $n=2$ ($n>2$), $\gamma(x)$ is a function of the same number of variables as (fewer variables than) $\lambda_\gamma(x,y)$.

For the simplest materials, called isotropic materials, if you apply a voltage $u(x)$, then the current at x is in the direction opposite to the voltage gradient, $\nabla u(x)$, and has magnitude proportional to the magnitude of the voltage gradient, with the constant of proportionality called the conductivity and denoted $\gamma(x)$. So, for isotropic materials, $i(x) = -\gamma(x)\nabla u(x)$. But there are more complicated, anisotropic, materials where the current at x need not be parallel to $\nabla u(x)$ and the magnitude of the current depends on the direction as well as the magnitude of $\nabla u(x)$. For these materials, $i(x) = -\gamma(x)\nabla u(x)$, but with $\gamma(x)$ being an $n \times n$ matrix, rather than just a number. In general, $\gamma(x)$ is a positive definite, symmetric, $n \times n$ matrix. We shall prove in Theorem 4.3.1 and Corollary 4.4.21 that, for $n \geq 2$, Λ_γ does indeed determine an isotropic conductivity. However, it cannot possibly determine anisotropic conductivities for the following obvious reason. Let $\Psi : \bar{\Omega} \to \bar{\Omega}$ be any diffeomorphism that is the identity map in some neighbourhood of $\partial\Omega$ and set

$$\tilde{\gamma} = \Big[\tfrac{1}{|\det(D\Psi)|}(D\Psi)\gamma(D\Psi)^t\Big] \circ \Psi^{-1} \qquad \tilde{u} = u \circ \Psi^{-1}$$

where $D\Psi$ is the Jacobian (matrix of first partial derivatives) of Ψ. In the next paragraph, we shall show, by a change of variables, that

$$(1.1.3) \qquad \left.\begin{array}{rl} \nabla\cdot\big[\gamma(x)\nabla u(x)\big] = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{array}\right\} \iff \left\{\begin{array}{rl} \nabla\cdot\big[\tilde{\gamma}(x)\nabla \tilde{u}(x)\big] = 0 & \text{in } \Omega \\ \tilde{u} = f & \text{on } \partial\Omega \end{array}\right.$$

Let u_f and $\tilde{u}_f$ denote the solutions of the left and right hand boundary value problems of (1.1.3), respectively. By definition, $\Lambda_\gamma(f)(x) = \hat{n}(x)\cdot\gamma(x)\nabla u_f(x) \restriction \partial\Omega$ and

$$\Lambda_{\tilde{\gamma}}(f)(x) = \hat{n}(x)\cdot\tilde{\gamma}(x)\nabla\tilde{u}_f(x) \restriction \partial\Omega$$

Since Ψ is the identity map in some neighbourhood of $\partial\Omega$, $D\Psi(x) = \mathbb{1}$, $\tilde{\gamma}(x) = \gamma(x)$ and $\tilde{u}_f(x) = u_f\big(\Psi^{-1}(x)\big) = u_f(x)$ for all x in that neighbourhood of $\partial\Omega$. Thus $\Lambda_\gamma = \Lambda_{\tilde{\gamma}}$.

In Theorem , we prove that, for $n = 2$, Λ_γ determines anisotropic conductivities up to diffeomorphisms like this. We expect that this is also true for $n > 2$.

A simple way to derive (1.1.3) is to observe that

$$\begin{aligned}\nabla \cdot \big[\gamma(x)\nabla u(x)\big] = 0 \text{ on } \Omega &\iff \int_\Omega v(x)\nabla \cdot \big[\gamma(x)\nabla u(x)\big]\ dx = 0 \text{ for all } v \in C_0^\infty(\Omega)\\ &\iff \int_\Omega \big[\nabla v(x)\big] \cdot \big[\gamma(x)\nabla u(x)\big]\ dx = 0 \text{ for all } v \in C_0^\infty(\Omega)\end{aligned}$$

by integration by parts (i.e. the divergence theorem). Making the change of variables $x = \Psi^{-1}(y)$.

$$\int_\Omega \nabla v(x) \cdot \gamma(x)\nabla u(x)\ dx = \int_\Omega (\nabla v)\big(\Psi^{-1}(y)\big) \cdot \gamma\big(\Psi^{-1}(y)\big)(\nabla u)\big(\Psi^{-1}(y)\big)\ \tfrac{dy}{|\det(D\Psi)|}$$

Set $\tilde{v} = v \circ \Psi^{-1}$ and apply the chain rule to $u(x) = \tilde{u}\big(\Psi(x)\big)$ and $v(x) = \tilde{v}\big(\Psi(x)\big)$.

$$\begin{aligned}\tfrac{\partial u}{\partial x_i}(x) &= \sum_{j=1}^n \tfrac{\partial \tilde{u}}{\partial x_j}\big(\Psi(x)\big)\tfrac{\partial \Psi_j}{\partial x_i}(x) \quad \Longrightarrow \quad (\nabla u)\big(\Psi^{-1}(y)\big) = (D\Psi)^t\big(\Psi^{-1}(y)\big)\nabla\tilde{u}(y)\\ \tfrac{\partial v}{\partial x_i}(x) &= \sum_{j=1}^n \tfrac{\partial \tilde{v}}{\partial x_j}\big(\Psi(x)\big)\tfrac{\partial \Psi_j}{\partial x_i}(x) \quad \Longrightarrow \quad (\nabla v)\big(\Psi^{-1}(y)\big) = (D\Psi)^t\big(\Psi^{-1}(y)\big)\nabla\tilde{v}(y)\end{aligned}$$

Thus

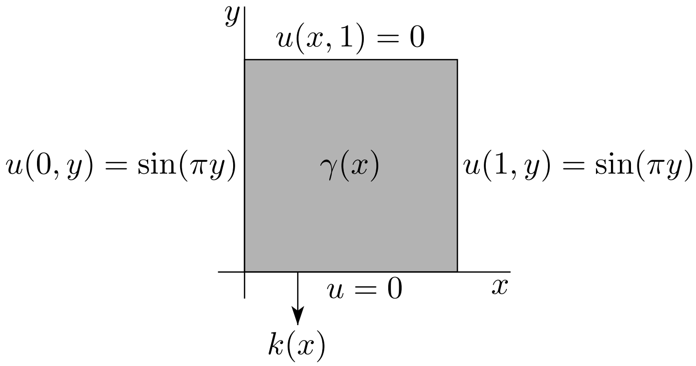
$$\int_\Omega \nabla v(x) \cdot \gamma(x)\nabla u(x)\ dx = \int_\Omega \nabla\tilde{v}(y) \cdot \tilde{\gamma}(y)\nabla\tilde{u}(y)\ dy$$

Integrating by parts once more (and observing that, as $v(x)$ runs over $C_0^\infty(\Omega)$, $\tilde{v}(y)$ also runs over $C_0^\infty(\Omega)$) gives that

$$\nabla \cdot \big[\gamma(x)\nabla u(x)\big] = 0 \iff \nabla \cdot \big[\tilde{\gamma}(x)\nabla\tilde{u}(x)\big] = 0$$

as desired.

Example 1.1.1 Here is a much simplified example (taken from [KV]) in which an isotropic conductivity is computed from a Dirichlet to Neumann map. The region is the square $\Omega = \big\{\ (x,y) \in \mathbb{R}^2\ \big|\ 0 < x, y < 1\ \big\}$. To reduce the number of variables that we are dealing with, we assume that the conductivity is a function of x only. As $\gamma(x)$ is a function only of a single variable, we hope to be able to determine it by measuring just one function of a single variable. We choose to measure the current $k(x) = \gamma(x)\frac{\partial u}{\partial y}\big|_{y=0}$ at the base of the square that results from applying the boundary voltage function specified in the figure



So our boundary value problem is

$$(1.1.4)\qquad \begin{aligned} &\text{(a)} \quad \nabla\cdot\big[\gamma(x)\nabla u(x,y)\big]=0 && \text{in } \Omega \\ &\text{(b)} \quad u(0,y)=u(1,y)=\sin(\pi y) && \text{for all } 0\le y\le 1 \\ &\text{(c)} \quad u(x,0)=u(x,1)=0 && \text{for all } 0\le x\le 1 \end{aligned}$$

The standard technique for solving the boundary value problem (1.1.4) is to Fourier expand $u(x,y)=\sum_{n=1}^{\infty} a_n(x)\sin(n\pi y)$. From the boundary condition (b), we would expect to only need the $n=1$ term. So we look for a solution of the form $u(x,y)=a(x)\sin(\pi y)$. Boundary condition (c) is satisfied for all functions $a(x)$. Boundary condition (b) is satisfied if and only if $a(0)=a(1)=1$. The differential equation (a) is satisfied if and only if

$$\begin{aligned} 0 &= \nabla\cdot\big(\gamma(x)a'(x)\sin(\pi y)\,,\ \pi\gamma(x)a(x)\cos(\pi y)\big) \\ &= \sin(\pi y)\big[\big(\gamma(x)a'(x)\big)'-\pi^2\gamma(x)a(x)\big] \end{aligned}$$

which is the case if and only if

$$(1.1.5)\qquad \big(\gamma(x)a'(x)\big)'-\pi^2\gamma(x)a(x)=0 \qquad \text{for all } 0<x<1$$

We imagine that we have measured

$$k(x)=\gamma(x)\tfrac{\partial u}{\partial y}\Big|_{y=0}=\gamma(x)\pi a(x)\cos(\pi y)\ \Big|_{y=0}=\pi\gamma(x)a(x)$$

and that we wish to determine $\gamma(x)$. We can do so by subbing $\gamma(x)=\frac{k(x)}{\pi a(x)}$ into (1.1.5) and solving for a.

$$\begin{aligned} \big(k(x)\tfrac{a'(x)}{a(x)}\big)'=\pi^2 k(x) &\implies \tfrac{d}{dx}\big[k(x)\tfrac{d}{dx}\ln a(x)\big]=\pi^2 k(x) \\ &\implies k(x)\tfrac{d}{dx}\ln a(x)=\pi^2\int_0^x k(t)\ dt-\pi^2 C \\ &\implies \ln a(x)=\pi^2\int_0^x \tfrac{1}{k(s)}\Big[\int_0^s k(t)\ dt - C\Big]\ ds + D \end{aligned}$$

for some constants C and D. To satisfy the boundary condition $a(0) = 1$, we need $D = 0$ and to satisfy $a(1) = 1$, we need

$$C = \left[\int_0^1 \tfrac{ds}{k(s)} \right]^{-1} \left[\int_0^1 \tfrac{ds}{k(s)} \int_0^s k(t)\ dt \right]$$

This determines[1] $a(x)$ and hence $\gamma(x) = \frac{k}{\pi a(x)}$.

Problem 1.1.1 Find the Dirichlet to Neumann map when $\Omega = \left\{ x \in \mathbb{R}^2 \;\middle|\; |x| < 1 \right\}$ and the conductivity $\gamma(x) \equiv 1$.

Problem 1.1.2 Let $\Omega = (-\infty, 0) \times S^1$. Functions on Ω can be identified with those functions $u(x, \theta)$ that are defined for $x < 0$ and all $\theta \in \mathbb{R}$ and that are periodic of period 2π in θ. The gradient operator for Ω is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}\right)$. Find the Dirichlet to Neumann map when the conductivity $\gamma(x, \theta) \equiv 1$. Assume that potentials $u(x, \theta)$ must remain bounded in the limit $x \to -\infty$.

Problem 1.1.3 Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the divergence theorem is applicable to Ω. Let $\gamma(x)$ be a real–valued C^∞ function on Ω all of whose derivatives are bounded. Suppose that the complex numbers λ, μ and the, not identically zero, functions $\varphi,\ \psi \in C^2(\overline{\Omega})$ obey

$$\begin{aligned}
&\nabla \cdot [\gamma(x)\nabla\varphi(x)] = \lambda\varphi(x) && \text{for all } \ x \in \Omega \\
&\varphi(x) = 0 && \text{for all } x \in \partial\Omega \\
&\nabla \cdot [\gamma(x)\nabla\psi(x)] = \mu\psi(x) && \text{for all } \ x \in \Omega \\
&\psi(x) = 0 && \text{for all } x \in \partial\Omega
\end{aligned}$$

We say that φ and ψ are eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. The numbers λ and μ are the corresponding eigenvalues.

(a) Prove that $\lambda, \mu \in \mathbb{R}$.

(b) Prove that if $\lambda \neq \mu$ then φ and ψ are orthogonal in $L^2(\Omega)$. In other words, prove that $\int_\Omega \varphi(x)\overline{\psi(x)}\ d^n x = 0$.

(c) Suppose that $\gamma(x) > 0$ for all $x \in \Omega$. Prove that $\lambda, \mu < 0$.

(d) Let $\mathcal{H}$ be the closure of the subspace of $L^2(\Omega)$ spanned by the eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Prove that there is an orthonormal basis for $\mathcal{H}$ consisting of real–valued eigenfunctions.

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[1] If you are worried about dividing by k in the integrals, you shouldn't be. We know that $0 \le u \le 1$ on $\partial\Omega$. By the maximum principle, this implies that $0 < u < 1$ in the interior of Ω. This in turn forces $\frac{\partial u}{\partial y} \ge 0$ when $y = 0$. In fact, by the strong maximum principle [Ev, §6.4.2], $\frac{\partial u}{\partial y} > 0$ for $y = 0$, which ensures that $k(x) > 0$ for all $0 \le x \le 1$.

Problem 1.1.4 Let Ω and γ be as in Problem 1.1.3. Assume that we already know

- an orthonormal basis for $L^2(\Omega)$ consisting of C^2 eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma \nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Call the eigenfunctions and corresponding eigenvalues $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.
- a linear map $E : C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ such that $(Ef)(x) = f(x)$ for all $x \in \partial\Omega$).

Find the Dirichlet to Neumann map for conductivity γ.

Problem 1.1.5 Apply the method of Problem 1.1.4 to find the Dirichlet to Neumann map for $\left\{\ x \in \mathbb{R}^2 \ \middle|\ |x| < 1\ \right\}$ with conductivity $\gamma \equiv 1$. You may assume that a suitable orthonormal basis exists.

Problem 1.1.6 Let Ω be an open subset of $\mathbb{R}^n$. Let $\gamma \in C^1(\Omega)$ be bounded away from zero. Find $q, \beta \in C^1(\Omega)$ such that

$$\nabla \cdot [\gamma \nabla u] = 0 \iff (-\Delta + q)v = 0 \text{ for } v = \beta u$$

Problem 1.1.7 Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\Lambda_\gamma(f)$ denote the Dirichlet to Neumann map for the conductivity $\gamma(x)$. Let $\beta(x)$ be a C^∞ function on Ω all of whose derivatives are bounded. Compute $\frac{d}{dt}\Lambda_{\gamma+t\beta}(f)\big|_{t=0}$. Assume that we already know

- a complete orthonormal basis for $L^2(\Omega)$ consisting of C^2 eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma \nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Call the eigenfunctions and corresponding eigenvalues $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.
- the solution to the boundary value problem $\nabla \cdot [\gamma \nabla u] = 0$ in Ω, $u = f$ on $\partial\Omega$. Call the solution $u_0(x)$.

§1.2. X–ray Tomography

Suppose that you shine a light beam (or beam of x-rays) down the length of a long rod. Suppose further that as the light passes through an infinitesimal hunk of rod it loses a fraction of its intensity that is proportional to the mass of the hunk. If we denote by $I(x)$ the intensity of the light at x and by $\rho(x)$ the mass density of the rod at x, then

$$\tfrac{I(x+dx)-I(x)}{I(x)} = -\gamma\, \rho(x)\, dx$$

where γ is the (positive) constant of proportionality. Dividing across by dx and taking the limit as $dx \to 0$ gives

$$\tfrac{I'(x)}{I(x)} = -\gamma\rho(x) \implies \tfrac{d}{dx} \ln I(x) = -\gamma\rho(x)$$

By integrating, we see that the (natural logarithm of the) fraction of the light intensity that survives the trip down the rod is

$$\ln \tfrac{I(\infty)}{I(-\infty)} = -\gamma \int_{-\infty}^{\infty} \rho(x)\, dx$$

In x–ray tomography, you shine thin beams through different parts of a body and wish to recover the density $\rho(x)$ of the body from the various values of $\int_{\text{beam path}} \rho(x)\, dx$ measured.

We now formulate a mathematical statement of this problem. It is no harder to consider general dimensions, so we do so. Let $\mathbb{P}^n$ denote the set of all hyperplanes in $\mathbb{R}^n$. By definition, a hyperplane in $\mathbb{R}^n$ is a translate of an $(n-1)$–dimensional linear subspace of $\mathbb{R}^n$. A hyperplane may be specified by giving its direction (say a vector perpendicular to it) and one point on it. Thus each hyperplane can be written

$$\xi = \left\{ \ x \in \mathbb{R}^n \ \middle| \ \hat{\omega} \cdot x = p \ \right\}$$

where $\hat{\omega}$ is a unit vector in $\mathbb{R}^n$ and $p \in \mathbb{R}$. Of course, if $\hat{\omega}$ is normal to a hyperplane, so is $-\hat{\omega}$ and indeed

- $(\hat{\omega}, p)$ and $(\hat{\omega}', p')$ give the same hyperplane if and only if $(\hat{\omega}, p) = \pm(\hat{\omega}', p')$
- Thus the map $(\hat{\omega}, p) \in S^{n-1} \times \mathbb{R} \mapsto \xi$ is a double cover of $\mathbb{P}^n$ and can be used to define coordinate patches on $\mathbb{P}^n$, turning it into a manifold.
- Hence a function φ on $\mathbb{P}^n$ can be identified with a function on $S^{n-1} \times \mathbb{R}$ obeying $\varphi(\hat{\omega}, p) = \varphi(-\hat{\omega}, -p)$. We shall typically use the same symbol (for example, φ) to denote both the function on $\mathbb{P}^n$ and the corresponding function on $S^{n-1} \times \mathbb{R}$.

Definition 1.2.1 (Radon Transform) The Radon transform associates to each function $f : \mathbb{R}^n \to \mathbb{R}$ the function $Rf : \mathbb{P}^n \to \mathbb{R}$ defined by

$$Rf(\xi) = \int_{\xi} f(x)\, dm(x)$$

where $dm(x)$ is the standard Euclidean measure on ξ. For example, if $\hat{\omega} = (1, 0, 0, \cdots, 0)$, then

$$\int_{\xi} f(x)\, dm(x) = \int_{\mathbb{R}^{n-1}} f(p, x_2, \cdots, x_n)\ dx_2 \cdots dx_n$$

We are not going to worry about what regularity should be required of $f(x)$ and what regularity the resulting $Rf(\xi)$ has. These details are dealt with in [Ep, H]. We now pose the

Question: Given Rf, can we determine f? The short answer is

Answer: Yes, if $n > 1$.

We shall justify this short answer by deriving two separate algorithms for determining f. Before doing so, note that once we know how to determine f from its integrals over linear spaces of codimension one (i.e. hyperplanes), we also know how to determine f from its integrals over linear spaces of higher codimension. For example, the algorithms with $n = 2$ allow us to recover a function defined on a planar region from its integrals over lines. As any three dimensional body is a union of planar regions, we can also recover a function on a three dimensional body from its integrals over lines, just by treating planar slices of the body individually.

The first algorithm is based on the following Theorem, which shows us how to compute the conventional Fourier transform $\hat{f}(k) = \int e^{-ik\cdot x} f(x)\, d^n x$ of f from its Radon transform Rf. Once $\hat{f}$ is known, you find f by applying the inverse Fourier transform to $\hat{f}$.

Theorem 1.2.2 (Central Slice Theorem) *For all $s \in \mathbb{R}$ and $\hat{\omega} \in S^{n-1}$.*

$$\hat{f}(s\hat{\omega}) = \int_{-\infty}^{\infty} Rf(\hat{\omega}, r) e^{-isr}\, dr$$

Proof: By definition

$$\begin{aligned}\hat{f}(s\hat{\omega}) &= \int_{\mathbb{R}^n} f(x) e^{-is\hat{\omega}\cdot x} d^n x \\ &= \int_{-\infty}^{\infty} dr \int_{x\cdot\hat{\omega}=r} dm(x)\ f(x) e^{-is\hat{\omega}\cdot x}\end{aligned}$$

We have just rewritten the integral over $\mathbb{R}^n$ as an iterated integral. For, example, if $n = 3$ and $\hat{\omega} = (0, 0, 1)$, then $dr = dx_3$ and $dm(x) = dx_1\, dx_2$. Subbing in the definition of Rf,

$$\begin{aligned}\hat{f}(s\hat{\omega}) &= \int_{-\infty}^{\infty} dr\ e^{-isr} \int_{x\cdot\hat{\omega}=r} dm(x)\ f(x) \\ &= \int_{-\infty}^{\infty} e^{-isr} Rf(\hat{\omega}, r)\, dr\end{aligned}$$

■

In preparation for the second algorithm we define a dual transform, which is sometimes called "back projection".

Definition 1.2.3 (Dual Radon Transform) The dual Radon transform associates to each function $\varphi : \mathbb{P}^n \to \mathbb{R}$ the function $R^*\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$R^*\varphi(x) = \int_{x\in\xi} \varphi(\xi)\, d\mu(\xi)$$

where $d\mu(\xi)$ is the unique measure on $\{ \xi \in \mathbb{P}^n \mid x \in \xi \}$ which is invariant under rotations around x and has mass one.

Let's take a closer look at the measure $d\mu(\xi)$. By translating, it suffices to consider $x = 0$. Then $\{ \xi \in \mathbb{P}^n \mid 0 \in \xi \}$ is just the sphere S^{n-1} with antipodal points identified. A function ψ on $\{ \xi \in \mathbb{P}^n \mid 0 \in \xi \}$ is identified with an even function $\psi(\hat{\omega})$ on S^{n-1}. Then, using $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ to denote the surface area of a unit sphere in $\mathbb{R}^n$,

$$\begin{aligned} \int_{0\in\xi} \psi(\xi)\, d\mu(\xi) &= \tfrac{1}{\Omega_n} \int_{S^{n-1}} \psi(\hat{\omega})\, d\hat{\omega} \\ &= \int_{SO(n)} \psi(\mathcal{O}\hat{\omega}_0)\, d\mathcal{O} \end{aligned} \tag{1.2.1}$$

where $d\hat{\omega}$ is the standard Euclidean measure on S^{n-1}, $SO(n)$ is the group of rotations in $\mathbb{R}^n$, $\hat{\omega}_0$ is any fixed unit vector in $\mathbb{R}^n$ and $d\mathcal{O}$ is Haar measure on $SO(n)$. When $n = 2$,

- S^{n-1} is just the unit circle in $\mathbb{R}^2$ centred on the origin,
- $d\hat{\omega} = d\theta$, where θ is the usual polar angle,
- $\Omega_2 = 2\pi$,
- the elements of $SO(2)$ are matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $0 \le \theta < 2\pi$ and may be labeled by the angle of rotation θ.
- $d\mathcal{O} = \frac{d\theta}{2\pi}$.

For general n

- $SO(n)$ is the set of $n \times n$ real matrices that are orthogonal (i.e. $R^t R = RR^t = \mathbb{1}$) and have determinant one.
- The Haar measure on $SO(n)$ is a normalized, rotation invariant measure on $SO(n)$. It is characterized by the requirements that $\int_{SO(n)} d\mathcal{O} = 1$ and

$$\int_{SO(n)} \psi(\mathcal{O})\, d\mathcal{O} = \int_{SO(n)} \psi(R\mathcal{O})\, d\mathcal{O}$$

for all continuous functions $\psi : SO(n) \to \mathbb{R}$ and all $R \in SO(n)$.

To prove that the two right hand sides of (1.2.1) are equal, it suffices to observe that both of the maps $\psi \mapsto \frac{1}{\Omega_n} \int_{S^{n-1}} \psi(\hat{\omega})\, d\hat{\omega}$ and $\psi \mapsto \int_{SO(n)} \psi(\mathcal{O}\hat{\omega}_0)\, d\mathcal{O}$ are continuous, rotation[2] invariant, mass one linear functionals on the space, $C_{\mathbb{R}}(S^{n-1})$, of real–valued continuous functions[3] on S^{n-1}. By Problem 1.2.1, there exists only one such linear functional.

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[2] To rotate a function, replace its argument x by $\mathcal{O}^{-1}x$ for some $\mathcal{O} \in SO(n)$.

[3] This is a Banach space with the norm $\|f\| = \sup_{x\in S^{n-1}} |f(x)|$.

Problem 1.2.1 The purpose of this problem is to prove that there is a unique continuous, rotation invariant, mass one linear functional on $C_{\mathbb{R}}(S^{n-1})$. The argument is similar to that used to prove the existence of a Haar measure on a compact topological group. For each $f \in C_{\mathbb{R}}(S^{n-1})$ set

$$\mathcal{F}(f) = \Big\{ \sum_{i=1}^{m} a_i L_{\alpha_i} f \ \Big|\ m \in \mathbb{N}, \alpha_i \in SO(n), a_i > 0, \sum_{i=1}^{m} a_i = 1 \Big\} \subset C_{\mathbb{R}}(S^{n-1})$$

where, for $f \in C_{\mathbb{R}}(S^{n-1})$ and $\alpha \in SO(n)$,

$$(L_\alpha f)(x) = f\big(\alpha^{-1} x\big)$$

Denote by $\overline{\mathcal{F}(f)}$ the closure of $\mathcal{F}(f)$ in $C_{\mathbb{R}}(S^{n-1})$.

(a) Use Arzelà–Ascoli to show that $\overline{\mathcal{F}(f)}$ is a compact subset of $C_{\mathbb{R}}(S^{n-1})$.

(b) Show that $\overline{\mathcal{F}(f)}$ contains a constant function.

(c) Show that the constant function in $\overline{\mathcal{F}(f)}$ is $c(f) = \frac{1}{\Omega_n} \int_{S^{n-1}} f(x)\, d\sigma(x)$ where $d\sigma(x)$ is the surface measure on S^{n-1} and $\Omega_n = \int_{s^{n-1}} 1\, d\sigma(x)$ is the volume of S^{n-1}.

(d) Let $\mathcal{L} : C_{\mathbb{R}}(S^{n-1}) \to \mathbb{R}$ be continuous and linear and obey $\mathcal{L}(1) = 1$ and

$$\mathcal{L}(L_\alpha f) = \mathcal{L}(f)$$

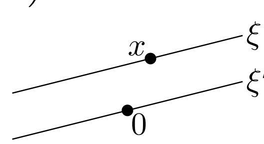
for all $f \in C_{\mathbb{R}}(S^{n-1})$ and $\alpha \in SO(n)$. Prove that $\mathcal{L}(f) = c(f)$ for all $f \in C_{\mathbb{R}}(S^{n-1})$.

Lemma 1.2.4

$$\big(R^* R f\big)(x) = \tfrac{\Omega_{n-1}}{\Omega_n} \int_{\mathbb{R}^n} |x - y|^{-1} f(y)\, d^n y$$

Proof: Subbing in the definitions of the Radon transform and the dual transform and then translating and applying the second part of (1.2.1),

$$\begin{aligned}
\big(R^* R\big)(x) &= \int_{x \in \xi} Rf(\xi)\, d\mu(\xi) = \int_{x \in \xi} \Big(\int_\xi f(y)\, dm(y) \Big)\, d\mu(\xi) \\
&= \int_{0 \in \xi'} \Big(\int_{\xi'} f(x + y)\, dm(y) \Big)\, d\mu(\xi') \\
&= \int_{SO(n)} \Big(\int_{\xi_0} f(x + \mathcal{O} y)\, dm(y) \Big)\, d\mathcal{O}
\end{aligned}$$



where ξ_0 is any fixed hyperplane through the origin. Observe that as $\mathcal{O}$ runs over $SO(n)$, $x + \mathcal{O}y$ runs over the surface of the sphere of radius $|y|$ centered on x. The equality of the two right hand sides of (1.2.1) says that the integral $\int_{SO(n)} \psi(\mathcal{O}\hat{\omega}_0)\, d\mathcal{O}$ is just the average value of ψ over the sphere S^{n-1}. Hence,

$$\begin{aligned}\big(R^*R\big)(x) &= \int_{\xi_0} \Big(\int_{SO(n)} f(x+\mathcal{O}y)\, d\mathcal{O} \Big)\, dm(y) \\ &= \int_{\xi_0} \Big(\tfrac{1}{\Omega_n} \int_{S^{n-1}} f(x+|y|\hat{\omega})\, d\hat{\omega} \Big)\, dm(y)\end{aligned}$$

Choose ξ_0 to be the hyperplane in $\mathbb{R}^n$ containing all points whose last coordinate is zero. Call this $\mathbb{R}^{n-1}$. Going to spherical coordinates on $\mathbb{R}^{n-1}$ and observing that the function $y \mapsto \int_{S^{n-1}} f(x+|y|\hat{\omega})\, d\hat{\omega}$ depends on y only through its length of $|y|$,

$$\begin{aligned}\big(R^*R\big)(x) &= \int_{\mathbb{R}^{n-1}} \Big(\tfrac{1}{\Omega_n} \int_{S^{n-1}} f(x+|y|\hat{\omega})\, d\hat{\omega} \Big)\, d^{n-1}y \\ &= \int_0^\infty dr\ r^{n-2}\Omega_{n-1} \Big(\tfrac{1}{\Omega_n} \int_{S^{n-1}} f(x+r\hat{\omega})\, d\hat{\omega} \Big) \\ &= \tfrac{\Omega_{n-1}}{\Omega_n} \int_0^\infty dr \int_{S^{n-1}} d\hat{\omega}\ r^{n-2} f(x+r\hat{\omega})\end{aligned}$$

This is almost spherical coordinates for $\mathbb{R}^n$. Only the power of r is wrong. So

$$\begin{aligned}\big(R^*R\big)(x) &= \tfrac{\Omega_{n-1}}{\Omega_n} \int_0^\infty dr \int_{S^{n-1}} d\hat{\omega}\ r^{n-1}\ \tfrac{1}{r} f(x+r\hat{\omega}) \\ &= \tfrac{\Omega_{n-1}}{\Omega_n} \int_{\mathbb{R}^n} \tfrac{1}{|y|} f(x+y)\ d^n y\end{aligned}$$

The change of variables $y \to y - x$ does it. ■

We now show that f can be recovered from $F = Rf$ by applying $\big(-\Delta\big)^{\frac{n-1}{2}}$ to a suitable constant times $R^*F = \frac{\Omega_{n-1}}{\Omega_n} \int_{\mathbb{R}^n} |x-y|^{-1} f(y)\, d^{n-1}y$. First we define what is meant by "applying $\big(-\Delta\big)^{\frac{n-1}{2}}$ to a function". When the dimension n is odd, $\frac{n-1}{2}$ is a positive integer and defining $\big(-\Delta\big)^{\frac{n-1}{2}}$ is not a problem. In particular, when $n = 3$, $\big(-\Delta\big)^{\frac{n-1}{2}} = -\Delta$ and Theorem 1.2.5, below, just says that applying the Laplacian is, up to a constant, the inverse operation to convolving with $\frac{1}{|x|}$. This should not be a surprise. It is just another way of saying that $\frac{1}{|x|}$ is, up to a constant, the Green's function for Laplacian in dimension 3. See Problem 1.3.1, below.

The Fourier transform of $-\Delta f$ is $k^2 \hat{f}(k)$. When n is odd, $n-1 = 2m$ is even and the Fourier transform of $\big(-\Delta\big)^{\frac{n-1}{2}} f = \big(-\Delta\big)^m f$ is $k^{2m}\hat{f}(k) = |k|^{n-1}\hat{f}(k)$. So even when n is even, we define $\big(-\Delta\big)^{\frac{n-1}{2}} f$ to be the function whose Fourier transform is $|k|^{n-1}\hat{f}(k)$.

Theorem 1.2.5 *If* $g(x) = \frac{\Omega_{n-1}}{\Omega_n}\int_{\mathbb{R}^n} |x-y|^{-1} f(y)\, d^n y$, *then*

$$cf = \big(-\Delta\big)^{\frac{n-1}{2}} g$$

where $c = (4\pi)^{\frac{n-1}{2}} \frac{\Gamma(n/2)}{\Gamma(1/2)}$.

"Proof": We have put in quotes because we are going to ignore the risks associated with manipulating conditionally convergent integrals. For a more careful statement and proof, see [LL, §5.9, 5.10]. The claim "$cf(x) = \big(-\Delta\big)^{\frac{n-1}{2}} g(x)$" is equivalent, by Fourier transforming, to the claim

$$c\hat{f}(k) = |k|^{n-1}\hat{g}(k) = |k|^{n-1}\frac{\Omega_{n-1}}{\Omega_n}\widehat{\big(\tfrac{1}{|x|}\big)}(k)\hat{f}(k) \qquad \text{or} \qquad \widehat{\big(\tfrac{1}{|x|}\big)}(k) = c\frac{\Omega_n}{\Omega_{n-1}}|k|^{n-1}$$

So we want to show that

$$\int_{\mathbb{R}^n} \frac{e^{ix\cdot k}}{|k|^{n-1}}\frac{d^n k}{(2\pi)^n} = \frac{\Omega_{n-1}}{c\Omega_n}\frac{1}{|x|} \qquad \text{or} \qquad \int_{\mathbb{R}^n} \frac{e^{ix\cdot k}}{|k|^{n-1}} d^n k = (2\pi)^n\frac{\Omega_{n-1}}{c\Omega_n}\frac{1}{|x|} = 2\pi^{n/2}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}\frac{1}{|x|}$$

Make the change of variables $p = |x|\mathcal{O}k$, where the rotation $\mathcal{O}$ is chosen so that $\mathcal{O}x = \big(|x|, 0, 0, \cdots, 0\big)$ and hence $x\cdot k = \mathcal{O}x\cdot\mathcal{O}k = |x|(\mathcal{O}k)_1 = p_1$ and

$$\int_{\mathbb{R}^n} \frac{e^{ix\cdot k}}{|k|^{n-1}} d^n k = |x|^{n-1-n}\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}} d^n p = \frac{1}{|x|}\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}} d^n p$$

Since $\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}} d^n p$ is a constant, we are done, up to evaluation of the constant, which is Problem 1.2.2. ■

Problem 1.2.2

(a) Use residues to show that $\int \frac{e^{ip_1}}{p_1^2+q^2}\, dp_1 = \pi\frac{e^{-|q|}}{|q|}$ for all $0 \neq q \in \mathbb{R}$.

(b) Let $n = 3$. Show that $\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\, d^n p = 2\pi^2$.

(c) Let $n = 2$. Use $\int \frac{dp_3}{p_3^2+q^2} = \frac{\pi}{|q|}$ to show that

$$\int \frac{e^{ip_1}}{\sqrt{p_1^2+p_2^2}}\, d^2 p = \frac{1}{\pi}\int \frac{e^{ip_1}}{p_1^2+p_2^2+p_3^2}\, d^3 p = 2\pi$$

(d) Let $n \geq 3$. Show that

$$\int_{\mathbb{R}^n} \frac{e^{ip_1}}{|p|^{n-1}}\, d^n p = \Omega_{n-2}\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}\frac{\sqrt{\pi}}{2}\int \frac{e^{ip_1}}{\sqrt{p_1^2+p_2^2}}\, dp_1 dp_2 = 2\pi^{n/2}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}$$

§1.3. Inverse Scattering

Suppose that we are interested in a system in which sound waves, for example, scatter off of some obstacle. Let $p(x,t)$ be the pressure at position x and time t. In (a somewhat idealized) free space, p obeys the wave equation $\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$, where c is the speed of sound. We shall assume that in most of the world, c takes a constant value c_0. But we introduce an obstacle by allowing c to depend on position in some compact region. We further allow for some absorption in that region. Then p obeys

$$\frac{\partial^2 p}{\partial t^2} + \gamma(x)\frac{\partial p}{\partial t} = c(x)^2 \Delta p$$

where $\gamma(x)$ is the damping coefficient of the medium at x. For solutions of fixed (temporal) frequency, $p(x,t) = \mathrm{Re}\left[u(x)e^{-i\omega t}\right]$ with

$$\Delta u + \frac{\omega^2}{c(x)^2}\left[1 + i\frac{\gamma(x)}{\omega}\right]u = 0$$

Outside of some compact region

$$\frac{\omega^2}{c(x)^2}\left[1 + i\frac{\gamma(x)}{\omega}\right] = \frac{\omega^2}{c_0^2} = k^2 \qquad \text{where} \qquad k = \frac{\omega}{c_0} > 0$$

If we define the index of refraction by

$$n(x) = \frac{c_0^2}{c(x)^2}\left[1 + i\frac{\gamma(x)}{\omega}\right]$$

then

$$\Delta u + k^2 n(x) u = 0 \tag{1.3.1}$$

with $n(x) = 1$ outside of some compact region. For concreteness, we restrict to three dimensions for the rest of this section. We first consider two special cases.

Example 1.3.1 (Free Space) In the absence of any obstacle $\Delta u + k^2 u = 0$ on all of $\mathbb{R}^3$. Then we can solve just by Fourier transforming. The general solution is a mixture of solutions of the form $u = e^{ik\hat{\theta}\cdot x}$ where $\hat{\theta}$ is a unit vector. This represents a plane wave moving in direction $\hat{\theta}$.

Example 1.3.2 (Point Source) If we have free space everywhere except at the origin and we have a unit point source at the origin, then

$$\Delta u + k^2 u = \delta(x)$$

where the Dirac delta function $\delta(x)$ is a distribution (generalized function) that is determined formally by the properties that $\delta(x) = 0$ for all $x \neq 0$ and $\int \delta(x)\ d^3x = 1$. A rigorous version of "$\Delta u + k^2 u = \delta(x)$" is provided in Problem 1.3.1 below. Except at the origin, where there is a singularity, we still have $\Delta u + k^2 u = 0$. The point source generates expanding spherical waves. So u should be a function of $r = |x|$ only and obey

$$u''(r) + \tfrac{2}{r}u'(r) + k^2 u(r) = 0$$

This is easily solved by changing variables to $v(r) = ru(r)$, which obeys

$$v''(r) + k^2 v(r) = 0$$

So $v(r) = \alpha \sin(kr) + \beta \cos(kr)$ and $u(r) = \alpha \frac{\sin(kr)}{r} + \beta \frac{\cos(kr)}{r}$. To be an outgoing (rather than incoming) wave $u(r) = \alpha' \frac{e^{ikr}}{r}$. (Note that $e^{ikr}e^{-i\omega t}$ is constant on $r = \frac{\omega}{k}t$, which is a sphere that is expanding with speed c_0.) To give the Dirac delta function on the right hand side of $\Delta u + k^2 u = \delta(x)$ coefficient one, we need $u(x) = -\frac{e^{ik|x|}}{4\pi|x|}$.

Problem 1.3.1 Set $\Phi(x) = -\frac{e^{ik|x|}}{4\pi|x|}$.

(a) Prove that $\Delta\Phi(x) + k^2\Phi(x) = 0$ for all $x \neq 0$.

(b) Let S_ε be the sphere of radius ε centered on the origin and let $d\sigma_\varepsilon$ be the surface measure on S_ε. Prove that, for any continuous function $\psi(x)$,

$$\lim_{\varepsilon \to 0+} \iint_{S_\varepsilon} \tfrac{\psi(x)}{|x|^p}\ d\sigma_\varepsilon = \begin{cases} 4\pi\psi(0) & \text{if } p = 2 \\ 0 & \text{if } p < 2 \end{cases}$$

(c) Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$. Prove that

$$\iiint \Phi(x)\big[\Delta\psi(x) + k^2\psi(x)\big]\ d^3x = \psi(0)$$

Now let's return to the general case. We want to think of a physical situation in which we send a plane wave $u^i(x) = e^{ik\hat{\theta}\cdot x}$ in from infinity. This plane wave shakes up the obstacle which then emits a bunch of expanding spherical waves $\frac{e^{ik|x-y|}}{|x-y|}$ emanating from various points y in the obstacle. So the full solution is of the form

$$u(x) = u^i(x) + u^s(x)$$

where the scattered wave, u^s, obeys the "radiation condition"

$$\frac{\partial}{\partial r}u^s(x) - iku^s(x) = O\big(\tfrac{1}{|x|^2}\big) \qquad \text{as} \qquad |x| \to \infty \tag{1.3.2}$$

This condition is chosen to allow outgoing waves $\frac{e^{ik|x-y|}}{|x-y|}$ but not incoming waves $\frac{e^{-ik|x-y|}}{|x-y|}$.

Let, as in Problem 1.3.1,

$$\Phi(x) = -\frac{e^{ik|x|}}{4\pi|x|}$$

Since $\delta(x-y)$ is the kernel of the identity operator,

$$(\Delta_x + k^2)\Phi(x-y) = \delta(x-y)$$

says, roughly, that $u(x) \mapsto \int \Phi(x-y)u(y)\,dy$ is the inverse of the map $u(x) \mapsto (\Delta+k^2)u(x)$ for functions that obey the radiation condition. We can exploit this to convert (1.3.1), (1.3.2) into an equivalent integral equation

$$\begin{aligned} \Delta u + k^2 n(x) u = 0 &\implies \Delta u + k^2 u = k^2\big(1-n(x)\big)u \\ &\implies \Delta u^s + k^2 u^s = k^2\big(1-n(x)\big)u \end{aligned}$$

since $\Delta u^i + k^2 u^i = 0$. As u^s obeys the radiation condition

$$u^s(x) = k^2 \int \Phi(x-y)\big(1-n(y)\big)u(y)\,d^3y$$

so that

$$u(x) = u^i(x) + k^2 \int \big(1-n(y)\big)\Phi(x-y)u(y)\,d^3y \tag{1.3.3}$$

This is called the Lippmann–Schwinger equation. Observe that it is of the form $u = u^i + Fu$ or $(\mathbb{1}+F)u = u^i$ where F is the linear operator $u(x) \mapsto k^2\int \Phi(x-y)\big(1-n(y)\big)u(y)\,d^3y$. This operator is compact (if you impose the appropriate norms) and so behaves much like a finite dimensional matrix. If F has operator norm smaller than one, which is the case if $k^2(1-n)$ is small enough, then $\mathbb{1}+F$ is trivially invertible and the equation $(\mathbb{1}+F)u = u^i$ has a unique solution. Even if F has operator norm larger than or equal to one, $(\mathbb{1}+F)u = u^i$ fails to have a unique solution only if F has eigenvalue minus one. One can show that this is impossible in the present setting. Thus, one can prove (see, for example, [])

Theorem 1.3.3 *If $n \in C^2(\mathbb{R}^3)$, $n(x)-1$ has compact support and $\operatorname{Re} n(x), \operatorname{Im} n(x) \geq 0$, then (1.3.1), (1.3.2) has a unique solution.*

For $|y|$ bounded and $|x|$ large, $\Phi(x-y)$ has the asymptotic behaviour

$$\Phi(x-y) = -\frac{e^{ik|x|}}{4\pi|x|}e^{-ik\hat{x}\cdot y} + O\big(\tfrac{1}{|x|^2}\big) \tag{1.3.4}$$

so that, when the incoming plane wave is moving in direction $\hat{\theta}$,

$$u(x;\hat{\theta}) = u^i(x;\hat{\theta}) + \frac{e^{ik|x|}}{4\pi|x|}u_\infty(\hat{x};\hat{\theta}) + O\big(\tfrac{1}{|x|^2}\big) \tag{1.3.5}$$

where

$$u_\infty(\hat{x};\hat{\theta}) = -k^2 \int e^{-ik\hat{x}\cdot y}\big(1-n(y)\big)u(y;\hat{\theta})\, d^3y$$

If we are observing the scattered wave from vantage points far from the obstacle, we will only be able to measure $u_\infty(\hat{x};\hat{\theta})$. The inverse problem then is
Question: Given $u_\infty(\hat{x};\hat{\theta})$, for all $\hat{x},\hat{\theta}\in S^2$, can we determine n? The short answer is
Answer: Yes, because we have the

Theorem 1.3.4 *If $n_1, n_2 \in C^2(\mathbb{R}^3)$ with n_1-1, n_2-1 of compact support and $u_{1,\infty}(\hat{x};\hat{\theta}) = u_{2,\infty}(\hat{x};\hat{\theta})$, for all $\hat{x},\hat{\theta}\in S^2$, then $n_1 = n_2$.*

We can get a rough idea why this Theorem is true by looking at the Born approximation. In this approximation u^s is ignored in the computation of u_∞ so that

$$\begin{aligned} u_\infty(\hat{x};\hat{\theta}) &\approx -k^2 \int e^{-ik\hat{x}\cdot y}\big(1-n(y)\big)u^i(y;\hat{\theta})\, d^3y \\ &= -k^2 \int e^{-ik(\hat{x}-\hat{\theta})\cdot y}\big(1-n(y)\big)\, d^3y \end{aligned}$$

If we measure $u_\infty(\hat{x};\hat{\theta})$, then, in this approximation, we know the Fourier transform of $1-n(y)$ on the set $\big\{\ k(\hat{x}-\hat{\theta})\ \big|\ \hat{x},\hat{\theta}\in S^2\ \big\}$ which is exactly the closed ball of radius $2k$ centered on the origin in $\mathbb{R}^3$. Since $1-n(y)$ is of compact support, its Fourier transform is analytic. So knowledge of the Fourier transform on any open ball uniquely determines it.

We shall discuss a very similar quantum mechanical analog of the above classical inverse scattering problem in §6 and §7.

Problem 1.3.2 Prove (1.3.4).

Problem 1.3.3 Let $f \in C_0^\infty(\mathbb{R}^3)$. Prove that

$$F(x) = \int \Phi(x-y)f(y)\ d^3y$$

obeys $\Delta F + k^2 F = f$ and the radiation condition.