Appendix SB: Problem Solutions for Appendix B

Problem B.1 Let $\mathcal{A}$ be an atlas for the Hausdorff space $\mathcal{M}$. Prove that there is a unique maximal atlas for $\mathcal{M}$ that contains $\mathcal{A}$.

Solution. Define $\mathfrak{A}$ to be the set of all charts $\{\mathcal{V}, \psi\}$ that are compatible with all of the charts of $\mathcal{A}$. If we can show that $\mathfrak{A}$ is an atlas, we are done, because (a) $\mathcal{A}$ is an atlas, so that $\mathcal{A} \subset \mathfrak{A}$ and (b) any chart in any atlas that contains $\mathcal{A}$ must be compatible with every chart of $\mathcal{A}$ and hence must be in $\mathfrak{A}$. Let $\{\mathcal{V}, \psi\}$ and $\{\mathcal{W}, \zeta\}$ be any two charts of $\mathfrak{A}$ with $\mathcal{V} \cap \mathcal{W} \neq \emptyset$. Let $x \in \mathcal{V} \cap \mathcal{W}$. We must show that $\zeta \circ \psi^{-1}$ is C^∞ in some neighbourhood of $\psi(x)$. Since $\mathcal{A}$ is an atlas, it contains a chart $\{\mathcal{U}, \phi\}$ with $x \in \mathcal{U}$. Since $\{\mathcal{V}, \psi\}$ and $\{\mathcal{W}, \zeta\}$ must both be compatible with $\{\mathcal{U}, \phi\}$, $\phi \circ \psi^{-1}$ must be C^∞ in some neighbourhood of $\psi(x)$ and $\zeta \circ \phi^{-1}$ must be C^∞ in some neighbourhood of $\phi(x)$. But then the composition $\zeta \circ \phi^{-1} \circ \phi \circ \psi^{-1}$ is C^∞ in some neighbourhood of $\psi(x)$. As $\zeta \circ \phi^{-1}\phi \circ \psi^{-1} = \zeta \circ \psi^{-1}$ in a neighbourhood of $\psi(x)$, we are done. ■

Problem B.2 Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of a Hausdorff space $\mathcal{M}$. Let φ be a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ and ψ be a homeomorphism from $\mathcal{V}$ to an open subset of $\mathbb{R}^m$. Prove that if $\mathcal{U} \cap \mathcal{V}$ is nonempty and

$$\begin{aligned} \psi \circ \varphi^{-1} &: \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \\ \varphi \circ \psi^{-1} &: \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \end{aligned}$$

are C^∞, then $m = n$.

Solution. Write $f(x) = \psi \circ \varphi^{-1}(x)$ and $g(y) = \varphi \circ \psi^{-1}(y)$. Fix any $p \in \mathcal{U} \cap \mathcal{V}$. Set

$$A = \left[\tfrac{\partial f_i}{\partial x_j}\big(\varphi(p)\big)\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}} \qquad B = \left[\tfrac{\partial g_i}{\partial y_j}\big(\psi(p)\big)\right]_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

Since $f\big(g(y)\big) = y$ for all y is a neighbourhood of $\psi(p)$, the chain rule gives that

$$\sum_{k=1}^{n} \tfrac{\partial f_i}{\partial x_k}\big(g(y)\big)\tfrac{\partial g_k}{\partial y_j}(y) = \delta_{i,j}$$

for all y is a neighbourhood of $\psi(p)$ and all $1 \le i, j \le m$. In particular $AB = \mathbb{1}_m$, the $m \times m$ identity matrix. Similarly, since $g\big(f(x)\big) = x$ for all x is a neighbourhood of $\varphi(p)$, $BA = \mathbb{1}_n$. So A is the inverse of the matrix B. But only square matrices have inverses, so $m = n$. ■

Problem B.3 Let $S^n = \{\ x \in \mathbb{R}^{n+1}\ \big|\ |x| = 1\ \}$ be the standard n–dimensional sphere.

(a) For each $1 \le j \le n+1$ and $\sigma \in \{-1, 1\}$ set

$$\mathcal{U}_{j,\sigma} = \{\ x \in S^n\ \big|\ \sigma x_j > 0\ \}$$

and define $\varphi_{j,\sigma}$ be the projection from $\mathcal{U}_{j,\sigma}$ into $\mathbb{R}^n$ that simply discards the $j^{\rm th}$ component x_j of each $x \in \mathcal{U}_{j,\sigma}$. Prove that $\big\{\ \{\mathcal{U}_{j,\sigma}, \varphi_{j,\sigma}\}\ \big|\ 1 \le j \le n+1,\ \sigma \in \{-1, 1\}\ \big\}$ is an atlas for S^n.

(b) Set

$$\mathcal{U} = S^n \setminus \{(0, \cdots, 0, 1)\} \qquad \mathcal{V} = S^n \setminus \{(0, \cdots, 0, -1)\}$$

and define the stereographic projections, $\varphi : \mathcal{U} \to \mathbb{R}^n$ and $\psi : \mathcal{V} \to \mathbb{R}^n$, by

$$\varphi(x) = \tfrac{2}{1-x_{n+1}}(x_1, \cdots, x_n) \qquad \psi(x) = \tfrac{2}{1+x_{n+1}}(x_1, \cdots, x_n)$$

Prove that $\big\{\{\mathcal{U}, \varphi\},\ \{\mathcal{V}, \psi\}\big\}$ is an atlas for S^n.

Solution. (a) Each $x \in S^n$ obeys $|x| = 1$ and so has at least one nonzero component x_j. If x_j, then $x \in \mathcal{U}_{j,1}$ and if $x_j < 0$, then $x \in \mathcal{U}_{j,-1}$. So $S^n \subset \bigcup_{\substack{1 \le j \le n+1 \\ \sigma=-1,1}} \mathcal{U}_{j,\sigma}$. We'll now verify that $\varphi_{1,1} \circ \varphi_{2,1}^{-1}$ and $\varphi_{1,1} \circ \varphi_{2,-1}^{-1}$ are C^∞. The other cases ares virtually identical. First observe that

$$\begin{aligned} \mathcal{U}_{1,1} \cap \mathcal{U}_{2,1} &= \{\ x \in S^n\ \big|\ x_1 > 0,\ x_2 > 0\ \} \\ \mathcal{U}_{1,1} \cap \mathcal{U}_{2,-1} &= \{\ x \in S^n\ \big|\ x_1 > 0,\ x_2 < 0\ \} \end{aligned}$$

so that

$$\begin{aligned} \varphi_{1,1}\big(\mathcal{U}_{1,1} \cap \mathcal{U}_{2,1}\big) &= \{\ (x_2, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_2 > 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \} \\ \varphi_{2,1}\big(\mathcal{U}_{1,1} \cap \mathcal{U}_{2,1}\big) &= \{\ (x_1, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \} \\ \varphi_{1,1}\big(\mathcal{U}_{1,1} \cap \mathcal{U}_{2,-1}\big) &= \{\ (x_2, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_2 < 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \} \\ \varphi_{2,-1}\big(\mathcal{U}_{1,1} \cap \mathcal{U}_{2,-1}\big) &= \{\ (x_1, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \} \end{aligned}$$

are in fact each one of two half disks in $\mathbb{R}^n$. Since

$$\begin{aligned} \varphi_{2,1}^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \big(t_1, \sqrt{1-t^2}, t_2, \cdots, t_n\big) \\ \varphi_{2,-1}^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \big(t_1, -\sqrt{1-t^2}, t_2, \cdots, t_n\big) \end{aligned}$$

(where $t^2 = t_1^2 + \cdots + t_n^2$) it is clear that

$$\begin{aligned} \varphi_{1,1} \circ \varphi_{2,1}^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \varphi_{1,1}\big(t_1, \sqrt{1-t^2}, t_2, \cdots, t_n\big) &&= \big(\sqrt{1-t^2}, t_2, \cdots, t_n\big) \\ \varphi_{1,1} \circ \varphi_{2,-1}^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \varphi_{-1,1}\big(t_1, -\sqrt{1-t^2}, t_2, \cdots, t_n\big) &&= \big(-\sqrt{1-t^2}, t_2, \cdots, t_n\big) \end{aligned}$$

are C^∞.

(b) Observe that

$$\varphi(x)^2 = \tfrac{4}{(1-x_{n+1})^2}\big(x_1^2 + \cdots + x_n^2\big) = 4\tfrac{1-x_{n+1}^2}{(1-x_{n+1})^2} = 4\tfrac{1+x_{n+1}}{1-x_{n+1}} \implies x_{n+1} = \tfrac{\varphi(x)^2-4}{\varphi(x)^2+4}$$

Since $\frac{1-x_{n+1}}{2} = \frac{4}{\varphi(x)^2+4}$, we have

$$\varphi^{-1}(t_1, \cdots, t_n) = \tfrac{4}{t^2+4}\big(t_1, \cdots, t_n, \tfrac{t^2-4}{4}\big)$$

and hence

$$\psi \circ \varphi^{-1}(t_1, \cdots, t_n) = 2\big[1 + \tfrac{t^2-4}{t^2+4}\big]^{-1} \tfrac{4}{t^2+4}\big(t_1, \cdots, t_n\big) = \tfrac{4}{t^2}\big(t_1, \cdots, t_n\big)$$

which is C^∞ except at $t = 0$, which corresponds to $x = (0, \cdots, 0, -1)$. Similarly,

$$\psi(x)^2 = 4\tfrac{1-x_{n+1}}{1+x_{n+1}} \implies x_{n+1} = \tfrac{4-\psi(x)^2}{4+\psi(x)^2} \implies \psi^{-1}(t_1, \cdots, t_n) = \tfrac{4}{4+t^2}\big(t_1, \cdots, t_n, \tfrac{4-t^2}{4}\big)$$

so that

$$\varphi \circ \psi^{-1}(t_1, \cdots, t_n) = 2\big[1 - \tfrac{4-t^2}{4+t^2}\big]^{-1} \tfrac{4}{4+t^2}\big(t_1, \cdots, t_n\big) = \tfrac{4}{t^2}\big(t_1, \cdots, t_n\big)$$

which is C^∞ except at $t = 0$, which this time corresponds to $x = (0, \cdots, 0, 1)$. ■

Problem B.4 Prove that the two atlases of Example B.6 are not compatible.

Solution. Since

$$\varphi_2^{-1}(x, y) = \begin{cases} (x, y) & \text{if } 0 \le x < \frac{1}{4} \\ (x+1, -y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

we have

$$\psi_2 \circ \varphi_2^{-1}(x, y) = \left\{ \begin{matrix} \psi_2(x, y) & \text{if } 0 \le x < \frac{1}{4} \\ \psi_2(x+1, -y) & \text{if } -\frac{1}{4} < x < 0 \end{matrix} \right\} = \begin{cases} (x, y) & \text{if } 0 \le x < \frac{1}{4} \\ (x, -y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

This is not continuous across $x = 0$.

■

Problem B.5 Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. Prove that $f : \mathcal{M} \to \mathcal{N}$ is C^∞ at $m \in \mathcal{M}$ if and only if $\psi \circ f \circ \phi^{-1}$ is C^∞ at $\phi(m)$ for every chart $(\mathcal{U}, \phi)$ for $\mathcal{M}$ with $m \in \mathcal{U}$ and every chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ with $f(m) \in \mathcal{V}$.

Solution. The "if" part is trivial. To prove the "only if" part, observe that, by definition, there are charts $\{\tilde{\mathcal{U}}, \tilde{\phi}\}$ and $\{\tilde{\mathcal{V}}, \tilde{\psi}\}$ such that $m \in \tilde{\mathcal{U}}$, $f(m) \in \tilde{\mathcal{V}}$ and $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ being C^∞ at $\tilde{\phi}(m)$. Let $\{\mathcal{U}, \phi\}$ and $\{\mathcal{V}, \psi\}$ be any charts with $m \in \mathcal{U}$ and $f(m) \in \mathcal{V}$. Then $m \in \tilde{\mathcal{U}} \cap \mathcal{U}$ and $f(m) \in \tilde{\mathcal{V}} \cap \mathcal{V}$. By compatibility $\tilde{\phi} \circ \phi^{-1}$ and $\psi \circ \tilde{\psi}^{-1}$ are C^∞ at $\phi(m)$ and $\tilde{\psi}(f(m))$ respectively. Consequently,

$$\psi \circ f \circ \phi^{-1} = \left(\psi \circ \tilde{\psi}^{-1}\right) \circ \left(\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}\right) \circ \left(\tilde{\phi} \circ \phi^{-1}\right)$$

is C^∞ at $\phi(m)$. ■

Problem B.6 Prove that $\mathbb{R}^n$ is diffeomorphic to $\left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i^2 < 1 \right\}$.

Solution. The map

$$\Phi(x) = \frac{x}{\sqrt{1 - |x|^2}}$$

is a diffeomorphism from $\left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i^2 < 1 \right\}$ to $\mathbb{R}^n$. The inverse map is

$$\Phi^{-1}(y) = \frac{y}{\sqrt{1 + |y|^2}}$$

■

Problem B.7 Prove that $\mathbb{R}^n$ is not diffeomorphic to S^n.

Solution. Suppose that $\Phi : S^n \to \mathbb{R}^n$ were a diffeomorphism. Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x) = x_1$. Then $g \circ \Phi : S^n \to \mathbb{R}$ is C^∞ and is onto $\mathbb{R}$. But that is impossible because S^n is compact so that every C^∞ function on S^n is bounded. ■

Problem B.8 Prove that the disk $\left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < 2 \right\}$ is not diffeomorphic to the annulus $\left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; 1 < x^2 + y^2 < 2 \right\}$.

Solution. The disk $\mathcal{M} = \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < 2 \right\}$ has the property that every C^∞ closed curve may be continuously deformed to a point. To see this parametrize any curve by a function $f : \mathbb{R} \to \mathcal{M}$ that has period one. Then $f_s(t) = (1-s)f(t)$ implements the deformation, since $f_0(t) = f(t)$ and $f_1(t) = 0$ has range the single point $\{0\}$. If the disk and annulus were diffeomorphic, the annulus would also have this property. It doesn't. For example, the circle $C = \big\{ \ (x,y) \ \big| \ x^2 + y^2 = 1 \ \big\}$ cannot be deformed to a point in the annulus. If it could, Green's theorem would yield that $\oint_C \big[- \frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \big] = 0$, which is false. ■

Problem B.9 In this problem $G = SO(3)$.

a) Fix any $a \in G$. Denote by $I = \big\{ \ (i,j) \in \mathbb{N}^2 \ \big| \ 1 \le i \le 3, \ 1 \le j \le 3 \ \big\}$ the set of indices for the matrix elements of the matrices in G. Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element $g_\delta, \ \delta \in I$ is a C^∞ function of $g_\alpha, \ g_\beta, \ g_\gamma$ for matrices $g \in G$ in a neighbourhood of a.

b) Prove that a curve $q : (c,d) \to G$ is C^∞ if and only if every matrix element $q(t)_{i,j}$ is C^∞.

c) Prove that matrix multiplication $(a,b) \mapsto ab$ is a C^∞ function from $G \times G$ to G.

d) Prove that the inverse function $a \mapsto a^{-1}$ is a C^∞ function from G to G.

Solution. (a) Name the matrix elements of $g \in G$ by

$$\begin{bmatrix} a_1(g) & a_2(g) & a_3(g) \\ b_1(g) & b_2(g) & b_3(g) \\ c_1(g) & c_2(g) & c_3(g) \end{bmatrix}$$

and set

$$\begin{aligned} f_1 &= a_1^2 + a_2^2 + a_3^2 & f_2 &= b_1^2 + b_2^2 + b_3^2 & f_3 &= c_1^2 + c_2^2 + c_3^2 \\ f_4 &= a_1b_1 + a_2b_2 + a_3b_3 & f_5 &= a_1c_1 + a_2c_2 + a_3c_3 & f_6 &= b_1c_1 + b_2c_2 + b_3c_3 \end{aligned}$$

We have to show that, with $\nabla = \big(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3}, \frac{\partial}{\partial c_1}, \frac{\partial}{\partial c_2}, \frac{\partial}{\partial c_3}, \big)$,

$$\begin{aligned} \nabla f_1 &= \big(2a_1, 2a_2, 2a_3, 0, 0, 0, 0, 0, 0\big) & \nabla f_2 &= \big(0, 0, 0, 2b_1, 2b_2, 2b_3, 0, 0, 0\big) \\ \nabla f_3 &= \big(0, 0, 0, 0, 0, 0, 2c_1, 2c_2, 2c_3\big) & \nabla f_4 &= \big(b_1, b_2, b_3, a_1, a_2, a_3, 0, 0, 0\big) \\ \nabla f_5 &= \big(c_1, c_2, c_3, 0, 0, 0, a_1, a_2, a_3\big) & \nabla f_6 &= \big(0, 0, 0, c_1, c_2, c_3, b_1, b_2, b_3\big) \end{aligned}$$

are linearly independent vectors at any point obeying $f_1 = f_2 = f_3 = 1$ and $f_4 = f_5 = f_6 = 0$. In other words, we have to show that

$$\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$$

at every such point. This follows from

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2}\big(2|a|^2\,\alpha_1 + a\cdot b\,\alpha_4 + a\cdot c\,\alpha_5\big)\\
&= \tfrac{1}{2}\big(a_1, a_2, a_3, 0, 0, 0, 0, 0, 0\big)\cdot\big(\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6\big)\\
\alpha_2 &= \tfrac{1}{2}\big(2|b|^2\,\alpha_2 + a\cdot b\,\alpha_4 + b\cdot c\,\alpha_6\big)\\
&= \tfrac{1}{2}\big(b_1, b_2, b_3, 0, 0, 0\big)\cdot\big(\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6\big)\\
\alpha_3 &= \tfrac{1}{2}\big(2|c|^2\,\alpha_3 + a\cdot c\,\alpha_5 + b\cdot c\,\alpha_6\big)\\
&= \tfrac{1}{2}\big(0, 0, 0, 0, 0, 0, c_1, c_2, c_3\big)\cdot\big(\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6\big)\\
\alpha_4 &= 2a\cdot b\,\alpha_1 + |b|^2\,\alpha_4 + b\cdot c\,\alpha_5\\
&= \big(b_1, b_2, b_3, 0, 0, 0, 0, 0, 0\big)\cdot\big(\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6\big)\\
\alpha_5 &= 2a\cdot c\,\alpha_1 + b\cdot c\,\alpha_4 + |c|^2\,\alpha_5\\
&= \big(c_1, c_2, c_3, 0, 0, 0, 0, 0, 0\big)\cdot\big(\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6\big)\\
\alpha_6 &= 2b\cdot c\,\alpha_2 + a\cdot c\,\alpha_4 + |c|^2\,\alpha_6\\
&= \big(0, 0, 0, c_1, c_2, c_3, 0, 0, 0\big)\cdot\big(\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6\big)
\end{aligned}$$

(b) Pick any $c < t_0 < d$. Pick three matrix elements as coordinates in some neighbourhood of $q_0(t)$. By Problem B.5, $q(t)$ is C^∞ at t_0 if and only if those three matrix elements are C^∞ at t_0. If those three matrix elements are C^∞ at t_0, the remaining matrix elements are also C^∞ by part (a).

(c) Every matrix element of ab is a polynomial in the matrix elements of a and the matrix elements of b and hence is C^∞.

(d) Every matrix element of a^{-1} is a polynomial in the matrix elements of a (since $\det a = 1$) and hence is C^∞.

■

Problem B.2.1 Let $\mathcal{M}$ be a manifold, ω be a 1–form on $\mathcal{M}$ and $c(t) : [0,1] \to \mathcal{M}$ be a path in $\mathcal{M}$. Prove that the definition of $\int_c \omega$ given in part (c) of Definition B.2.1 is independent of the decomposition of c into finitely many pieces and of the choice of coordinate patches.

Solution. "Decomposing c into finitely many pieces" means picking $0 < t_1 < t_2 < \cdots < t_m$ with $m \in \mathbb{N}$ and $\big\{\ c(t)\ \big|\ t_{\ell-1} \le t \le t_\ell\ \big\}$ contained in a single coordinate patch for each $\ell = 1,\ 2,\ \cdots,\ m$ and then applying the single patch algorithm of Definition B.2.1 to the part of $c(t)$ with $t_{\ell-1} \le t \le t_\ell$ for each $\ell = 1,\ 2,\ \cdots,\ m$.

Make any two such decompositions and choices of coordinate patches for each piece. Since

$$\int_{t_{\ell-1}}^{t_\ell} h(t)\ dt = \int_{t_{\ell-1}}^{s} h(t)\ dt + \int_{s}^{t_\ell} h(t)\ dt$$

for all $t_{\ell-1} < s < t_\ell$ we are free to add any finite number of decomposition points. So we may assume that the two sets of decomposition times are identical. Thus it suffices to prove that if $0 \le t_\ell < t_{\ell+1} \le 1$ and

- if $\{U, \zeta\}$ and $\{\tilde U, \tilde\zeta\}$ are two patches with $c(t) \in U \cap \tilde U$ for all $t_\ell \le t \le t_{\ell+1}$ and
- if ω assigns $\{U, \zeta\}$ the pair of functions (f, g) and assigns $\{\tilde U, \tilde\zeta\}$ the pair of functions $(\tilde f, \tilde g)$

then

$$\int_0^1 \Big[f\big(\zeta(c(t))\big)\tfrac{dx(c(t))}{dt} + g\big(\zeta(c(t))\big)\tfrac{dy(c(t))}{dt}\Big]\ dt = \int_0^1 \Big[\tilde f\big(\tilde\zeta(c(t))\big)\tfrac{d\tilde x(c(t))}{dt} + \tilde g\big(\tilde\zeta(c(t))\big)\tfrac{d\tilde y(c(t))}{dt}\Big]\ dt$$

But this is the computation of part (c) of Remark B.2.2. ■

Problem B.5.1 Prove that Definition B.5.1 is independent of the choice of coordinate patch.

Solution. Assume that

- $\{U, \zeta\}$ and $\{\tilde U, \tilde\zeta\}$ are two charts with $U \cap \tilde U \ne \emptyset$ and
- the transition function $\tilde\zeta \circ \zeta^{-1}$ $\big($from $\zeta(U \cap \tilde U) \subset \mathbb{R}^2$ to $\tilde\zeta(U \cap \tilde U) \subset \mathbb{R}^2\big)$ is $\big(\tilde x(x, y), \tilde y(x, y)\big)$

(a) Let $F : \mathcal{M} \to \mathbb{C}$ be a C^1 0–form and set $\varphi = F \circ \zeta^{-1}$ and $\tilde\varphi = F \circ \tilde\zeta^{-1}$. Then $\varphi(x, y) = \tilde\varphi\big(\tilde x(x, y), \tilde y(x, y)\big)$ and

$$\begin{aligned}
dF\big|_{\{\tilde U,\tilde\zeta\}} &= \tilde f(\tilde x, \tilde y)\, d\tilde x + \tilde g(\tilde x, \tilde y)\, d\tilde y \text{ with } \tilde f(\tilde x, \tilde y) = \tfrac{\partial\tilde\varphi}{\partial\tilde x}(\tilde x, \tilde y),\ \tilde g(\tilde x, \tilde y) = \tfrac{\partial\tilde\varphi}{\partial\tilde y}(\tilde x, \tilde y)\\
dF\big|_{\{U,\zeta\}} &= \tfrac{\partial\varphi}{\partial x}(x, y)\, dx + \tfrac{\partial\varphi}{\partial y}(x, y)\, dy\\
&= \Big\{\tfrac{\partial\tilde\varphi}{\partial\tilde x}\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde x}{\partial x}(x, y) + \tfrac{\partial\tilde\varphi}{\partial\tilde y}\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde y}{\partial x}(x, y)\Big\}\, dx\\
&\quad + \Big\{\tfrac{\partial\tilde\varphi}{\partial\tilde x}\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde x}{\partial y}(x, y) + \tfrac{\partial\tilde\varphi}{\partial\tilde y}\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde y}{\partial y}(x, y)\Big\}\, dy\\
&= f(x, y)\, dx + g(x, y)\, dy
\end{aligned}$$

with

$$\begin{aligned}
f(x, y) &= \tilde f\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde x}{\partial x}(x, y) + \tilde g\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde y}{\partial x}(x, y)\\
g(x, y) &= \tilde f\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde x}{\partial y}(x, y) + \tilde g\big(\tilde x(x, y), \tilde y(x, y)\big)\tfrac{\partial\tilde y}{\partial y}(x, y)
\end{aligned}$$

This agrees with the coordinate transformation rule of Definition B.2.1.

(b) If ω is a C^1 1–form with

$$\begin{aligned} \omega\big|_{\{U,\zeta\}} &= f(x,y)\,dx + g(x,y)\,dy \\ \omega\big|_{\{\tilde U,\tilde\zeta\}} &= \tilde f(\tilde x,\tilde y)\,d\tilde x + \tilde g(\tilde x,\tilde y)\,d\tilde y \end{aligned}$$

then

$$\begin{aligned} f(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y) \\ g(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y) \end{aligned}$$

Since

$$\begin{aligned} \tfrac{\partial g}{\partial x}(x,y) &= \tfrac{\partial\tilde f}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial x}\tfrac{\partial\tilde x}{\partial y} + \tfrac{\partial\tilde f}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial x}\tfrac{\partial\tilde x}{\partial y} + \tilde f\tfrac{\partial^2\tilde x}{\partial x\partial y} + \tfrac{\partial\tilde g}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial x}\tfrac{\partial\tilde y}{\partial y} + \tfrac{\partial\tilde g}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial x}\tfrac{\partial\tilde y}{\partial y} + \tilde g\tfrac{\partial^2\tilde y}{\partial x\partial y} \\ \tfrac{\partial f}{\partial y}(x,y) &= \tfrac{\partial\tilde f}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial y}\tfrac{\partial\tilde x}{\partial x} + \tfrac{\partial\tilde f}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial y}\tfrac{\partial\tilde x}{\partial x} + \tilde f\tfrac{\partial^2\tilde x}{\partial x\partial y} + \tfrac{\partial\tilde g}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial y}\tfrac{\partial\tilde y}{\partial x} + \tfrac{\partial\tilde g}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial y}\tfrac{\partial\tilde y}{\partial x} + \tilde g\tfrac{\partial^2\tilde y}{\partial x\partial y} \end{aligned}$$

we have that

$$\begin{aligned} d\omega\big|_{\{U,\zeta\}} &= \big[\tfrac{\partial g}{\partial x}(x,y) - \tfrac{\partial f}{\partial y}(x,y)\big]\,dx\wedge dy \\ &= \big[\tfrac{\partial\tilde g}{\partial\tilde x} - \tfrac{\partial\tilde f}{\partial y}\big]\ \big[\tfrac{\partial\tilde x}{\partial x}\tfrac{\partial\tilde y}{\partial y} - \tfrac{\partial\tilde x}{\partial y}\tfrac{\partial\tilde y}{\partial x}\big]\,dx\wedge dy \end{aligned}$$

which agrees with the coordinate transformation rule of Definition B.3.1.

(c) The 2–form case is trivial. ■

Problem B.5.2 Prove the product rule that if ω is a k–form and ω' is a k'–form, then

$$d(\omega\wedge\omega') = (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega')$$

Solution. It suffices to consider $k+k'\le 1$, since all of $d(\omega\wedge\omega')$, $(d\omega)\wedge\omega'$ and $\omega\wedge(d\omega')$ vanish if $k+k'\ge 2$ because the manifold $\mathcal{M}$ is assumed to be two dimensional. It suffices to verify the specified formula in any chart $\{U,\zeta\}$,

(a) If $k=k'=0$, $\omega\circ\zeta^{-1}(x,y)=f(x,y)$ and $\omega'\circ\zeta^{-1}(x,y)=g(x,y)$, then

$$\begin{aligned} d\omega &= \tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy \\ d\omega' &= \tfrac{\partial g}{\partial x}(x,y)\,dx + \tfrac{\partial g}{\partial y}(x,y)\,dy \\ (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega') &= \tfrac{\partial f}{\partial x}(x,y)g(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)g(x,y)\,dy \\ &\quad + f(x,y)\tfrac{\partial g}{\partial x}(x,y)\,dx + f(x,y)\tfrac{\partial g}{\partial y}(x,y)\,dy \end{aligned}$$

and

$$d(\omega\wedge\omega') = \tfrac{\partial}{\partial x}(fg)\,dx + \tfrac{\partial}{\partial y}(fg)\,dy$$

agree by the product rule.

(b) If $k=0$, $k'=1$, $\omega\circ\zeta^{-1}(x,y)=f(x,y)$ and $\omega'\big|_{\{\mathcal{U},\zeta\}}=g(x,y)\,dx+h(x,y)\,dy$, then

$$\begin{aligned}
d\omega &= \tfrac{\partial f}{\partial x}(x,y)\,dx+\tfrac{\partial f}{\partial y}(x,y)\,dy\\
d\omega' &= \big[\tfrac{\partial h}{\partial x}(x,y)-\tfrac{\partial g}{\partial y}(x,y)\big]\,dx\wedge dy\\
(d\omega)\wedge\omega'+(-1)^k\omega\wedge(d\omega') &= \big[\tfrac{\partial f}{\partial x}(x,y)\,dx+\tfrac{\partial f}{\partial y}(x,y)\,dy\big]\wedge\big[g(x,y)\,dx+h(x,y)\,dy\big]\\
&\qquad+f(x,y)\big[\tfrac{\partial h}{\partial x}(x,y)-\tfrac{\partial g}{\partial y}(x,y)\big]\,dx\wedge dy\\
&=\big[h\tfrac{\partial f}{\partial x}-g\tfrac{\partial f}{\partial y}+f\tfrac{\partial h}{\partial x}-f\tfrac{\partial g}{\partial y}\big]\,dx\wedge dy
\end{aligned}$$

and

$$\begin{aligned}
d(\omega\wedge\omega') &= d\big(f(x,y)g(x,y)\,dx+f(x,y)h(x,y)\,dy\big)\\
&=\big[-\tfrac{\partial}{\partial y}\big(f(x,y)g(x,y)\big)+\tfrac{\partial}{\partial x}\big(f(x,y)h(x,y)\big)\big]\,dx\wedge dy
\end{aligned}$$

agree by the product rule.

(c) If $k=1$ and $k'=0$, then, by the previous case

$$d(\omega\wedge\omega')=d(\omega'\wedge\omega)=(d\omega')\wedge\omega+\omega'\wedge(d\omega)=(d\omega)\wedge\omega'-\omega\wedge(d\omega')$$

■