

## Appendix B: Integration on Manifolds

This is intended as a lightning fast introduction to integration on manifolds. For a more thorough, but still elementary discussion see [O, Chapter 4], [R, Chapter 10] or [Sp]. We shall explicitly define integrals over 0-, 1- and 2-dimensional regions of a two dimensional manifold. The same ideas also work for higher dimensions. For an  $n$ -dimensional integral, the domain of integration will be called an  $n$ -chain and the object integrated will be called an  $n$ -form. The definitions are chosen so that (a) we can use coordinate charts to express our integrals in terms of ordinary first and second year calculus integrals for evaluation, but at the same time (b) the answer to the integral so obtained does not depend on which coordinates are used.

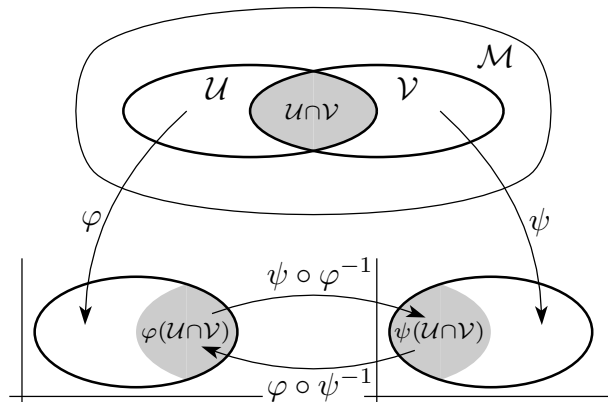
Roughly speaking, an  $n$ -dimensional manifold is a set that looks locally like  $\mathbb{R}^n$ . It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of  $\mathbb{R}^n$ . Here is a precise definition.

**Definition B.1** Let  $\mathcal{M}$  be a topological space. This means that  $\mathcal{M}$  is a set together with a designated family of subsets called “open sets” that obeys some simple axioms [F]. Assume that  $\mathcal{M}$  is a Hausdorff space. This means that, given any two different points  $x, y \in \mathcal{M}$ , there are two open sets  $\mathcal{O}_x, \mathcal{O}_y$  such that  $x \in \mathcal{O}_x, y \in \mathcal{O}_y$  and  $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$ . We now define what is meant by the statement that  $\mathcal{M}$  is an  $n$ -dimensional  $C^\infty$  manifold.

- (a) A *chart* on  $\mathcal{M}$  is a pair  $\{\mathcal{U}, \varphi\}$  with  $\mathcal{U}$  an open subset of  $\mathcal{M}$  and  $\varphi$  a homeomorphism (a 1-1, onto, continuous function with continuous inverse) from  $\mathcal{U}$  to an open subset of  $\mathbb{R}^n$ . Think of  $\varphi$  as assigning coordinates to each point of  $\mathcal{U}$ .
- (b) Two charts  $\{\mathcal{U}, \varphi\}$  and  $\{\mathcal{V}, \psi\}$  are said to be *compatible* if the transition functions

$$\begin{aligned} \psi \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n &\rightarrow \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \\ \varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n &\rightarrow \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \end{aligned}$$

are  $C^\infty$ .



- (c) An *atlas* for  $\mathcal{M}$  is a family  $\mathcal{A} = \{ \{ \mathcal{U}_i, \varphi_i \} \mid i \in \mathcal{I} \}$  of charts on  $\mathcal{M}$  such that  $\{ \mathcal{U}_i \}_{i \in \mathcal{I}}$  is an open cover of  $\mathcal{M}$  and such that every pair of charts in  $\mathcal{A}$  are compatible. An atlas  $\mathcal{A}$  is called *maximal* if every chart  $\{ \mathcal{U}, \varphi \}$  on  $\mathcal{M}$  that is compatible with every chart of  $\mathcal{A}$  is itself in  $\mathcal{A}$ .
- (d) An  $n$ -dimensional manifold consists of a Hausdorff space  $\mathcal{M}$  together with a maximal atlas  $\mathcal{A}$ .

**Problem B.1** Let  $\mathcal{A}$  be an atlas for the Hausdorff space  $\mathcal{M}$ . Prove that there is a unique maximal atlas for  $\mathcal{M}$  that contains  $\mathcal{A}$ .

**Problem B.2** Let  $\mathcal{U}$  and  $\mathcal{V}$  be open subsets of a Hausdorff space  $\mathcal{M}$ . Let  $\varphi$  be a homeomorphism from  $\mathcal{U}$  to an open subset of  $\mathbb{R}^n$  and  $\psi$  be a homeomorphism from  $\mathcal{V}$  to an open subset of  $\mathbb{R}^m$ . Prove that if  $\mathcal{U} \cap \mathcal{V}$  is nonempty and

$$\begin{aligned} \psi \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n &\rightarrow \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \\ \varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m &\rightarrow \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \end{aligned}$$

are  $C^\infty$ , then  $m = n$ .

Thanks to Problem B.1, it suffices to supply any, not necessarily maximal, atlas for a Hausdorff space to turn it into a manifold. We do exactly that in each of the following examples.

**Example B.2** Let  $\mathbb{1}_n$  be the identity map on  $\mathbb{R}^n$ . Then  $\{ \{ \mathbb{R}^n, \mathbb{1}_n \} \}$  is an atlas for  $\mathbb{R}^n$ . Indeed, if  $\mathcal{U}$  is any open subset of  $\mathbb{R}^n$ , then  $\{ \{ \mathcal{U}, \mathbb{1}_n \} \}$  is an atlas for  $\mathcal{U}$ . So every open subset of  $\mathbb{R}^n$  is naturally a  $C^\infty$  manifold.

**Problem B.3** Let  $S^n = \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \}$  be the standard  $n$ -dimensional sphere.

- (a) For each  $1 \leq j \leq n+1$  and  $\sigma \in \{-1, 1\}$  set

$$\mathcal{U}_{j,\sigma} = \{ x \in S^n \mid \sigma x_j > 0 \}$$

and define  $\varphi_{j,\sigma}$  be the projection from  $\mathcal{U}_{j,\sigma}$  into  $\mathbb{R}^n$  that simply discards the  $j^{\text{th}}$  component  $x_j$  of each  $x \in \mathcal{U}_{j,\sigma}$ . Prove that  $\{ \{ \mathcal{U}_{j,\sigma}, \varphi_{j,\sigma} \} \mid 1 \leq j \leq n+1, \sigma \in \{-1, 1\} \}$  is an atlas for  $S^n$ .

- (b) Set

$$\mathcal{U} = S^n \setminus \{(0, \dots, 0, 1)\} \quad \mathcal{V} = S^n \setminus \{(0, \dots, 0, -1)\}$$

and define the stereographic projections,  $\varphi : \mathcal{U} \rightarrow \mathbb{R}^n$  and  $\psi : \mathcal{V} \rightarrow \mathbb{R}^n$ , by

$$\varphi(x) = \frac{2}{1-x_{n+1}}(x_1, \dots, x_n) \quad \psi(x) = \frac{2}{1+x_{n+1}}(x_1, \dots, x_n)$$

Prove that  $\{\mathcal{U}, \varphi\}, \{\mathcal{V}, \psi\}$  is an atlas for  $S^n$ .

**Example B.3** Let  $g_1, \dots, g_m$  be  $C^\infty$  functions on  $\mathbb{R}^{n+m}$  and set

$$\mathcal{M} = \{ x \in \mathbb{R}^{n+m} \mid g_1(x) = g_2(x) = \dots = g_m(x) = 0 \}$$

Assume that, at each point  $\tilde{x} \in \mathcal{M}$ , the vectors  $\nabla g_1(\tilde{x}), \nabla g_2(\tilde{x}), \dots, \nabla g_m(\tilde{x})$  are linearly independent. Then it is possible to choose  $m$  distinct indices (possibly depending on  $\tilde{x}$ ),  $1 \leq i_1 < i_2 < \dots < i_m \leq n+m$  such that the matrix  $\left[ \frac{\partial g_i}{\partial x_{i_j}}(\tilde{x}) \right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$  is invertible. Denote by  $1 \leq j_1 < j_2 < \dots < j_n \leq n+m$  the remaining indices. Then, by the implicit function theorem, we can solve  $g_1(x) = g_2(x) = \dots = g_m(x) = 0$  for  $x_{i_1}, x_{i_2}, \dots, x_{i_m}$  in terms of  $x_{j_1}, x_{j_2}, \dots, x_{j_n}$  at least in some neighbourhood  $\mathcal{O}_{\tilde{x}}$  of  $\tilde{x}$ . That is, we may express  $\mathcal{M} \cap \mathcal{O}_{\tilde{x}}$  as

$$\begin{aligned} x_{i_1} &= f_1(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \\ x_{i_2} &= f_2(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \\ &\vdots \\ x_{i_m} &= f_m(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \end{aligned}$$

for some  $C^\infty$  functions  $f_1, \dots, f_m$ , and use  $x_{j_1}, x_{j_2}, \dots, x_{j_n}$  as coordinates for  $\mathcal{M}$  in  $\mathcal{M} \cap \mathcal{O}_{\tilde{x}}$ . This turns  $\mathcal{M}$  into a manifold.

**Example B.4** As a special case of Example B.3 we have the groups

$$\begin{aligned} SO(3) &= \{ 3 \times 3 \text{ real matrices } R \mid R^t R = \mathbb{1}_3, \det R = 1 \} \\ O(3) &= \{ 3 \times 3 \text{ real matrices } R \mid R^t R = \mathbb{1}_3 \} \end{aligned}$$

of rotations and rotations/reflections in  $\mathbb{R}^3$ . We can identify the set of all  $3 \times 3$  real matrices with  $\mathbb{R}^9$ . The restriction that

$$R = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in O(3)$$

is given by six equations.

$$\begin{aligned} (R^t R)_{1,1} &= a_1^2 + a_2^2 + a_3^2 = 1 && \text{i.e. } |a| = 1 \\ (R^t R)_{2,2} &= b_1^2 + b_2^2 + b_3^2 = 1 && \text{i.e. } |b| = 1 \\ (R^t R)_{3,3} &= c_1^2 + c_2^2 + c_3^2 = 1 && \text{i.e. } |c| = 1 \\ (R^t R)_{1,2} &= (R^t R)_{2,1} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0 && \text{i.e. } a \perp b \\ (R^t R)_{1,3} &= (R^t R)_{3,1} = a_1 c_1 + a_2 c_2 + a_3 c_3 = 0 && \text{i.e. } a \perp c \\ (R^t R)_{2,3} &= (R^t R)_{3,2} = b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 && \text{i.e. } b \perp c \end{aligned}$$

We can verify the independence conditions of Example B.3 directly. Or we can argue geometrically. In a neighbourhood of any element,  $R$ , of  $SO(3)$ , we may use two of the three  $a$ -components as coordinates. (In fact we may use any two  $a$ -coordinates whose magnitude at  $R$  is not one.) The third  $a$ -component is determined up to a sign by the requirement that  $|a| = 1$ . The sign is chosen so as to remain in the neighbourhood. Then  $b$  is determined up to a single rotation angle by the requirements that  $b \perp a$  and  $|b| = 1$ . That rotation angle is the third coordinate. Finally  $c$  is determined up to a sign by the requirements that  $c \perp a, b$  and  $|c| = 1$ . Again, the sign is chosen so as to remain in the neighbourhood. So  $O(3)$  is a manifold of dimension 3. Any element of  $O(3)$  automatically obeys

$$(\det R)^2 = \det R^t R = \det \mathbb{1}_3 = 1 \implies \det R = \pm 1$$

So  $SO(3)$  is just one of the two connected components of  $O(3)$ . It is an important example of a Lie group, which is, by definition, a  $C^\infty$  manifold that is also a group with the operations of multiplication and taking inverses continuous.

**Example B.5** Define an equivalence relation on  $\mathbb{R}^n$  by

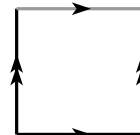
$$x \sim y \iff x - y \in \mathbb{Z}^n$$

The equivalence class of  $x \in \mathbb{R}^n$  under this equivalence relation is  $[x] = \{ y \in \mathbb{R}^n \mid y \sim x \}$ . The set of equivalence classes is denoted  $\mathbb{R}^n / \mathbb{Z}^n = \{ [x] \mid x \in \mathbb{R}^n \}$ . Each equivalence class  $[x]$  contains exactly one representative  $\tilde{x} \in [x]$  obeying  $0 \leq \tilde{x}_j < 1$  for each  $1 \leq j \leq n$ . So we can also think of  $\mathbb{R}^n / \mathbb{Z}^n$  as being

$$\{ x \in \mathbb{R}^n \mid 0 \leq x_j < 1 \text{ for all } 1 \leq j \leq n \}$$

But then we should also identify, for each  $1 \leq j \leq n$ , the edges

$$\begin{aligned} & \{ x \in \mathbb{R}^n \mid x_j = 1, 0 \leq x_i \leq 1 \forall i \neq j \} \\ \text{and } & \{ x \in \mathbb{R}^n \mid x_j = 0, 0 \leq x_i \leq 1 \forall i \neq j \} \end{aligned}$$



We can turn the set  $\mathbb{R}^n / \mathbb{Z}^n$ , which is called a torus, into a metric space by imposing the metric

$$\rho([x], [y]) = \min \{ |\tilde{x} - \tilde{y}| \mid \tilde{x} \in [x], \tilde{y} \in [y] \}$$

All metric spaces are Hausdorff, so we only need an atlas to turn the torus into a manifold. If  $\mathcal{U}$  is any open subset of  $\mathbb{R}^n$  with the property that no two points of  $\mathcal{U}$  are equivalent (any open ball of radius at most  $\frac{1}{2}$  has this property), then  $[\mathcal{U}] = \{ [x] \mid x \in \mathcal{U} \}$  is an open subset of  $\mathbb{R}^n / \mathbb{Z}^n$  and each element of  $[\mathcal{U}]$  contains a unique representative  $\tilde{x} \in [x]$  that is in  $\mathcal{U}$ . Define

$$\Phi_{\mathcal{U}} : [\mathcal{U}] \rightarrow \mathbb{R}^n$$

$$[x] \mapsto \tilde{x} \text{ with } \tilde{x} \in [x], \tilde{x} \in \mathcal{U}$$

Then  $\{[\mathcal{U}], \Phi_{\mathcal{U}}\}$  is a chart and the set of all such charts is an atlas.

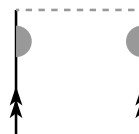
**Example B.6** We are now going to turn the set

$$M = [0, 1) \times (-1, 1)$$

into two very different manifolds by assigning two different, incompatible, atlases. Both atlases will contain two charts with

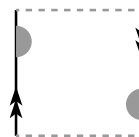
$$\mathcal{U}_1 = \left(\frac{1}{8}, \frac{7}{8}\right) \times (-1, 1) \quad \mathcal{U}_2 = \left[0, \frac{1}{4}\right) \times (-1, 1) \cup \left(\frac{3}{4}, 1\right) \times (-1, 1)$$

The first atlas attaches each point  $(0, t)$  on the left hand side to the point  $(1, t)$  on the right hand side by using the coordinate functions

$$\begin{aligned} \psi_1(x, y) &= (x, y) \\ \psi_2(x, y) &= \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x - 1, y) & \text{if } \frac{3}{4} < x < 1 \end{cases} \end{aligned}$$


The figure above displays the inverse image under  $\psi_2$  of the disk  $x^2 + (y - \frac{1}{2})^2 < \frac{1}{4^2}$ . So the disk displayed is connected in the manifold with atlas  $\{\{\mathcal{U}_1, \psi_1\}, \{\mathcal{U}_1, \psi_2\}\}$ . This manifold may be constructed from a strip of paper by gluing the left and right hand edges together.

The second atlas attaches each point  $(0, t)$  on the left hand side to the point  $(1, -t)$  on the right hand side by using the coordinate functions

$$\begin{aligned} \varphi_1(x, y) &= (x, y) \\ \varphi_2(x, y) &= \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x - 1, -y) & \text{if } \frac{3}{4} < x < 1 \end{cases} \end{aligned}$$


The figure above displays the inverse image under  $\varphi_2$  of the disk  $x^2 + (y - \frac{1}{2})^2 < \frac{1}{4^2}$ . So the disk displayed is connected in the manifold with atlas  $\{\{\mathcal{U}_1, \varphi_1\}, \{\mathcal{U}_1, \varphi_2\}\}$ . This manifold may be constructed from a strip of paper by gluing the left and right hand edges together, after putting a half twist in the strip. It is called a Möbius band.

**Problem B.4** Prove that the two atlases of Example B.6 are not compatible.

**Definition B.7**

- (a) A function  $f$  from a manifold  $\mathcal{M}$  to a manifold  $\mathcal{N}$  (it is traditional to omit the atlas from the notation) is said to be  $C^\infty$  at  $m \in \mathcal{M}$  if there exists a chart  $\{\mathcal{U}, \varphi\}$  for  $\mathcal{M}$  and a chart  $\{\mathcal{V}, \psi\}$  for  $\mathcal{N}$  such that  $m \in \mathcal{U}$ ,  $f(m) \in \mathcal{V}$  and  $\psi \circ f \circ \varphi^{-1}$  is  $C^\infty$  at  $\varphi(m)$ .
- (b) Two manifolds  $\mathcal{M}$  and  $\mathcal{N}$  are *diffeomorphic* if there exists a function  $f : \mathcal{M} \rightarrow \mathcal{N}$  that is 1-1 and onto with  $f$  and  $f^{-1}$   $C^\infty$  everywhere.

**Problem B.5** Let  $\mathcal{M}$  and  $\mathcal{N}$  be manifolds. Prove that  $f : \mathcal{M} \rightarrow \mathcal{N}$  is  $C^\infty$  at  $m \in \mathcal{M}$  if and only if  $\psi \circ f \circ \phi^{-1}$  is  $C^\infty$  at  $\phi(m)$  for every chart  $(\mathcal{U}, \phi)$  for  $\mathcal{M}$  with  $m \in \mathcal{U}$  and every chart  $(\mathcal{V}, \psi)$  for  $\mathcal{N}$  with  $f(m) \in \mathcal{V}$ .

**Problem B.6** Prove that  $\mathbb{R}^n$  is diffeomorphic to  $\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1 \}$ .

**Problem B.7** Prove that  $\mathbb{R}^n$  is not diffeomorphic to  $S^n$ .

**Problem B.8** Prove that the disk  $\{ \mathbf{x} \in \mathbb{R}^2 \mid x^2 + y^2 < 2 \}$  is not diffeomorphic to the annulus  $\{ \mathbf{x} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \}$ .

**Problem B.9** In this problem  $G = SO(3)$ .

- Fix any  $a \in G$ . Denote by  $I = \{ (i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq 3, 1 \leq j \leq 3 \}$  the set of indices for the matrix elements of the matrices in  $G$ . Prove that there exist  $\alpha, \beta, \gamma \in I$  such that every matrix element  $g_\delta$ ,  $\delta \in I$  is a  $C^\infty$  function of  $g_\alpha, g_\beta, g_\gamma$  for matrices  $g \in G$  in a neighbourhood of  $a$ .
- Prove that a curve  $q : (c, d) \rightarrow G$  is  $C^\infty$  if and only if every matrix element  $q(t)_{i,j}$  is  $C^\infty$ .
- Prove that matrix multiplication  $(a, b) \mapsto ab$  is a  $C^\infty$  function from  $G \times G$  to  $G$ .
- Prove that the inverse function  $a \mapsto a^{-1}$  is a  $C^\infty$  function from  $G$  to  $G$ .

For the rest of this appendix, assume that  $\mathcal{M}$  is a two dimensional  $C^\infty$  manifold with maximal atlas  $\mathcal{A}$ .

## §B.1. 0-dimensional Integrals

### Definition B.1.1

- A 0-form is a (complex valued) continuous function  $F$  on  $\mathcal{M}$ .
- A 0-chain is an expression of the form  $n_1 P_1 + \cdots + n_k P_k$  with  $P_1, \dots, P_k$  distinct points of  $\mathcal{M}$  and  $n_1, \dots, n_k \in \mathbb{Z}$ .
- If  $F$  is a 0-form and  $n_1 P_1 + \cdots + n_k P_k$  is a 0-chain, then

$$\int_{n_1 P_1 + \cdots + n_k P_k} F = n_1 F(P_1) + \cdots + n_k F(P_k)$$

The definition of a chain given in part (b) is somewhat casual. Under a more formal definition, a 0-chain is a function  $\sigma : \mathcal{M} \rightarrow \mathbb{Z}$  for which  $\sigma(P)$  is zero for all but finitely many  $P \in \mathcal{M}$ . The function  $\sigma : \mathcal{M} \rightarrow \mathbb{Z}$  which corresponds to  $n_1 P_1 + \cdots + n_k P_k$  has

$\sigma(P) = n_i$  when  $P = P_i$  for some  $1 \leq i \leq k$  and  $\sigma(P) = 0$  if  $P \notin \{P_1, \dots, P_k\}$ . Addition of 0-chains and multiplication of a 0-chain by an integer are defined by

$$(\sigma + \sigma')(P) = \sigma(P) + \sigma'(P) \quad (n\sigma)(P) = n\sigma(P)$$

## §B.2. 1-dimensional Integrals

### Definition B.2.1

- (a) A 1-form  $\omega$  is a rule which assigns to each coordinate chart  $\{U, \zeta = (x, y) : \mathcal{M} \rightarrow \mathbb{R}^2\}$  a pair  $(f, g)$  of (complex valued) continuous functions on  $\zeta(U)$  such that
- if  $\{U, \zeta\}$  and  $\{\tilde{U}, \tilde{\zeta}\}$  are two charts with  $U \cap \tilde{U} \neq \emptyset$  and
  - if  $\omega$  assigns  $\{U, \zeta\}$  the pair of functions  $(f, g)$  and assigns  $\{\tilde{U}, \tilde{\zeta}\}$  the pair of functions  $(\tilde{f}, \tilde{g})$  and
  - if the transition function  $\tilde{\zeta} \circ \zeta^{-1}$  (from  $\zeta(U \cap \tilde{U}) \subset \mathbb{R}^2$  to  $\tilde{\zeta}(U \cap \tilde{U}) \subset \mathbb{R}^2$ ) is  $(\tilde{x}(x, y), \tilde{y}(x, y))$ ,

then

$$\begin{aligned} f(x, y) &= \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial x}(x, y) + \tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial x}(x, y) \\ g(x, y) &= \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial y}(x, y) + \tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial y}(x, y) \end{aligned}$$

We write

$$\omega|_{\{U, \zeta\}} = f dx + g dy$$

to indicate that  $\omega$  assigns the pair  $(f, g)$  to the chart  $\{U, \zeta\}$  and we say that  $\omega$  is invariant under coordinate transformations. Motivation, and a memory aid, for the above coordinate transformation rule is provided in Remark B.2.2, below.

- (b) The standard 1-simplex is  $Q^1 = [0, 1]$ . A path is a  $C^1$  map  $C : [0, 1] \rightarrow \mathcal{M}$ .
- (c) Let  $\{U, \zeta = (x, y)\}$  be a coordinate chart for  $\mathcal{M}$  and let  $\omega|_{U, \zeta} = f dx + g dy$ . If  $c(t) : [0, 1] \rightarrow U \subset \mathcal{M}$  is a path with range in  $U$ , then

$$\int_c \omega = \int_0^1 \left[ \underbrace{f\left(\underbrace{\zeta\left(\underbrace{c(t)}_{\in \mathcal{M}}\right)}_{\in \mathbb{R}^2}\right)}_{\in \mathbb{C}} \frac{dx(c(t))}{dt} + g\left(\zeta(c(t))\right) \frac{dy(c(t))}{dt} \right] dt$$

If  $c$  does not have range in a single chart, split it up into a finite number of pieces, each with range in a single chart. This can always be done, since the range of  $c$  is always compact. The answer is independent of choice of chart(s), because  $\omega$  is invariant under coordinate transformations. See part (b) of Remark B.2.2 and Problem B.2.1.

- (d) A 1-chain is an expression of the form  $n_1 C_1 + \dots + n_k C_k$  with  $C_1, \dots, C_k$  distinct paths and  $n_1, \dots, n_k \in \mathbb{Z}$ .

(e) If  $\omega$  is a 1-form and  $n_1 C_1 + \cdots + n_k C_k$  is a 1-chain, then

$$\int_{n_1 C_1 + \cdots + n_k C_k} \omega = n_1 \int_{C_1} \omega + \cdots + n_k \int_{C_k} \omega$$

**Remark B.2.2**

- (a) For now think of  $f dx + g dy$  just as a piece of notation which specifies the two functions  $(f, g)$  that  $\omega$  assigns to the chart  $\{U, \zeta = (x, y)\}$ . We will later define an operator  $d$  that maps  $n$ -forms to  $(n + 1)$ -forms. In particular, it will map the coordinate function  $x$ , which is a zero form (but which is only defined on part of the manifold) to the 1-form  $1dx + 0dy$ .
- (b) The integral of part (c) is a generalization of the calculus definition of an integral along a parameterized line.
- (c) The motivation for the definition of a 1-form is the ordinary change of variables rule for an integral along a curve. Suppose that  $(\tilde{x}(x, y), \tilde{y}(x, y))$  expresses  $(\tilde{x}, \tilde{y})$ -coordinates as a function of  $(x, y)$ -coordinates. If  $(X(t), Y(t))$  is a parameterized curve in  $(x, y)$ -coordinates, then  $\tilde{X}(t) = \tilde{x}(X(t), Y(t))$ ,  $\tilde{Y}(t) = \tilde{y}(X(t), Y(t))$  provides a parameterization of the same curve in  $(\tilde{x}, \tilde{y})$  coordinates. By definition, the line integral

$$\begin{aligned} & \int \tilde{f}(\tilde{x}, \tilde{y}) d\tilde{x} + \tilde{g}(\tilde{x}, \tilde{y}) d\tilde{y} = \int \tilde{f}(\tilde{X}(t), \tilde{Y}(t)) \frac{d\tilde{X}}{dt}(t) dt + \tilde{g}(\tilde{X}(t), \tilde{Y}(t)) \frac{d\tilde{Y}}{dt}(t) dt \\ & = \int \left\{ \tilde{f}(\tilde{x}(X(t), Y(t)), \tilde{y}(X(t), Y(t))) \left[ \frac{\partial \tilde{x}}{\partial x}(X(t), Y(t)) \frac{dX}{dt}(t) + \frac{\partial \tilde{x}}{\partial y}(X(t), Y(t)) \frac{dY}{dt}(t) \right] \right. \\ & \quad \left. + \tilde{g}(\tilde{x}(X(t), Y(t)), \tilde{y}(X(t), Y(t))) \left[ \frac{\partial \tilde{y}}{\partial x}(X(t), Y(t)) \frac{dX}{dt}(t) + \frac{\partial \tilde{y}}{\partial y}(X(t), Y(t)) \frac{dY}{dt}(t) \right] \right\} dt \\ & = \int \left\{ \left[ \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial x}(x, y) + \tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial x}(x, y) \right]_{x=X(t), y=Y(t)} \frac{dX}{dt}(t) \right. \\ & \quad \left. + \left[ \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial y}(x, y) + \tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial y}(x, y) \right]_{x=X(t), y=Y(t)} \frac{dY}{dt}(t) \right\} dt \\ & = \int f(x, y) dx + g(x, y) dy \end{aligned}$$

with

$$\begin{aligned} f(x, y) &= \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial x}(x, y) + \tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial x}(x, y) \\ g(x, y) &= \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial y}(x, y) + \tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial y}(x, y) \end{aligned}$$

which is exactly the coordinate transformation rule of part (a) of Definition B.2.1. To remember this rule, just remember

$$\begin{aligned} d\tilde{x} &= \frac{\partial \tilde{x}}{\partial x}(x, y) dx + \frac{\partial \tilde{x}}{\partial y}(x, y) dy \\ d\tilde{y} &= \frac{\partial \tilde{y}}{\partial x}(x, y) dx + \frac{\partial \tilde{y}}{\partial y}(x, y) dy \end{aligned}$$

Substituting this into

$$\tilde{f}d\tilde{x} + \tilde{g}d\tilde{y} = \tilde{f}\frac{\partial\tilde{x}}{\partial x}dx + \tilde{f}\frac{\partial\tilde{x}}{\partial y}dy + \tilde{g}\frac{\partial\tilde{y}}{\partial x}dx + \tilde{g}\frac{\partial\tilde{y}}{\partial y}dy$$

and putting in the only arguments that make sense, gives the detailed coordinate transformation rule.

**Problem B.2.1** Let  $\mathcal{M}$  be a manifold,  $\omega$  be a 1-form on  $\mathcal{M}$  and  $c(t) : [0, 1] \rightarrow \mathcal{M}$  be a path in  $\mathcal{M}$ . Prove that the definition of  $\int_c \omega$  given in part (c) of Definition B.2.1 is independent of the decomposition of  $c$  into finitely many pieces and of the choice of coordinate charts.

### §B.3. 2-dimensional Integrals

#### Definition B.3.1

(a) A 2-form  $\Omega$  is a rule which assigns to each chart  $\{U, \zeta\}$  a continuous function  $f$  on  $\zeta(U)$  such that  $\Omega|_{\{U, \zeta\}} = f dx \wedge dy$  is invariant under coordinate transformations.

This means that

- if  $\{U, \zeta\}$  and  $\{\tilde{U}, \tilde{\zeta}\}$  are two charts with  $U \cap \tilde{U} \neq \emptyset$  and
- if  $\Omega$  assigns  $\{U, \zeta\}$  the function  $f$  and assigns  $\{\tilde{U}, \tilde{\zeta}\}$  the function  $\tilde{f}$  and
- if the transition function  $\tilde{\zeta} \circ \zeta^{-1}$  (from  $\zeta(U \cap \tilde{U}) \subset \mathbb{R}^2$  to  $\tilde{\zeta}(U \cap \tilde{U}) \subset \mathbb{R}^2$ ) is  $(\tilde{x}(x, y), \tilde{y}(x, y))$ ,

then

$$f(x, y) = \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \left[ \frac{\partial\tilde{x}}{\partial x}(x, y) \frac{\partial\tilde{y}}{\partial y}(x, y) - \frac{\partial\tilde{x}}{\partial y}(x, y) \frac{\partial\tilde{y}}{\partial x}(x, y) \right]$$

(b) The standard 2-simplex is

$$Q^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \geq 0, x + y \leq 1 \}$$

A surface is a  $C^1$  map  $D : Q^2 \rightarrow \mathcal{M}$ .

(c) Let  $\{U, \zeta = (x, y)\}$  be a chart and let  $\Omega|_{U, \zeta} = f(x, y) dx \wedge dy$ . If  $D : Q^2 \rightarrow U \subset \mathcal{M}$  is a surface with range in  $U$ , then

$$\int_D \Omega = \iint_{Q^2} f(\zeta(D(s, t))) \left[ \frac{\partial}{\partial s} x(D(s, t)) \frac{\partial}{\partial t} y(D(s, t)) - \frac{\partial}{\partial t} x(D(s, t)) \frac{\partial}{\partial s} y(D(s, t)) \right] ds dt$$

If  $D$  does not have range in a single chart, split it up into a finite number of pieces, each with range in a single chart. This can always be done, since the range of  $D$  is always compact. The answer is independent of choice of chart(s).

- (d) A 2-chain is an expression of the form  $n_1 D_1 + \cdots + n_k D_k$  with  $D_1, \dots, D_k$  surfaces and  $n_1, \dots, n_k \in \mathbb{Z}$ .
- (e) If  $\Omega$  is a 2-form and  $n_1 D_1 + \cdots + n_k D_k$  is a 2-chain, then

$$\int_{n_1 D_1 + \cdots + n_k D_k} \Omega = n_1 \int_{D_1} \Omega + \cdots + n_k \int_{D_k} \Omega$$

### Remark B.3.2

- (a) Once again think, for now, of  $f dx \wedge dy$  as just a piece of notation which specifies the function  $f$  that  $\Omega$  assigns to the chart  $\{U, \zeta = (x, y)\}$ . We will later define a wedge product  $\wedge$ . Then  $dx \wedge dy$  will really be the wedge product of the 1-forms  $dx$  and  $dy$  and  $f dx \wedge dy$  will be the wedge product of the 0-form  $f$  and the 2-form  $dx \wedge dy$ . Under the normal notation convention, the wedge product of a 0-form  $f$  and any form  $\omega$  is written  $f\omega$ , rather than  $f \wedge \omega$ .
- (b) The motivation for the definition of a 2-form is the ordinary change of variables rule

$$\begin{aligned} \int \tilde{f}(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} &= \int \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \left| \det \begin{bmatrix} \frac{\partial \tilde{x}}{\partial x}(x, y) & \frac{\partial \tilde{y}}{\partial x}(x, y) \\ \frac{\partial \tilde{x}}{\partial y}(x, y) & \frac{\partial \tilde{y}}{\partial y}(x, y) \end{bmatrix} \right| dx dy \\ &= \int \tilde{f} \left| \frac{\partial \tilde{x}}{\partial x} \frac{\partial \tilde{y}}{\partial y} - \frac{\partial \tilde{x}}{\partial y} \frac{\partial \tilde{y}}{\partial x} \right| dx dy \end{aligned}$$

for an integral on a region in  $\mathbb{R}^2$ , except for the absolute value signs. So we are dealing with oriented (i.e. signed) areas.

- (c) The integral over  $Q^2$  in part (c) of Definition B.3.1 is the standard multivariable calculus expression for an integral over a parametrized surface.

## §B.4. The Wedge Product

**Definition B.4.1** If  $\omega$  is a  $k$ -form and  $\omega'$  is a  $k'$ -form then  $\omega \wedge \omega'$  is the  $(k + k')$ -form that is determined by  $\omega \wedge \omega' = (-1)^{kk'} \omega' \wedge \omega$  (so that  $\omega \wedge \omega' = \omega' \wedge \omega$  if at least one of  $k$  and  $k'$  is even and  $\omega \wedge \omega' = -\omega' \wedge \omega$  if both  $k$  and  $k'$  are odd) and

- (a) if  $k = k' = 0$  and  $(\omega \wedge \omega')(P) = \omega(P)\omega'(P)$ .
- (b) if  $k = 0$  and  $\omega'|_{U, \zeta} = f dx + g dy$  then

$$\omega \wedge \omega'|_{U, \zeta} = (\omega \circ \zeta^{-1})f dx + (\omega \circ \zeta^{-1})g dy$$

- (c) if  $k = 0$  and  $\omega'|_{U, \zeta} = f dx \wedge dy$  then

$$\omega \wedge \omega'|_{U, \zeta} = (\omega \circ \zeta^{-1})f dx \wedge dy$$

(d) if  $k = k' = 1$  and  $\omega|_{U,\zeta} = f dx + g dy$  then  $\omega'|_{U,\zeta} = f' dx + g' dy$  then

$$\omega \wedge \omega'|_{U,\zeta} = [fg' - gf'] dx \wedge dy$$

In particular  $dx \wedge dx = dy \wedge dy = 0$  and  $dx \wedge dy = -dy \wedge dx$ .

(e) If  $k + k' > 2$ ,  $\omega \wedge \omega' = 0$ .

## §B.5. The Differential Operator $d$

**Definition B.5.1** Let  $\mathcal{M}$  be a two real dimensional manifold. If  $\{U, \zeta\}$  is a coordinate chart on  $\mathcal{M}$  and

(a) if  $F : \mathcal{M} \rightarrow \mathbb{C}$  is a  $C^1$  0-form, then

$$dF|_{\{U,\zeta\}} = \frac{\partial}{\partial x}(F \circ \zeta^{-1})(x, y) dx + \frac{\partial}{\partial y}(F \circ \zeta^{-1})(x, y) dy$$

(b) if  $\omega$  is a  $C^1$  1-form with  $\omega|_{\{U,\zeta\}} = f(x, y) dx + g(x, y) dy$ , then

$$d\omega|_{\{U,\zeta\}} = \left[ \frac{\partial g}{\partial x}(x, y) - \frac{\partial f}{\partial y}(x, y) \right] dx \wedge dy$$

(c) if  $\Omega$  is a  $C^1$  2-form, then  $d\Omega = 0$

**Lemma B.5.2** *The differential operator  $d$  maps  $n$ -forms to  $n + 1$  forms and obeys*

$$d^2 = 0$$

**Proof:** In the case  $n = 0$ , (writing  $f = F \circ \zeta^{-1}$ )

$$d^2 F = d\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy \wedge dx + \frac{\partial}{\partial x} \frac{\partial f}{\partial y} dx \wedge dy = \left[-\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y}\right] dx \wedge dy = 0$$

The cases  $n = 1, 2$  are trivial, since  $d$  applied to any 2-form is zero. ■

**Problem B.5.1** Prove that Definition B.5.1 is independent of the choice of coordinate chart.

**Problem B.5.2** Prove the product rule that if  $\omega$  is a  $k$ -form and  $\omega'$  is a  $k'$ -form, then

$$d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^k \omega \wedge (d\omega')$$



## §B.7. Stokes' Theorem

**Theorem B.7.1 (Stokes' Theorem)** *If  $\omega$  is a  $C^1$   $k$ -form and  $D$  is a  $(k + 1)$ -chain, then*

$$\int_{\delta D} \omega = \int_D d\omega$$

**Proof:** We give the proof for a manifold of dimension two. For  $k = 0$  and  $D$  being the path  $C$ , this is the fundamental theorem of calculus.

$$F(C(1)) - F(C(0)) = \int_C dF$$

For  $k = 1$  and  $D$  a surface, this is Green's Theorem.

$$\int_{\delta D} f dx + g dy = \iint_D \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dx \wedge dy$$

For  $k \geq 2$ , both sides are zero. ■

Here are two consequences of Stokes' Theorem. Firstly, if  $\omega$  is a compactly supported 1-form,  $\iint_{\mathcal{M}} d\omega = 0$ . To see this, let  $S$  be a surface in  $\mathcal{M}$  that contains the support of  $\omega$ . Then  $\iint_{\mathcal{M}} d\omega = \iint_S d\omega = \int_{\partial S} \omega = 0$ , since  $\omega$  vanishes on  $\partial S$ . Secondly, if  $\omega$  is a closed 1-form (meaning that  $d\omega = 0$ ) and if  $C_1$  and  $C_2$  are two 1-chains with  $C_1 - C_2 = \delta D$  for some 2-chain  $D$ , then  $\int_{C_1} \omega = \int_{C_2} \omega$ , since

$$\int_{C_1} \omega - \int_{C_2} \omega = \int_{C_1 - C_2} \omega = \int_D d\omega = 0$$

