Appendix S3: Problem Solutions for §3

Problem 3.1 Let γ be a symmetric, real, positive definite matrix. Assume that there are constants $C_1, C_2 > 0$ such that $C_1|\xi|^2 \le \sum_{i,j=1}^n \gamma^{i,j}\xi_i\xi_j \le C_2|\xi|^2$ for all $\xi \in \mathbb{R}^n$. Prove that

$$C_1|\xi|^2 \le \sum_{i,j=1}^n \gamma^{i,j}\xi_i\overline{\xi_j} \le C_2|\xi|^2$$

for all $\xi \in \mathbb{C}^n$.

Solution. Write $\xi = x + iy$ with $x, y \in \mathbb{R}^n$. Then $|\xi|^2 = |x|^2 + |y|^2$ and, since γ is symmetric,

$$\begin{aligned}\sum_{\ell,m=1}^n \gamma^{\ell,m}\xi_\ell\overline{\xi_m} &= \sum_{\ell,m=1}^n \gamma^{\ell,m}x_\ell x_m + i\sum_{\ell,m=1}^n \gamma^{\ell,m}y_\ell x_m - i\sum_{\ell,m=1}^n \gamma^{\ell,m}x_\ell y_m + \sum_{\ell,m=1}^n \gamma^{\ell,m}y_\ell y_m \\ &= \sum_{\ell,m=1}^n \gamma^{\ell,m}x_\ell x_m + \sum_{\ell,m=1}^n \gamma^{\ell,m}y_\ell y_m\end{aligned}$$

Adding together

$$C_1|x|^2 \le \sum_{\ell,m=1}^n \gamma^{\ell,m}x_\ell x_m \le C_2|x|^2 \quad \text{and} \quad C_1|y|^2 \le \sum_{\ell,m=1}^n \gamma^{\ell,m}y_\ell y_m \le C_2|y|^2$$

which are true by hypothesis, gives the desired bounds. ■

Problem 3.1.1 Prove that, under Hypotheses 3.1.1, there are constants $C_1 > 0$ and C_2 such that

$$C_1|\xi|^2 \le \sum_{i,j=1}^n \gamma^{i,j}(x)\xi_i\overline{\xi_j} \le C_2|\xi|^2$$

for all $\xi \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$.

Solution. For any $n \times n$ matrix, $\big[\Gamma^{i,j}\big]_{1\le i,j\le n}$, there is a constant C such that

$$\Big|\sum_{i,j=1}^n \Gamma^{i,j}\xi_i\overline{\xi_j}\Big| \le C|\xi|^2$$

for all $\xi \in \mathbb{C}^n$. Since $|\xi_i|\,|\overline{\xi_j}| \le \frac{1}{2}(|\xi_i|^2 + |\xi_j|^2) \le |\xi|^2$, one obvious, but crude, choice of C is $\sum_{i,j=1}^n |\Gamma^{i,j}|$. This choice depends continuously on the matrix elements of Γ. The matrix elements of $\gamma(x)$ depend continuously on x and have finite limits, namely the matrix elements of γ_∞, as $|x| \to \infty$. Hence there is a constant C_2 such that

$$\sum_{i,j=1}^n \gamma^{i,j}(x)\xi_i\overline{\xi_j} \le C_2|\xi|^2$$

for all $\xi \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$. Since each $\gamma(x)$ is strictly positive definite, $\gamma(x)^{-1}$ exists for each x, depends continuously on x and approaches γ_∞^{-1}, which also exists, in the limit $|x| \to \infty$. Hence there is a constant $C_1 > 0$ such that

$$\sum_{i,j=1}^n \left(\gamma(x)^{-1}\right)^{i,j}\xi_i\overline{\xi_j} \le \tfrac{1}{C_1}|\xi|^2$$

and hence

$$C_1|\xi|^2 \le \sum_{i,j=1}^n \gamma^{i,j}(x)\xi_i\overline{\xi_j} \le C_2|\xi|^2$$

for all $\xi \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$. ■

Problem 3.1.2 Prove that there is a unique continuous sesquilinear form

$$\langle\ \cdot\ ,\ \cdot\ \rangle_{\gamma,0} : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{C}$$

such that

$$\langle u, v\rangle_{\gamma,0} = \sum_{i,j=1}^n \int \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial v}{\partial x_j}(x)}\ d^n x$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$. Is it an inner product? If so, is the associated norm equivalent to $|\!|\ \cdot\ |\!|_{1,n}$?

Solution. That $\langle u, v\rangle_{\gamma,0}$ is linear in u and conjugate linear in v and $\langle u, u\rangle_{\gamma,0} \ge 0$ are obvious. By Problem 3.1.1,

$$\begin{aligned} C_1\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \le \langle u, u\rangle_{\gamma,0} &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^n x \\ &\le C_2\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \le C_2|\!|u|\!|^2_{1,n} \end{aligned}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. So if $\langle u, u\rangle_{\gamma,0} = 0$, then $\|\nabla u\|_{L^2(\mathbb{R}^n)} = 0$. But this forces $k\hat{u}(\mathbf{k})$ to be zero for almost all k, which in turn forces $\hat{u}(k)$ to be zero for almost all k. Thus $\langle\ \cdot\ ,\ \cdot\ \rangle_{\gamma,0}$ is an inner product and is continuous in the topology of $H^1(\mathbb{R}^n)$. So it has a unique continuous extension to an inner product on $H^1(\mathbb{R}^n)$. But the associated norm is not equivalent to $|\!|\ \cdot\ |\!|_{1,n}$, because the inequality $\langle u, u\rangle_{\gamma,0} \ge \text{const}\,|\!|u|\!|^2_{1,n}$ fails. If this inequality were true, we would have $\|\nabla u\|_{L^2(\mathbb{R}^n)} \ge \text{const}\,|\!|u|\!|_{1,n}$. ■

Problem 3.1.3 Prove that there is a unique bounded linear map $L_{\gamma,0} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ such that

$$(L_{\gamma,0}u)(x) = -\sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big)$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Solution. We shall show that there is a constant C such that

$$\big|\left\langle v, L_{\gamma,0}\bar{u}\right\rangle_{L^2(\mathbb{R}^n)}\big| \le C|u|_1|v|_1$$

for all $u, v \in C_0^\infty(\mathbb{R}^n)$. Since, by Problem 2.1.8, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, this will imply that $\big|\left\langle v, L_{\gamma,0}\bar{u}\right\rangle_{L^2(\mathbb{R}^n)}\big| \le C|u|_1|v|_1$ for all $u \in C_0^\infty(\mathbb{R}^n)$ and $v \in H^1(\mathbb{R}^n)$ and hence that $|L_{\gamma,0}u|_{-1} \le C|u|_1$, by part (b) of Problem 2.1.16. Since $C_0^\infty(\mathbb{R}^n)$ is still dense in $H^1(\mathbb{R}^n)$, this will in turn imply that $L_{\gamma,0}$ has a unique continuous extension to all of $H^1(\mathbb{R}^n)$ by the B.L.T. theorem.

By the divergence theorem

$$\begin{aligned}
\left\langle v, L_{\gamma,0}\bar{u}\right\rangle_{L^2(\mathbb{R}^n)} &= -\int_{\mathbb{R}^n} v(x)\Big[\sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big)\Big]\, d^n x \\
&= \sum_{i,j=1}^{n}\int_{\mathbb{R}^n} \gamma^{i,j}(x)\tfrac{\partial v}{\partial x_i}(x)\tfrac{\partial u}{\partial x_j}(x)\ d^n x \\
&= \left\langle v, \bar{u}\right\rangle_{\gamma,0}
\end{aligned}$$

By Problem 3.1.2, $\left\langle v, \bar{u}\right\rangle_{\gamma,0}$ is an inner product and $\|u\|_{\gamma,0} \le c|u|_1$ so that, by Cauchy–Schwarz,

$$\big|\left\langle v, L_{\gamma,0}\bar{u}\right\rangle_{L^2(\mathbb{R}^n)}\big| \le \|v\|_{\gamma,0}\|u\|_{\gamma,0} \le c^2|u|_1|v|_1$$

■

Problem 3.1.4 Let $h, h' \in \mathbb{R}$ be nonzero and $\alpha, \alpha' \in \mathbb{N}_0^n$ with $|\alpha| = |\alpha'| = 1$. Prove that the operators Δ_h^α and $\Delta_{h'}^{\alpha'}$ commute.

Solution. By definition

$$\begin{aligned}
\Delta_h^\alpha\big(\Delta_{h'}^{\alpha'}u\big)(x) &= \tfrac{1}{h}\big[\big(\Delta_{h'}^{\alpha'}u\big)(x+h\alpha) - \big(\Delta_{h'}^{\alpha'}u\big)(x)\big] \\
&= \tfrac{1}{hh'}\big[u(x+h\alpha+h'\alpha') - u(x+h\alpha) - u(x+h'\alpha') + u(x)\big]
\end{aligned}$$

Exchanging $(\alpha, h) \leftrightarrow (\alpha', h')$ gives the same result. ■

Problem 3.1.5 Let $\gamma(x)$ obey Hypotheses 3.1.1 and let $\ell \in \mathbb{N}$. Prove that there is a constant C, depending only on ℓ and γ such that, for all $u \in H^1(\mathbb{R}^n)$ and $F \in H^{\ell-2}(\mathbb{R}^n)$ obeying $L_{\gamma,0}u = F$ (setting $\mu = 0$ eliminates the $\mu(x)u(x)$ term from the differential equation) we have $u \in H^\ell(\mathbb{R}^n)$ and

$$\|u\|_\ell \le C\Big(\|F\|_{\ell-2} + \|u\|_{-1}\Big) \le C\Big(\|F\|_{\ell-2} + \|u\|_1\Big)$$

Solution. Set $\mu(x) = 1$ for all $x \in \mathbb{R}^n$. Then u obeys $L_{\gamma,\mu}u = F + u$. By Proposition 3.1.7, with $\ell = 1$,

$$\|u\|_1 \le c_1\|F+u\|_{-1} \le c_1\Big(\|F\|_{-1} + \|u\|_{-1}\Big)$$

Now by Proposition 3.1.7, with $\ell = 2$, $u \in H^2(\mathbb{R}^n)$ and

$$\|u\|_2 \le c_2\|F+u\|_0 \le c_2\Big(\|F\|_0 + \|u\|_1\Big) \le c_2\|F\|_0 + c_1c_2\Big(\|F\|_{-1} + \|u\|_{-1}\Big) \le C_2\Big(\|F\|_0 + \|u\|_{-1}\Big)$$

with $C_2 = c_2 + c_1c_2$. Now just proceed by induction. ■

Problem 3.1.6 In this problem we explore the need for the $\mu(x)u(x)$ term in $L_{\gamma,\mu}u = F$.

(a) Find a $\gamma(x)$ obeying Hypotheses 3.1.1 and an $F \in H^{-1}(\mathbb{R}^n)$ such that **no** $u \in H^1(\mathbb{R}^n)$ obeys $L_{\gamma,0}u = F$.

(b) Find a $\gamma(x)$ obeying Hypotheses 3.1.1 such that there does **not** exist a constant C obeying $\|u\|_{1,n} \le C\|F\|_{-1,n}$ for all $F \in H^{-1}(\mathbb{R}^n)$ and $u \in H^1(\mathbb{R}^n)$ with $L_{\gamma,0}u = F$.

(c) Let $\gamma(x)$ obey Hypotheses 3.1.1. Prove that the map $L_{\gamma,0} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ is bounded and injective, but that it is NOT surjective. Prove that the inverse map $L_{\gamma,0}^{-1}$ (defined on the range of $L_{\gamma,0}$) is NOT bounded.

(d) Let $\gamma(x)$ obey Hypotheses 3.1.1, let $\ell \in \mathbb{N}$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Prove that there is a constant C, depending only on ℓ, γ and Ω such that, for all $F \in H^{\ell-2}(\mathbb{R}^n)$ and all $u \in H^1(\mathbb{R}^n)$ that vanish outside of Ω and obey $L_{\gamma,0}u = F$ we have $u \in H^\ell(\mathbb{R}^n)$ and

$$\|u\|_\ell \le C\|F\|_{\ell-2}$$

Solution. For parts (a) and (b), we take $\gamma(x)$ to be the identity matrix for all x. Then $L_{\gamma,0}$ is just $-\Delta$ so that $\widehat{L_{\gamma,0}u}(k) = k^2\hat{u}(k)$ and $L_{\gamma,0}u = F$ if and only if $k^2\hat{u}(k) = \hat{F}(k)$.

(a) Take $\hat{F}(k) = |k|^{-\alpha}e^{-k^2}$ with $\frac{n}{2} - 2 < \alpha < \frac{n}{2}$. As $|k|^{-2\alpha}$ is locally integrable at $k = 0$, $F \in H^s(\mathbb{R}^n)$ for all s, including $s = -1$. As $|k|^{-2\alpha-4}$ is not locally integrable at $k = 0$, $u \notin H^s(\mathbb{R}^n)$ for any s, including $s = 1$.

(b) Set $\hat{F}_j(k) = |k|^{-\frac{n}{2}+2+\frac{1}{j}} e^{-k^2}$. As $|k|^{-n+4}$ is locally integrable at $k=0$, $F_j \in H^{-1}(\mathbb{R}^n)$ for all $j \in \mathbb{N}$ and $\sup_{j\in\mathbb{N}} |\!|\!|F_j|\!|\!|_{-1}$ is finite. The solution to $L_{\gamma,0}u = F_j$ is the inverse Fourier transform of $\hat{u}_j(k) = |k|^{-\frac{n}{2}+\frac{1}{j}} e^{-k^2}$. As $|k|^{-n+\frac{2}{j}}$ is locally integrable at $k=0$, $u_j \in H^1(\mathbb{R}^n)$ for all $j \in \mathbb{N}$. But $|k|^{-n}$ is not locally integrable at $k=0$, so $\lim_{j\to\infty} |\!|\!|u_j|\!|\!|_1$ is infinite.

(c) In the solution to Problem 3.1.2, we showed that $\langle\ ,\ \rangle_{\gamma,0}$ is a well–defined inner product on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and that

$$C_1 \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \le \langle u, u\rangle_{\gamma,0} \le C_2 \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \tag{S3.1}$$

for all $u \in H^1(\mathbb{R}^n)$, though $\|\ \cdot\ \|_{\gamma,0}$ is no longer equivalent to $|\!|\!|\ \cdot\ |\!|\!|_{1,n}$. In Problem 3.1.3, we showed that there is a unique bounded linear map $L_{\gamma,0} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ such that

$$(L_{\gamma,0}u)(x) = -\sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big)$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. Then the argument of Lemma 3.1.3 still shows that

$$\mathcal{L}_{L_{\gamma,0}u}(v) = \langle v, \bar{u}\rangle_{\gamma,0}$$

for all $u, v \in H^1(\mathbb{R}^n)$.

Injectiveness: If $u \in H^1(\mathbb{R}^n)$ and $L_{\gamma,0}u = 0$, then,

$$0 = \mathcal{L}_{L_{\gamma,0}u}(\bar{u}) = \langle \bar{u}, \bar{u}\rangle_{\gamma,0} \ge C_1 \|\nabla\bar{u}\|^2_{L^2(\mathbb{R}^n)}$$

so that $\nabla\bar{u} = 0$ and hence $\nabla u = 0$. Thus $k\hat{u}(k) = 0$. Since k vanishes only on a set of measure zero, $\hat{u}$ vanishes almost everywhere and u is the zero element of $H^1(\mathbb{R}^n)$.

Unboundedness of the inverse: If the inverse of $L_{\gamma,0}$ were bounded, there would be a constant C such that $|\!|\!|u|\!|\!|_1 \le C|\!|\!|L_{\gamma,0}u|\!|\!|_{-1}$ for all $u \in H^1(\mathbb{R}^n)$. Since $\mathcal{L}$ is norm preserving, this would imply that

$$|\!|\!|u|\!|\!|_1 \le C|\!|\!|L_{\gamma,0}u|\!|\!|_{-1} = C \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\ne 0}} \frac{|\mathcal{L}_{L_{\gamma,0}u}(v)|}{|\!|\!|v|\!|\!|_1} = C \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\ne 0}} \frac{|\,\langle v, \bar{u}\rangle_{\gamma,0}\,|}{|\!|\!|v|\!|\!|_1} \le C \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\ne 0}} \frac{\|v\|_{\gamma,0}\|\bar{u}\|_{\gamma,0}}{|\!|\!|v|\!|\!|_1}$$

But, by (S3.1),

$$\|v\|_{\gamma,0}\|\bar{u}\|_{\gamma,0} \le C_2 \|\nabla v\|_{L^2(\mathbb{R}^n)} \|\nabla\bar{u}\|_{L^2(\mathbb{R}^n)} \le C_2 |\!|\!|v|\!|\!|_1 \|\nabla u\|_{L^2(\mathbb{R}^n)}$$

So boundedness of the inverse would imply the existence of a constant C such that $|\!|\!|u|\!|\!|_1 \le CC_2\|\nabla u\|_{L^2(\mathbb{R}^n)}$ for all $u \in H^1(\mathbb{R}^n)$.

As in part (b), the sequence $\{u_j\}_{j\in\mathbb{N}}$ determined by $\hat{u}_j(k) = |k|^{-\frac{n}{2}+\frac{1}{j}} e^{-k^2}$ obeys

$$\lim_{j\to\infty} \frac{|\!|\!|u_j|\!|\!|_1}{\|\nabla u_j\|_{L^2(\mathbb{R}^n)}} = \infty$$

and provides our contradiction. To see this, observe that, as $|k|^{-n+2}$ is locally integrable at $k=0$, $\nabla u_j \in L^2(\mathbb{R}^n)$ for all $j \in \mathbb{N}$ and $\sup_{j\in\mathbb{N}} \|\nabla u_j\|_{L^2(\mathbb{R}^n)}$ is finite. On the other hand, even though $|k|^{-n+\frac{2}{j}}$ is locally integrable at $k=0$, so that $u_j \in H^1(\mathbb{R}^n)$ for all $j \in \mathbb{N}$, $|k|^{-n}$ is not locally integrable at $k=0$, so $\lim_{j\to\infty} |u_j|_1$ is infinite.

Nonsurjectiveness: If $L_{\gamma,0}$ were surjective, it would be a continuous bijection of the Banach space $H^1(\mathbb{R}^n)$ onto the Banach space $H^{-1}(\mathbb{R}^n)$. The inverse mapping theorem [RS, Theorem III.11] would then imply that the inverse of $L_{\gamma,0}$ is continuous, which we already know not to be the case.

(d) By Problem 3.1.5, we already know that $u \in H^\ell(\mathbb{R}^n)$ and $|u|_\ell \le C\big(|F|_{\ell-2} + |u|_1\big)$. So it suffices to prove that $|u|_1$ is bounded by a constant times $|F|_{-1}$.

We now show that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \le \tfrac{1}{C_1}|u|_1 |L_{\gamma,0}u|_{-1} \tag{S3.2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where C_1 is the constant of Problem 3.1.1. Since, by Problem 2.1.8, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, and since $L_{\gamma,0}$ is a bounded map from $H^1(\mathbb{R}^n)$ to $H^{-1}(\mathbb{R}^n)$ and ∇ is a bounded map from $H^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, this will imply (S3.2) for all $u \in H^1(\mathbb{R}^n)$. If $u \in C_0^\infty(\mathbb{R}^n)$, then by Problem 3.1.1, followed by the divergence theorem

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\mathbb{R}^n)} &\le \tfrac{1}{C_1}\sum_{i,j=1}^n \int_{\mathbb{R}^n} \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^nx \\
&= -\tfrac{1}{C_1}\int_{\mathbb{R}^n} u(x)\Big[\sum_{i,j=1}^n \overline{\tfrac{\partial}{\partial x_i}\big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\big)}\Big]\ d^nx \\
&= \tfrac{1}{C_1}\ \langle u, L_{\gamma,0}u\rangle_{L^2(\mathbb{R}^n)} \\
&\le \tfrac{1}{C_1}|u|_1|L_{\gamma,0}u|_{-1} \qquad \text{(by part (a) of Proposition 2.1.10)}
\end{aligned}$$

By Poincaré's inequality, Proposition 2.3.7, applied in a sufficiently large region that the boundary contribution $Ru = 0$,

$$\|u\|^2_{L^2(\mathbb{R}^n)} \le C_\Omega \|\nabla u\|^2_{L^2(\mathbb{R}^n)}$$

Hence

$$|u|_1^2 = \|u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \le (1+C_\Omega)\tfrac{1}{C_1}|u|_1|L_{\gamma,0}u|_{-1} = (1+C_\Omega)\tfrac{1}{C_1}|u|_1|F|_{-1}$$

so that

$$|u|_1 \le (1+C_\Omega)\tfrac{1}{C_1}|F|_{-1}$$

■

Problem 3.2.1 Let $\gamma(x)$ obey Hypotheses 3.1.1 and let $\ell \in \mathbb{N}$.

(a) Prove that there is a constant C, depending only on ℓ and γ such that for all $u \in H^1(\mathbb{R}^n_+)$, $F \in H^{\ell-2}(\mathbb{R}^n_+)$ and $f \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$ obeying

$$L_{\gamma,0,+}u = F \qquad Ru = f$$

we have $u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|u\|_{\ell,\mathbb{R}^n_+} \le C\Big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1} + \|u\|_{-1,\mathbb{R}^n_+}\Big)$$

(b) Let Ω be a bounded open subset of $\mathbb{R}^n$. Prove that there is a constant C, depending only on ℓ, γ and Ω such that, for all $F \in H^{\ell-2}(\mathbb{R}^n_+)$, all $f \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1}_+)$ and all $u \in H^1(\mathbb{R}^n_+)$ that vanish outside of Ω and obey

$$L_{\gamma,0,+}u = F \qquad Ru = f$$

we have $u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|u\|_{\ell,\mathbb{R}^n_+} \le C\Big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1}\Big)$$

Solution. (a) Set $\mu(x) = 1$ for all $x \in \mathbb{R}^n$. Then u obeys $L_{\gamma,\mu,+}u = F + u$. By Proposition 3.2.7, with $\ell = 1$,

$$\|u\|_{1,\mathbb{R}^n_+} \le c_1\Big(\|F+u\|_{-1,\mathbb{R}^n_+} + |f|_{\frac{1}{2},n-1}\Big) \le c_1\Big(\|F\|_{-1,\mathbb{R}^n_+} + |f|_{\frac{1}{2},n-1} + \|u\|_{-1,\mathbb{R}^n_+}\Big)$$

Now by Proposition 3.2.7, with $\ell = 2$, $u \in H^2(\mathbb{R}^n_+)$ and

$$\begin{aligned}\|u\|_{2,\mathbb{R}^n_+} &\le c_2\big(\|F+u\|_{0,\mathbb{R}^n_+} + |f|_{\frac{3}{2},n-1}\big) \le c_2\big(\|F\|_{0,\mathbb{R}^n_+} + |f|_{\frac{3}{2},n-1} + \|u\|_{1,\mathbb{R}^n_+}\big)\\ &\le c_2\big(\|F\|_{0,\mathbb{R}^n_+} + |f|_{\frac{3}{2},n-1}\big) + c_1c_2\Big(\|F\|_{-1,\mathbb{R}^n_+} + |f|_{\frac{1}{2},n-1} + \|u\|_{-1,\mathbb{R}^n_+}\Big)\\ &\le C_2\Big(\|F\|_{0,\mathbb{R}^n_+} + |f|_{\frac{3}{2},n-1} + \|u\|_{-1,\mathbb{R}^n_+}\Big)\end{aligned}$$

with $C_2 = c_2 + c_1c_2$. Proceed by induction.

(b) We already know that $u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|u\|_{\ell,\mathbb{R}^n_+} \le C\big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1} + \|u\|_{-1,\mathbb{R}^n_+}\big)$$

by part (a). So it suffices to prove that $\|u\|_{-1,\mathbb{R}^n_+} \le \|u\|_{1,\mathbb{R}^n_+}$ is bounded by a constant times $\|F\|_{-1,\mathbb{R}^n_+} + |f|_{\frac{1}{2},n-1}$.

We first consider the case $f = 0$. By Proposition 2.2.11, $u \in H_0^1(\mathbb{R}^n_+)$. We shall show that

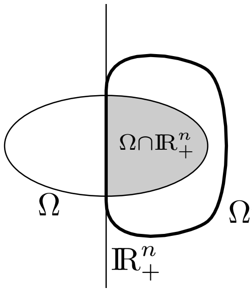
$$\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)} \le \tfrac{1}{C_1}\|u\|_{1,\mathbb{R}^n_+}\|L_{\gamma,0,+}u\|_{-1,\mathbb{R}^n_+} \tag{S3.3}$$

for all $u \in C_0^\infty(\mathbb{R}^n_+)$. Here C_1 is the constant of Problem 3.1.1. Since, by definition, $C_0^\infty(\mathbb{R}^n_+)$ is dense in $H_0^1(\mathbb{R}^n_+)$, and since $L_{\gamma,0,+}$ is a bounded map from $H^1(\mathbb{R}^n_+)$ to $H^{-1}(\mathbb{R}^n_+)$ (even though $\mu = 0$) and ∇ is a bounded map from $H^1(\mathbb{R}^n_+)$ to $L^2(\mathbb{R}^n_+)$, this will imply (S3.3) for all $u \in H_0^1(\mathbb{R}^n_+)$. If $u \in C_0^\infty(\mathbb{R}^n_+)$, then by Problem 3.1.1, followed by the divergence theorem

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)} &\le \tfrac{1}{C_1}\sum_{i,j=1}^n \int_{\mathbb{R}^n_+} \gamma^{i,j}(x)\overline{\tfrac{\partial u}{\partial x_i}(x)}\tfrac{\partial u}{\partial x_j}(x)\ d^nx \\
&= -\tfrac{1}{C_1}\int_{\mathbb{R}^n_+} \overline{u(x)}\Big[\sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\big)\Big]\ d^nx \\
&= \tfrac{1}{C_1}(L_{\gamma,0,+}u)(\bar u) \le \tfrac{1}{C_1}\|L_{\gamma,0,+}u\|_{-1,\mathbb{R}^n_+}\|\bar u\|_{1,\mathbb{R}^n_+} \\
&= \tfrac{1}{C_1}\|L_{\gamma,0,+}u\|_{-1,\mathbb{R}^n_+}\|u\|_{1,\mathbb{R}^n_+}
\end{aligned}$$

By Poincaré's inequality, Proposition 2.3.7, applied in a region Ω' like that in the figure below

$$\|u\|^2_{L^2(\mathbb{R}^n_+)} \le C_\Omega\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)}$$



Hence

$$\begin{aligned}
\|u\|^2_{1,\mathbb{R}^n_+} &= \|u\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n_+)} \le (1+C_\Omega)\tfrac{1}{C_1}\|u\|_{1,\mathbb{R}^n_+}\|L_{\gamma,0,+}u\|_{-1,\mathbb{R}^n_+} \\
&= (1+C_\Omega)\tfrac{1}{C_1}\|u\|_{1,\mathbb{R}^n_+}\|F\|_{-1,\mathbb{R}^n_+}
\end{aligned}$$

so that

$$\|u\|_{1,\mathbb{R}^n_+} \le (1+C_\Omega)\tfrac{1}{C_1}\|F\|_{-1,\mathbb{R}^n_+}$$

which is the desired bound when $f = 0$.

Now we consider general $f \in H^{1/2}(\mathbb{R}^{n-1})$. By part (iv) of Proposition 2.2.10 with $\ell = 1$, there is a $u_f \in H^1(\mathbb{R}^n_+)$ such that $Ru_f = f$ and $\|u_f\|_{1,\mathbb{R}^n_+} \le C'|f|_{1/2,n-1}$. Then $v = u - u_f$ obeys

$$L_{\gamma,0,+}v = F - L_{\gamma,0,+}u_f \qquad Rv = Ru - Ru_f = 0$$

By the $f = 0$ bound just proven

$$\|v\|_{1,\mathbb{R}^n_+} \le (1 + C_\Omega)\tfrac{1}{C_1}\|F - L_{\gamma,0,+}u_f\|_{-1,\mathbb{R}^n_+}$$

so that, using L to denote the operator norm of $L_{\gamma,0,+}$ as a map from $H^1(\mathbb{R}^n_+)$ to $H^{-1}(\mathbb{R}^n_+)$,

$$\begin{aligned}\|u\|_{1,\mathbb{R}^n_+} &\le \|v\|_{1,\mathbb{R}^n_+} + \|u_f\|_{1,\mathbb{R}^n_+}\\ &\le (1 + C_\Omega)\tfrac{1}{C_1}\|F\|_{-1,\mathbb{R}^n_+} + (1 + C_\Omega)\tfrac{1}{C_1}\mathrm{L}\|u_f\|_{1,\mathbb{R}^n_+} + \|u_f\|_{1,\mathbb{R}^n_+}\\ &\le (1 + C_\Omega)\tfrac{1}{C_1}\|F\|_{-1,\mathbb{R}^n_+} + (1 + C_\Omega)\tfrac{1}{C_1}\mathrm{L}C'\|f\|_{\frac{1}{2},n-1} + C'\|f\|_{\frac{1}{2},n-1}\end{aligned}$$

as desired. ■

Problem 3.3.1 Prove that $\langle u, u\rangle_{\gamma,\Omega} = 0$ if and only if $u = 0$.

Solution. That $u = 0$ implies $\langle u, u\rangle_{\gamma,\Omega} = 0$ is obvious, so we just prove the converse. Assume that $\langle u, u\rangle_{\gamma,\Omega} = 0$. By (3.1.2)

$$\begin{aligned}0 = \langle u, u\rangle_{\gamma,\Omega} &= \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^n x + \int_{\partial\Omega} |u(x)|^2\ d^{n-1}\sigma\\ &\ge C_1 \int_\Omega \left|\nabla u(x)\right|^2\ d^n x + \int_{\partial\Omega} |u(x)|^2\ d^{n-1}\sigma\end{aligned}$$

This implies that both $\int_\Omega \left|\nabla u(x)\right|^2\ d^n x$ and $\int_{\partial\Omega} |u(x)|^2\ d^{n-1}\sigma$ are zero. Then Proposition 2.3.7 (Poincaré's inequality) implies that $\|u\|_{L^2(\Omega)} = 0$ and hence $u = 0$. ■

Problem 3.3.2 Prove that, for each $\ell \in \mathbb{N}$, L_γ is bounded as a map from $H^\ell(\Omega)$ to $H^{\ell-2}(\Omega)$.

Solution. The case $\ell = 1$ has been dealt with in Lemma 3.3.4, so consider $\ell \ge 2$. It suffices to combine the observations that

- $\frac{\partial}{\partial x_i}$ is a bounded map from $H^\ell(\Omega)$ to $H^{\ell-1}(\Omega)$ for all $1 \le i \le n$ and all $\ell \ge 2$
- By part (a) of Problem 2.1.3, multiplication by each $\gamma^{i,j}$ is a bounded map on $H^{\ell-1}(\Omega)$ for each $\ell \ge 1$.
- $\frac{\partial}{\partial x_i}$ is a bounded map from $H^{\ell-1}(\Omega)$ to $H^{\ell-2}(\Omega)$ for all $1 \le i \le n$ and all $\ell \ge 2$ ■

Problem 3.3.3 Let $\mathcal{B}$ and $\mathcal{B}'$ be Banach spaces and let the map

$$f : \mathcal{D} \subset \mathcal{B} \to \mathcal{B}'$$

be analytic at $x \in \mathcal{D}$.

(a) Prove that there are constants C and δ such that

$$\|f(x') - f(x)\|_{\mathcal{B}'} \le C\|x' - x\|_{\mathcal{B}}$$

for all $x' \in \mathcal{D}$ with $\|x' - x\|_{\mathcal{B}} < \delta$. In particular, f is continuous at x.

(b) Prove that there exists a bounded linear map $D : \mathcal{B} \to \mathcal{B}'$ such that

- for any curve $x(t)$ that is defined for t in a neighbourhood of 0, takes values in $\mathcal{D}$, obeys $x(0) = x$ and is differentiable in the sense that there is an $X \in \mathcal{B}$ such that

$$\lim_{t\to 0}\big\|\tfrac{1}{t}[x(t) - x] - X\big\|_{\mathcal{B}} = 0$$

we have

$$\lim_{t\to 0}\big\|\tfrac{1}{t}[f\big(x(t)\big) - f(x)] - DX\big\|_{\mathcal{B}'} = 0$$

The linear map D is called the Fréchet derivative of f at x.

Solution. (a) We set $z = x' - x$ and use the notation of Definition 3.3.7. Since

$$f(x') - f(x) = f(x + z) - f(x) = \sum_{\ell=1}^{\infty} F_\ell(z, \cdots, z)$$

we have

$$\|f(x') - f(x)\|_{\mathcal{B}'} \le \sum_{\ell=1}^{\infty} \|F_\ell(z, \cdots, z)\|_{\mathcal{B}'} \le \sum_{\ell=1}^{\infty} Cc^\ell \|z\|_{\mathcal{B}}^\ell = Cc\tfrac{\|z\|_{\mathcal{B}}}{1-c\|z\|_{\mathcal{B}}}$$

so it suffices to take $\delta = \frac{1}{2c}$ and rename $2Cc$ back to C.

(b) We set $z(t) = x(t) - x$ and use the notation of Definition 3.3.7. Then $Dy \equiv F_1(y)$ is a bounded linear map from $\mathcal{B}$ to $\mathcal{B}'$ and

$$\begin{aligned}\tfrac{1}{t}\big[f\big(x(t)\big) - f(x)\big] - DX &= \tfrac{1}{t}\big[f\big((x + z(t)\big) - f(x)\big] - F_1(X)\\ &= F_1\big(\tfrac{1}{t}z(t) - X\big) + \sum_{\ell=2}^{\infty} \tfrac{1}{t}F_\ell\big(z(t), \cdots, z(t)\big)\end{aligned}$$

so that

$$\begin{aligned}\big\|\tfrac{1}{t}\big[f\big(x(t)\big) - f(x)\big] - DX\big\|_{\mathcal{B}'} &\le Cc\big\|\tfrac{1}{t}z(t) - X\big\|_{\mathcal{B}} + \sum_{\ell=2}^{\infty} Cc^\ell \tfrac{1}{t}\|z(t)\|_{\mathcal{B}}^\ell\\ &= Cc\big\|\tfrac{1}{t}z(t) - X\big\|_{\mathcal{B}} + Cc^2\tfrac{\|z(t)\|_{\mathcal{B}}^2}{t}\tfrac{1}{1-c\|z(t)\|_{\mathcal{B}}}\end{aligned} \tag{S3.4}$$

By hypothesis $\frac{1}{t}z(t)$ converges to X in $\mathcal{B}$ as $t \to 0$. In particular $\frac{1}{t}\|z(t)\|_{\mathcal{B}}$ remains bounded and $\|z(t)\|$ converges to zero as $t \to \infty$. So the right hand side of (S3.4) converges to zero in the limit $t \to 0$, as desired. ■

Problem 3.3.4 Let $m \in \mathbb{N}$, $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}_1, \cdots, \mathcal{B}_m$ be Banach spaces and let, for each $1 \le i \le m$, the map

$$f_i : \mathcal{D} \subset \mathcal{B} \to \mathcal{B}_i$$

be analytic at $x \in \mathcal{D}$. Let $M : \mathcal{B}_1 \times \cdots \mathcal{B}_m \to \mathcal{B}'$ be a bounded m–linear map. Prove that

$$\begin{aligned} f : \mathcal{D} \subset \mathcal{B} &\to \mathcal{B}' \\ x' &\mapsto M\big(f_1(x'), \cdots f_m(x')\big) \end{aligned}$$

is analytic at x.

Solution. For each $1 \le i \le m$, let

$$f_i(x+z) = \sum_{\ell=0}^{\infty} F_\ell^{(i)}(z, \cdots, z)$$

for all $z \in \mathcal{B}$ obeying $\|z\|_{\mathcal{B}} < \frac{1}{c}$ and $x + z \in D$ and let

$$\|F_\ell^{(i)}(z_1, \cdots, z_\ell)\|_{\mathcal{B}_i} \le C_i c_i^\ell \|z_1\|_{\mathcal{B}} \cdots \|z_\ell\|_{\mathcal{B}}$$

for all $z_1, \cdots, z_\ell \in \mathcal{B}$. Define, for each $\ell \in \mathbb{N}_0$,

$$\Phi_\ell(z_1, \cdots, z_\ell) = \sum_{\substack{\ell_1, \cdots, \ell_m \in \mathbb{N}_0 \\ \ell_1 + \cdots + \ell_m = \ell}} M\Big(F_{\ell_1}^{(1)}(z_1, \cdots, z_{\ell_1}), \cdots, F_{\ell_m}^{(m)}(z_{\ell-\ell_m+1}, z_\ell)\Big)$$

If the norm of M is bounded by C_M, then

$$\begin{aligned} \big\|\Phi_\ell(z_1, \cdots, z_\ell)\big\|_{\mathcal{B}'} &\le \sum_{\substack{\ell_1, \cdots, \ell_m \in \mathbb{N}_0 \\ \ell_1 + \cdots + \ell_m = \ell}} C_M \,\big\|F_{\ell_1}^{(1)}(z_1, \cdots, z_{\ell_1})\big\|_{\mathcal{B}_1} \cdots \big\|F_{\ell_m}^{(m)}(z_{\ell-\ell_m+1}, z_\ell)\big\|_{\mathcal{B}_\ell} \\ &\le \sum_{\substack{\ell_1, \cdots, \ell_m \in \mathbb{N}_0 \\ \ell_1 + \cdots + \ell_m = \ell}} C_M \, C_1 c_1^{\ell_1} \cdots C_m c_m^{\ell_m} \|z_1\|_{\mathcal{B}} \cdots \|z_\ell\|_{\mathcal{B}} \\ &\le C_M \, C_1 \cdots C_m \big(c_1 + \cdots + c_m\big)^\ell \, \|z_1\|_{\mathcal{B}} \cdots \|z_\ell\|_{\mathcal{B}} \end{aligned}$$

So every Φ_ℓ satisfies the prescribed bound. That the Φ_ℓ's are multilinear and

$$f(x+z) = M\big(f_1(x+z), \cdots f_m(x+z)\big) = \sum_{\ell=0}^{\infty} \Phi_\ell(z, \cdots, z)$$

are obvious. ■

Problem 3.3.5 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let γ satisfy Hypothesis 3.3.1, $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\Omega)$. Define, for any sequence $\{\gamma_m\}_{m\in\mathbb{N}} \subset L^\infty(\Omega)$, the sequence $\{U_\ell(\gamma_1,\cdots,\gamma_\ell)\}_{\ell\in\mathbb{N}_0} \subset H^1(\Omega)$ inductively by

$$\begin{aligned} U_0 &= u\big(\gamma,(F,f)\big) \\ L_\gamma U_\ell(\gamma_1,\cdots,\gamma_\ell) &= -\nabla\cdot\gamma_\ell\nabla U_{\ell-1}(\gamma_1,\cdots,\gamma_{\ell-1}) \qquad RU_\ell(\gamma_1,\cdots,\gamma_\ell)=0 \qquad \text{for } \ell\in\mathbb{N} \end{aligned}$$

(a) Prove that $U_\ell(\gamma_1,\cdots,\gamma_\ell)$ is linear in in each of $\gamma_1,\ \cdots,\ \gamma_\ell$ and that there are constants C and c that depend only on n, Ω and γ such that

$$\big\|U_\ell(\gamma_1,\cdots,\gamma_\ell)\big\|_{1,\Omega} \le Cc^\ell\big\|\gamma_1\big\|_{L^\infty(\Omega)}\cdots\big\|\gamma_\ell\big\|_{L^\infty(\Omega)}\ \big\|(F,f)\big\|_{H^{-1}(\Omega)\oplus H^{1/2}(\Omega)}$$

(b) Prove that if $\gamma'\in L^\infty(\Omega)$ obeys $\|\gamma'\|_{L^\infty(\Omega)} < \frac{1}{c}$, then

$$u' = \sum_{\ell=0}^{\infty} U_\ell\big(\gamma',\cdots,\gamma'\big)$$

converges in $H^1(\Omega)$ and obeys (3.3.1).

Solution. (a) By Theorem 3.3.5 and Lemma 2.1.16 the unique solution, V, to

$$L_\gamma V = -\nabla\cdot\gamma'\nabla U \qquad RV=0 \tag{S3.5}$$

obeys

$$\begin{aligned} \|V\|_{1,\Omega} &\le c_1\|\nabla\cdot\gamma'\nabla U\|_{-1,\Omega} \\ &\le c_1c_2\|\gamma'\nabla U\|_{L^2(\Omega)} \\ &\le c_1c_2c_3\|\gamma'\|_{L^\infty(\Omega)}\ \|\nabla U\|_{L^2(\Omega)} \\ &\le c_1c_2c_3c_4\|\gamma'\|_{L^\infty(\Omega)}\ \|U\|_{1,\Omega} \end{aligned}$$

In particular, setting $c=c_1c_2c_3c_4$,

$$\big\|U_\ell(\gamma_1,\cdots,\gamma_\ell)\big\|_{1,\Omega} \le c\big\|\gamma_\ell\big\|_{L^\infty(\Omega)}\ \big\|U_{\ell-1}(\gamma_1,\cdots,\gamma_{\ell-1})\big\|_{1,\Omega}$$

So, by induction,

$$\big\|U_\ell(\gamma_1,\cdots,\gamma_\ell)\big\|_{1,\Omega} \le c^\ell\big\|\gamma_1\big\|_{L^\infty(\Omega)}\cdots\big\|\gamma_\ell\big\|_{L^\infty(\Omega)}\ \big\|U_0\big\|_{1,\Omega}$$

By Theorem 3.3.5,

$$\big\|U_0\big\|_{1,\Omega} = \big\|u\big(\gamma,(F,f)\big)\big\|_{1,\Omega} \le C\big\|(F,f)\big\|_{H^{-1}(\Omega)\oplus H^{1/2}(\Omega)}$$

which proves the desired bound. The multilinearity follows by induction from the observation that the V of (S3.5) depends linearly on γ' and on U.

(b) Since

$$\big\|U_\ell(\gamma',\cdots,\gamma')\big\|_{1,\Omega} \le Cc^\ell \big\|\gamma'\big\|_{L^\infty(\Omega)}^\ell \ \big\|(F,f)\big\|_{H^{-1}(\Omega)\oplus H^{1/2}(\Omega)}$$

the series

$$u' = \sum_{\ell=0}^{\infty} U_\ell\big(\gamma',\cdots,\gamma'\big)$$

converges in $H^1(\Omega)$ for all $c\big\|\gamma'\big\|_{L^\infty(\Omega)} < 1$. Set

$$u_\ell = \sum_{m=0}^{\ell} U_m\big(\gamma',\cdots,\gamma'\big)$$

Then

$$L_\gamma(u_\ell - u_{\ell-1}) = -\nabla\cdot\gamma'\nabla(u_{\ell-1}-u_{\ell-2}) \qquad R(u_\ell - u_{\ell-1}) = 0$$

for $\ell \ge 1$, with $u_{-1}=0$, and

$$L_\gamma u_0 = F \qquad Ru_0 = f$$

Adding up,

$$L_\gamma u_\ell = F - \nabla\cdot\gamma'\nabla u_{\ell-1} \qquad Ru_\ell = f$$

Taking the limits $\ell\to\infty$ gives

$$L_\gamma u' = F - \nabla\cdot\gamma'\nabla u' \qquad Ru' = f$$

as desired. ■

Problem 3.3.6 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let γ satisfy Hypothesis 3.3.1. Prove that there is a constant $\delta > 0$ such that, for each $F \in H^{-1}(\Omega)$, $f \in H^{1/2}(\Omega)$ and $\gamma' \in L^\infty(\Omega)$ with $\|\gamma'\|_{L^\infty(\Omega)} < \delta$, there is a unique $u \in H^1(\Omega)$ obeying

$$L_\gamma u = F - \nabla\cdot\gamma'\nabla u \qquad Ru = f$$

Solution. The existence of a solution has been proven in part (b) of Problem 3.3.5, so it suffice to prove uniqueness. To do so, it suffices to prove that if $u \in H^1(\Omega)$ obeys

$$L_\gamma u = -\nabla\cdot\gamma'\nabla u \qquad Ru = 0$$

then $u = 0$. By Theorem 3.3.5 followed by Lemma 2.1.16,

$$\begin{aligned}\|u\|_{1,\Omega} &\le C\|\nabla\cdot\gamma'\nabla u\|_{-1,\Omega}\\ &\le Cc_1\|\gamma'\nabla u\|_{L^2(\Omega)}\\ &\le Cc_1c_2\|\gamma'\|_{L^\infty(\Omega)}\ \|\nabla u\|_{L^2(\Omega)}\\ &\le Cc_1c_2c_3\|\gamma'\|_{L^\infty(\Omega)}\ \|u\|_{1,\Omega}\end{aligned}$$

As long as $\delta < \frac{1}{Cc_1c_2c_3}$ and $\|\gamma'\|_{L^\infty(\Omega)} < \delta$, this forces $\|u\|_{1,\Omega} = 0$. ■

Problem 3.3.7 Let, for $i = 1, 2$, γ_i obey Hypothesis 3.3.1, $F_i \in H^{-1}(\Omega)$, $f_i \in H^{1/2}(\partial\Omega)$ and u_i be the solution of

$$L_{\gamma_i}u_i = F_i \qquad u_i\big|_{\partial\Omega} = f_i$$

Prove that there are constants C and $\varepsilon > 0$, depending only on n, Ω and γ_1, such that, if $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} < \varepsilon$, then

$$\|u_1 - u_2\|_{1,\Omega} \le C\Big[\|F_1 - F_2\|_{-1,\Omega} + \|f_1 - f_2\|_{\frac12,\partial\Omega} + \big(\|F_1\|_{-1,\Omega} + \|f_1\|_{\frac12,\Omega}\big)\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}\Big]$$

Solution. This is a fairly immediate consequence of Problems 3.3.5 and 3.3.3. Here is an alternative proof that does not use those Problems. Since $(u_1 - u_2)\big|_{\partial\Omega} = f_1 - f_2$ and

$$\begin{aligned}L_{\gamma_1}(u_1 - u_2) &= L_{\gamma_1}u_1 - L_{\gamma_2}u_2 + \nabla\cdot(\gamma_2 - \gamma_1)\nabla u_2\\ &= F_1 - F_2 + \nabla\cdot(\gamma_2 - \gamma_1)\nabla u_2\end{aligned}$$

there is, by Theorem 3.3.5, a constant C_1 depending only on n, Ω and γ_1 such that

$$\|u_1 - u_2\|_{1,\Omega} \le C_1\Big[\|F_1 - F_2 + \nabla\cdot(\gamma_2 - \gamma_1)\nabla u_2\|_{-1,\Omega} + \|f_1 - f_2\|_{\frac12,\Omega}\Big]$$

By Lemma 2.1.16

$$\begin{aligned}\|\nabla\cdot(\gamma_2 - \gamma_1)\nabla u_2\|_{-1,\Omega} &\le C_2\|(\gamma_2 - \gamma_1)\nabla u_2\|_{L^2(\Omega)} \le C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|\nabla u_2\|_{L^2(\Omega)}\\ &\le C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|u_2\|_{1,\Omega}\\ &\le C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|u_1\|_{1,\Omega} + C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|u_1 - u_2\|_{1,\Omega}\end{aligned}$$

So far, we have

$$\begin{aligned}\|u_1 - u_2\|_{1,\Omega} \le C_1\Big[&\|F_1 - F_2\|_{-1,\Omega} + \|f_1 - f_2\|_{\frac12,\Omega} + C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|u_1\|_{1,\Omega}\Big]\\ &+ C_1C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|u_1 - u_2\|_{1,\Omega}\end{aligned}$$

Choosing ε sufficiently small that $\varepsilon C_1C_2 \le \frac12$,

$$\|u_1 - u_2\|_{1,\Omega} \le 2C_1\Big[\|F_1 - F_2\|_{-1,\Omega} + \|f_1 - f_2\|_{\frac12,\Omega} + C_2\|\gamma_2 - \gamma_1\|_{L^\infty(\Omega)}\ \|u_1\|_{1,\Omega}\Big]$$

Now it suffices to apply Theorem 3.3.5, once more, to obtain

$$\|u_1\|_{1,\Omega} \le C_1\Big[\|F_1\|_{-1,\Omega} + \|f_1\|_{\frac12,\Omega}\Big]$$

■

Problem 3.3.8 Let, for each t in a neighbourhood of 0, γ_t obey Hypothesis 3.3.1, $F_t \in H^{-1}(\Omega)$ and $f_t \in H^{1/2}(\partial\Omega)$. Assume that γ_t is differentiable with respect to t at $t = 0$ in $L^\infty(\Omega)$. In other words, assume that there is a $\overset{\bullet}{\gamma}(x) = \big[\overset{\bullet}{\gamma}{}^{i,j}(x)\big]_{1\le i,j\le n} \in L^\infty(\Omega)$ such that

$$\lim_{t\to 0}\big\|\overset{\bullet}{\gamma} - \tfrac{1}{t}(\gamma_t - \gamma_0)\big\|_{L^\infty(\Omega)} = 0$$

Similarly, assume that F_t and f_t are differentiable with respect to t at $t = 0$ in $H^{-1}(\Omega)$ and $H^{1/2}(\partial\Omega)$, respectively. Let u_t be the solution of

$$L_{\gamma_t} u_t = F_t \qquad u_t\big|_{\partial\Omega} = f_t$$

Prove that u_t differentiable with respect to t at $t = 0$ in $H^1(\Omega)$ and that

$$L_{\gamma_0}\overset{\bullet}{u} = \overset{\bullet}{F} - \nabla\cdot\overset{\bullet}{\gamma}\nabla u_0 \qquad \overset{\bullet}{u}\big|_{\partial\Omega} = \overset{\bullet}{f}$$

Here $\overset{\bullet}{X} = \lim_{t\to 0}\frac{1}{t}(X_t - X_0)$ in $H^1(\Omega)$ when $X = u$, in $H^{-1}(\Omega)$ when $X = F$ and in $H^{1/2}(\partial\Omega)$ when $X = f$.

Solution. This is also a fairly immediate consequence of Problems 3.3.5 and 3.3.3. Here is an alternative, direct, proof. By Theorem 3.3.5, $u_t \in H^1(\Omega)$. Since, by hypothesis, $\overset{\bullet}{\gamma} \in L^\infty(\Omega)$,

$$u_0 \in H^1(\Omega) \implies \nabla u_0 \in H^0(\Omega) = L^2(\Omega) \implies \overset{\bullet}{\gamma}\nabla u_0 \in H^0(\Omega) \implies \nabla\cdot\overset{\bullet}{\gamma}\nabla u_0 \in H^{-1}(\Omega)$$

by Lemma 2.1.16. Hence, by Theorem 3.3.5,

$$L_{\gamma_0}\overset{\bullet}{u} = \overset{\bullet}{F} - \nabla\cdot\overset{\bullet}{\gamma}\nabla u_0 \qquad \overset{\bullet}{u}\big|_{\partial\Omega} = \overset{\bullet}{f}$$

has a unique solution $\overset{\bullet}{u} \in H^1(\Omega)$. So it suffices to prove that $\delta_t = \overset{\bullet}{u} - \frac{1}{t}(u_t - u_0)$ converges to zero in $H^1(\Omega)$ as t tends to zero. Since

$$\begin{aligned} L_{\gamma_0}\delta_t &= L_{\gamma_0}\overset{\bullet}{u} - \tfrac{1}{t}(L_{\gamma_0}u_t - L_{\gamma_0}u_0) \\ &= \overset{\bullet}{F} - \nabla\cdot\overset{\bullet}{\gamma}\nabla u_0 - \tfrac{1}{t}(L_{\gamma_0}u_t - F_0) \\ &= \overset{\bullet}{F} - \tfrac{1}{t}(F_t - F_0) - \nabla\cdot\overset{\bullet}{\gamma}\nabla u_0 - \tfrac{1}{t}(L_{\gamma_0}u_t - L_{\gamma_t}u_t) \end{aligned}$$

and

$$\delta_t\big|_{\partial\Omega} = \overset{\bullet}{f} - \tfrac{1}{t}(f_t - f_0)$$

we have, by Theorem 3.3.5, that

$$\|\delta_t\|_{1,\Omega}^2 \le C\Big[\big\|\overset{\bullet}{F}-\tfrac{1}{t}(F_t-F_0)-\nabla\cdot\overset{\bullet}{\gamma}\nabla u_0-\tfrac{1}{t}(L_{\gamma_0}u_t-L_{\gamma_t}u_t)\big\|_{-1,\Omega}^2+\big\|\overset{\bullet}{f}-\tfrac{1}{t}(f_t-f_0)\big\|_{-1/2,\partial\Omega}^2\Big]$$

By hypothesis,

$$\lim_{t\to 0}\big\|\overset{\bullet}{F}-\tfrac{1}{t}(F_t-F_0)\big\|_{-1,\Omega}=\lim_{t\to 0}\big\|\overset{\bullet}{f}-\tfrac{1}{t}(f_t-f_0)\big\|_{-1/2,\partial\Omega}=0$$

so it suffices to prove that

$$\nabla\cdot\overset{\bullet}{\gamma}\nabla u_0+\tfrac{1}{t}(L_{\gamma_0}u_t-L_{\gamma_t}u_t)=\nabla\cdot\overset{\bullet}{\gamma}\nabla(u_0-u_t)+\nabla\cdot\big[\overset{\bullet}{\gamma}-\tfrac{1}{t}(\gamma_t-\gamma_0)\big]\nabla u_t$$

converges to converges to zero in $H^{-1}(\Omega)$ as t tends to zero. But this follows from

$$\begin{aligned}
\big\|\nabla\cdot\overset{\bullet}{\gamma}\nabla(u_0-u_t)\big\|_{-1,\Omega} &\le C\big\|\overset{\bullet}{\gamma}\nabla(u_0-u_t)\big\|_{L^2(\Omega)}\le C\big\|\overset{\bullet}{\gamma}\big\|_{L^\infty(\Omega)}\,\big\|\nabla(u_0-u_t)\big\|_{L^2(\Omega)}\\
&\le C\big\|\overset{\bullet}{\gamma}\big\|_{L^\infty(\Omega)}\,\big\|u_0-u_t\big\|_{1,\Omega}\\
\big\|\nabla\cdot\big[\overset{\bullet}{\gamma}-\tfrac{1}{t}(\gamma_t-\gamma_0)\big]\nabla u_t\big\|_{-1,\Omega} &\le C\big\|\big[\overset{\bullet}{\gamma}-\tfrac{1}{t}(\gamma_t-\gamma_0)\big]\nabla u_t\big\|_{L^2(\Omega)}\\
&\le C\big\|\overset{\bullet}{\gamma}-\tfrac{1}{t}(\gamma_t-\gamma_0)\big\|_{L^\infty(\Omega)}\,\big\|\nabla u_t\big\|_{L^2(\Omega)}\\
&\le C\big\|\overset{\bullet}{\gamma}-\tfrac{1}{t}(\gamma_t-\gamma_0)\big\|_{L^\infty(\Omega)}\,\big\|u_t\big\|_{1,\Omega}
\end{aligned}$$

the hypothesis that $\overset{\bullet}{\gamma}-\frac{1}{t}(\gamma_t-\gamma_0)$ converges to zero in $L^\infty(\Omega)$ and the fact, from Problem 3.3.7, that u_t converges to u_0 in $H^1(\Omega)$. ■

Problem 3.3.9 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let γ obey Hypothesis 3.3.1. Prove that there is a γ' obeying Hypotheses 3.1.1 that coincides with γ on $\overline{\Omega}$.

Solution. The first step is to construct, for each $x\in\mathbb{R}^n$, a symmetric real matrix $\tilde\gamma(x)=\big[\tilde\gamma^{i,j}(x)\big]_{1\le i,j\le n}$ such that $\tilde\gamma(x)=\gamma(x)$ for all $x\in\overline{\Omega}$ and such that $\tilde\gamma^{i,j}\in C^\infty(\mathbb{R}^n)$ for all $1\le i,j\le n$. Let $\big(\mathcal{U}(p),\psi_p\big)$ be a coordinate system as in Notation 2.1.18 and let $\chi_\ell\in C_0^\infty\big(\mathcal{U}(p_\ell)\big)$, $1\le\ell\le N$ be a partition of unity as in Definition 2.1.20. Then for each $1\le\ell\le N$ and $1\le i\le j\le n$, $(\chi_\ell\gamma^{i,j})\circ\psi_{p_\ell}^{-1}\in C^\infty(\overline{\mathbb{R}^n_+})$ and hence has an extension $\tilde\gamma_\ell^{i,j}\in C^\infty(\mathbb{R}^n)$ by part (c) of Problem 2.2.8. We can always ensure that $\tilde\gamma_\ell^{i,j}$ is compactly supported by multiplying by a C_0^∞ function that is identically one on the support of $\chi_\ell\circ\psi_{p_\ell}^{-1}$. To complete the first step it suffices to set $\tilde\gamma^{i,j}=\sum_{\ell=1}^N\tilde\gamma_\ell^{i,j}\circ\psi_{p_\ell}$ for $1\le i\le j\le n$ and $\tilde\gamma^{i,j}=\sum_{\ell=1}^N\tilde\gamma_\ell^{j,i}\circ\psi_{p_\ell}$ for $1\le j\le i\le n$.

The second step is to find an open set $\Omega' \supset \overline{\Omega}$ such that $\tilde{\gamma}(x)$ is strictly positive definite at each $x \in \Omega'$. Let, for each real symmetric $n \times n$ matrix Γ, $\lambda_m(\Gamma)$ denote the smallest eigenvalue of Γ. By Hypothesis 3.3.1, $\lambda_0 = \inf_{x\in\overline{\Omega}} \lambda_m\big(\gamma(x)\big) > 0$. Since $\lambda_m(\Gamma)$ depends continuously on Γ, $\lambda(x) = \lambda_m\big(\tilde{\gamma}(x)\big)$ is continuous. To complete step two, it suffices to take $\Omega' = \lambda^{-1}\big((\lambda_0/2, \infty)\big)$.

Let $\psi \in C_0^\infty(\Omega')$ take values in $[0,1]$ and be identically one on $\overline{\Omega}$. To complete the problem, it now suffices to take $\gamma'(x) = \big(1-\psi(x)\big)\mathbb{1} + \psi(x)\tilde{\gamma}(x)$, where $\mathbb{1}$ is the identity matrix. ■

Problem 3.4.1 Let $2 \le n \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Denote by $\hat{n}(x)$ the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$ and by $d\sigma(x)$ the surface measure on $\partial\Omega$. Let $\vec{f}(x) = \big(f_j(x)\big)_{1\le j\le n}$ be a smooth vector field on $\overline{\Omega}$.

(a) Prove that

$$\sum_{j=1}^n \int_{\partial\Omega} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = (-1)^{j-1} \sum_{j=1}^n \int_{\partial\Omega} f_j(x)\ dx_1 \wedge \cdots \wedge \cancel{dx_j} \wedge \cdots \wedge dx_n$$

On the right hand side, the orientation of $\partial\Omega$ is such that, for an open neighbourhood $\mathcal{O} \subset \mathbb{R}^n$ in which $\partial\Omega \cap \mathcal{O}$ is given by the equation

$$x_\ell = \varphi(x_1, \cdots, \cancel{x_\ell}, \cdots, x_n), \qquad (x_1, \cdots, \cancel{x_\ell}, \cdots, x_n) \in \mathcal{R}$$

we have

$$\int_{\partial\Omega\cap\mathcal{O}} g(x)\ dx_1 \wedge \cdots \wedge \cancel{dx_\ell} \wedge \cdots \wedge dx_n = (-1)^\ell \int_{\mathcal{R}} g(x)\big|_{x_\ell=\varphi(x)} \prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i$$

when Ω is given by $x_\ell > \varphi(x_1, \cdots, \cancel{x_\ell}, \cdots, x_n)$ and

$$\int_{\partial\Omega\cap\mathcal{O}} g(x)\ dx_1 \wedge \cdots \wedge \cancel{dx_\ell} \wedge \cdots \wedge dx_n = (-1)^{\ell-1} \int_{\mathcal{R}} g(x)\big|_{x_\ell=\varphi(x)} \prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i$$

when Ω is given by $x_\ell < \varphi(x_1, \cdots, \cancel{x_\ell}, \cdots, x_n)$. For a brief introduction to wedge products and differential forms, see Chapter 7 of [Ar].

(b) Prove that

$$\sum_{j=1}^n \int_{\partial\Omega} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = \sum_{j=1}^n \int_\Omega \tfrac{\partial f_j}{\partial x^j}(x)\ d^n x$$

(c) Let $v \in H^1(\Omega)$ and, for each $1 \le j \le n$, $u_j \in H^1(\Omega)$. Prove that

$$\sum_{j=1}^n \int_\Omega v(x) \tfrac{\partial u_j}{\partial x^j}(x)\ d^n x = \sum_{j=1}^n \int_{\partial\Omega} v(x) u_j(x)\ \hat{n}^j(x)\ d\sigma(x) - \sum_{j=1}^n \int_\Omega \tfrac{\partial v}{\partial x^j}(x) u_j(x)\ d^n x$$

Solution. (a) By linearity in f, we may assume without loss of generality that only a single component of f, say f_j, is nonzero. Furthermore, since $\partial\Omega$ is smooth, it suffices to prove

$$\int_{\partial\Omega\cap\mathcal{O}} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = (-1)^{j-1}\int_{\partial\Omega\cap\mathcal{O}} f_j(x)\ dx_1\wedge\cdots\wedge \cancel{dx_j}\wedge\cdots\wedge dx_n$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ in which $\partial\Omega$ is given by the equation

$$x_\ell = \varphi(x_1,\cdots,\cancel{x_\ell},\cdots,x_n),\qquad (x_1,\cdots,\cancel{x_\ell},\cdots,x_n)\in\mathcal{R}$$

and Ω is given by $x_\ell > \varphi(x_1,\cdots,\cancel{x_\ell},\cdots,x_n)$. The case that Ω is given by the inequality $x_\ell < \varphi(x_1,\cdots,\cancel{x_\ell},\cdots,x_n)$ is similar. Since

$$dx_\ell = \sum_{\substack{1\le i\le n\\ i\ne\ell}} \tfrac{\partial\varphi}{\partial x_i}(x)\,dx_i$$

the right hand side is

$$\begin{aligned}(-1)^{j-1}\int_{\partial\Omega\cap\mathcal{O}} f_j(x)\ dx_1\wedge\cdots\wedge \cancel{dx_j}\wedge\cdots\wedge dx_n &= (-1)^{j-1}(-1)^\ell\int_{\mathcal{R}} f_j(x)\big|_{x_\ell=\varphi(x)}\prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i\\ &= -\int_{\mathcal{R}} f_j(x)\big|_{x_\ell=\varphi(x)}\prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i\end{aligned}$$

if $\ell=j$ and

$$(-1)^{j-1}\int_{\partial\Omega\cap\mathcal{O}} f_j(x)\ dx_1\wedge\cdots\wedge \cancel{dx_j}\wedge\cdots\wedge dx_n = \int_{\mathcal{R}} f_j(x)\big|_{x_\ell=\varphi(x)}\ \tfrac{\partial\varphi}{\partial x_j}(x)\prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i$$

if $\ell\ne j$. The sign on the right hand side is the product of $(-1)^{j-1}$, which came from the left hand side, $(-1)^{\ell-j-1}$, which came from moving $\frac{\partial\varphi}{\partial x_j}(x)\,dx_j$ from the dx_ℓ slot in the wedge product to the dx_j slot in the wedge product, and $(-1)^\ell$, which came from the orientation of $\partial\Omega$.

For the left hand side, we consider only $\ell=n$. The other cases are similar. The unit normal vector to the level surface $x_n-\varphi(x_1,\cdots,x_{n-1})=0$ at x that has negative x_n–component (so that it points out of Ω) is

$$\hat{n} = -\tfrac{\nabla(x_n-\varphi(x_1,\cdots,x_{n-1}))}{|\nabla(x_n-\varphi(x_1,\cdots,x_{n-1}))|} = -\tfrac{(\partial_{x_1}\varphi,\cdots,\partial_{x_{n-1}}\varphi,-1)}{|(\partial_{x_1}\varphi,\cdots,\partial_{x_{n-1}}\varphi,-1)|}$$

The cosine of the angle between $\hat{n}$ and the positive x_n–axis is $\hat{n}\cdot(0,\cdots,0,1) = |(\partial_{x_1}\varphi,\cdots,\partial_{x_{n-1}}\varphi,-1)|^{-1}$. So projecting an infinitesimal piece of $\partial\Omega$ at x onto the hyperplane $x_n=0$ reduces its area by $|(\partial_{x_1}\varphi,\cdots,\partial_{x_{n-1}}\varphi,-1)|$ and

$$d\sigma(x) = |(\partial_{x_1}\varphi,\cdots,\partial_{x_{n-1}}\varphi,-1)|\ dx_1\cdots dx_{n-1}$$

Thus

$$\int_{\partial\Omega\cap\mathcal{O}} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = -\int_{\mathcal{R}} f_j(x)\big|_{x_n=\varphi(x)} dx_1\cdots dx_{n-1}$$

if $j = n$ and

$$\int_{\partial\Omega\cap\mathcal{O}} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = \int_{\mathcal{R}} f_j(x)\big|_{x_n=\varphi(x)}\ \tfrac{\partial\varphi}{\partial x_j}(x)\ dx_1\cdots dx_{n-1}$$

if $j \neq n$, as desired.

(b) By part (a) and Stokes' theorem

$$\begin{aligned}
\sum_{j=1}^{n}\int_{\partial\Omega} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) &= (-1)^{j-1}\sum_{j=1}^{n}\int_{\partial\Omega} f_j(x)\ dx_1\wedge\cdots\wedge d\not{x}_j\wedge\cdots\wedge dx_n \\
&= (-1)^{j-1}\sum_{j=1}^{n}\int_{\Omega} d\big[f_j(x)\ dx_1\wedge\cdots\wedge d\not{x}_j\wedge\cdots\wedge dx_n\big] \\
&= (-1)^{j-1}\sum_{j=1}^{n}\int_{\Omega} \tfrac{\partial f_j}{\partial x^j}(x)\ dx_j\wedge dx_1\wedge\cdots\wedge d\not{x}_j\wedge\cdots\wedge dx_n \\
&= \sum_{j=1}^{n}\int_{\Omega} \tfrac{\partial f_j}{\partial x^j}(x)\ dx_1\wedge\cdots\wedge dx_j\wedge\cdots\wedge dx_n \\
&= \sum_{j=1}^{n}\int_{\Omega} \tfrac{\partial f_j}{\partial x^j}(x)\ d^n x
\end{aligned}$$

(c) For v and the u_j's in $C^\infty(\overline{\Omega})$, the identity follows immediately from part (b) with $f_j(x) = v(x)u_j(x)$ and the product rule. Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, $\frac{\partial}{\partial x^j}$ is bounded as a map from $H^1(\Omega)$ to $L^2(\Omega)$ and restriction to the boundary of Ω is bounded as a map from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ and hence to $L^2(\partial\Omega)$, the desired identity follows by continuity.

■

Problem 3.4.2

(a) Prove that Λ_γ is both symmetric (self–transpose) and self–adjoint in the sense that

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \mathcal{L}_{\Lambda_\gamma g}(f) \qquad \mathcal{L}_{\Lambda_\gamma f}(\bar{g}) = \overline{\mathcal{L}_{\Lambda_\gamma g}(\bar{f})}$$

for all $f, g \in H^{1/2}(\partial\Omega)$.

(b) Let A be any bounded linear operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. Define the operator norm

$$\|A\|_{1/2,-1/2} = \sup\left\{\ \big|\mathcal{L}_{Af}(g)\big|\ \big|\ f, g \in H^{1/2}(\partial\Omega),\ \|f\|_{1/2,\partial\Omega} = \|g\|_{1/2,\partial\Omega} = 1\ \right\}$$

Define the quadratic form $Q_A : H^{1/2}(\partial\Omega) \to \mathbb{C}$ associated with A by

$$Q_A(f) = \mathcal{L}_{Af}(\bar{f})$$

Prove that if A is self–adjoint, in the sense that $\mathcal{L}_{\Lambda_\gamma f}(\bar{g}) = \overline{\mathcal{L}_{\Lambda_\gamma g}(\bar{f})}$ for all $f, g \in H^{1/2}(\partial\Omega)$, then

$$\|A\|_{1/2,-1/2} = \sup\left\{\ \left|Q_A(f)\right|\ \middle|\ f \in H^{1/2}(\partial\Omega),\ \|f\|_{1/2,\partial\Omega} = 1\ \right\}$$

Solution. (a) Since $\gamma^{i,j}(x) = \gamma^{j,i}(x)$ for all $1 \le, i, j \le n$ and all $x \in \Omega$ and $\gamma^{i,j}(x)$ is real for all $x \in \Omega$,

$$\begin{aligned}
\mathcal{L}_{\Lambda_\gamma f}(g) &= \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial u_g}{\partial x_j}\ d^n x = \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_g}{\partial x_i} \tfrac{\partial u_f}{\partial x_j}\ d^n x = \mathcal{L}_{\Lambda_\gamma g}(f) \\
\mathcal{L}_{\Lambda_\gamma f}(\bar{g}) &= \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \overline{\tfrac{\partial u_g}{\partial x_j}}\ d^n x = \overline{\sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_g}{\partial x_i} \overline{\tfrac{\partial u_f}{\partial x_j}}\ d^n x} = \overline{\mathcal{L}_{\Lambda_\gamma g}(\bar{f})}
\end{aligned}$$

for all $f, g \in H^{1/2}(\partial\Omega)$. Here we used that

$$(L_\gamma \overline{u_g})(x) = \overline{(L_\gamma u_g)(x)} = 0 \qquad \text{if } x \in \Omega \qquad\qquad \overline{u_g(x)} = \overline{g(x)} \qquad \text{if } x \in \partial\Omega$$

so that $u_{\bar{g}} = \overline{u_g}$.

(b) Write

$$\begin{aligned}
\|A\| &= \sup\left\{\ \left|\mathcal{L}_{Af}(\bar{g})\right|\ \middle|\ f, g \in H^{1/2}(\partial\Omega),\ \|f\|_{1/2,\partial\Omega} = \|g\|_{1/2,\partial\Omega} = 1\ \right\} \\
\|A\|' &= \sup\left\{\ \left|Q_A(f)\right| = \left|\mathcal{L}_{Af}(\bar{f})\right|\ \middle|\ f \in H^{1/2}(\partial\Omega),\ \|f\|_{1/2,\partial\Omega} = 1\ \right\}
\end{aligned}$$

Clearly $\|A\|_{1/2,-1/2} = \|A\|$ and $\|A\|' \le \|A\|$, so it suffices to prove that $\|A\| \le \|A\|'$. Let $f, g \in H^{1/2}(\partial\Omega)$ with $\|f\|_{1/2,\partial\Omega} = \|g\|_{1/2,\partial\Omega} = 1$ and choose a complex number α of modulus one such that $\mathcal{L}_{Af}(\overline{\alpha g}) = \bar{\alpha}\mathcal{L}_{Af}(\bar{g}) \ge 0$. Since A is self–adjoint,

$$\begin{aligned}
\left|\mathcal{L}_{Af}(\bar{g})\right| &= \operatorname{Re}\mathcal{L}_{Af}(\overline{\alpha g}) = \tfrac{1}{2}\Big[\mathcal{L}_{Af}(\overline{\alpha g}) + \overline{\mathcal{L}_{Af}(\overline{\alpha g})}\Big] = \tfrac{1}{2}\Big[\mathcal{L}_{Af}(\overline{\alpha g}) + \mathcal{L}_{A(\alpha g)}(\bar{f})\Big] \\
&= \tfrac{1}{4}\Big[\mathcal{L}_{A(f+\alpha g)}(\overline{f+\alpha g}) - \mathcal{L}_{A(f-\alpha g)}(\overline{f-\alpha g})\Big] \\
&= \tfrac{1}{4}\|A\|'\Big[\|f+\alpha g\|_{1/2,\partial\Omega}^2 + \|f-\alpha g\|_{1/2,\partial\Omega}^2\Big]
\end{aligned}$$

By problem 2.1.20, the norm $\|\ \cdot\ \|_{1/2,\partial\Omega}$ arises from an inner product and so obeys the parallelogram law

$$\|u+v\|_{1/2,\partial\Omega}^2 + \|u-v\|_{1/2,\partial\Omega}^2 = 2\|u\|_{1/2,\partial\Omega}^2 + 2\|v\|_{1/2,\partial\Omega}^2$$

Consequently

$$\left|\mathcal{L}_{Af}(\bar{g})\right| \le \tfrac{1}{2}\|A\|'\Big[\|f\|_{1/2,\partial\Omega}^2 + \|\alpha g\|_{1/2,\partial\Omega}^2\Big] = \tfrac{1}{2}\|A\|'\Big[\|f\|_{1/2,\partial\Omega}^2 + \|g\|_{1/2,\partial\Omega}^2\Big] \le \|A\|'$$

as desired. ■

Problem 3.4.3 Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary. Denote by $\mathcal{BL}_{\frac{1}{2},-\frac{1}{2}}$ the Banach space of bounded linear maps from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ endowed with the operator norm.

(a) Let γ satisfy Hypothesis 3.3.1. Prove that the map $\gamma' \mapsto \Lambda_{\gamma+\gamma'}$ has an analytic continuation, as a map from a neighbourhood of the origin in $L^\infty(\Omega)$ to $\mathcal{BL}_{\frac{1}{2},-\frac{1}{2}}$.

(b) Let $E > 0$. Denote by Γ_E the set of γ's that obey Hypothesis 3.3.1 and also obey

$$\tfrac{1}{E}|\xi|^2 < \textstyle\sum\limits_{i,j=1}^{n} \gamma_1(x)^{i,j}\xi_i\xi_j < E|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Prove that the map $\gamma \mapsto \Lambda_\gamma$ has an analytic continuation, as a map from a neighbourhood of Γ_E in $L^\infty(\Omega)$ to $\mathcal{BL}_{\frac{1}{2},-\frac{1}{2}}$.

Solution. (a) Let, for each $f, g \in H^{1/2}(\partial\Omega)$ and $\gamma' \in L^\infty(\Omega)$ $u(\gamma'; f)$ and $v(\gamma'; g)$ be the unique solutions to

$$\begin{aligned} L_\gamma u(\gamma'; f) &= -\nabla\cdot\gamma'\nabla u(\gamma'; f) \quad & Ru(\gamma'; f) &= f \\ L_\gamma v(\gamma'; g) &= -\nabla\cdot\gamma'\nabla v(\gamma'; g) \quad & Rv(\gamma'; g) &= g; \end{aligned}$$

provided by Problem 3.3.6. (Yes, we know that $u(\gamma'; f) = v(\gamma'; f)$. We just want to make a notational connection to Theorem 3.4.1.) By Problem 3.3.5, with $F = 0$, $u(\gamma'; f)$ and $v(\gamma'; g)$ are analytic maps from $L^\infty(\Omega)$ to $H^1(\Omega)$ at $\gamma' = 0$ and the $\ell^{\rm th}$ terms in their respective Taylor expansions obey the bounds

$$\begin{aligned} \big\|u_\ell(\gamma_1, \cdots, \gamma_\ell; f)\big\|_{1,\Omega} &\le Cc^\ell \big\|\gamma_1\big\|_{L^\infty(\Omega)} \cdots \big\|\gamma_\ell\big\|_{L^\infty(\Omega)} \ \big\|f\big\|_{1/2,\Omega} \\ \big\|v_\ell(\gamma_1, \cdots, \gamma_\ell; g)\big\|_{1,\Omega} &\le Cc^\ell \big\|\gamma_1\big\|_{L^\infty(\Omega)} \cdots \big\|\gamma_\ell\big\|_{L^\infty(\Omega)} \ \big\|g\big\|_{1/2,\Omega} \end{aligned}$$

and are linear in f and g respectively. Since ∇ is a bounded map from $H^1(\Omega)$ to $L^2(\Omega)$, $\nabla u(\gamma'; f)$ and $\nabla v(\gamma'; g)$ are analytic maps from $L^\infty(\Omega)$ to $L^2(\Omega)$ at $\gamma' = 0$. The $\ell^{\rm th}$ terms in their respective Taylor expansions are ∇u_ℓ and ∇v_ℓ and obey the bounds

$$\begin{aligned} \big\|\nabla u_\ell(\gamma_1, \cdots, \gamma_\ell; f)\big\|_{L^2(\Omega)} &\le Cc^\ell \big\|\gamma_1\big\|_{L^\infty(\Omega)} \cdots \big\|\gamma_\ell\big\|_{L^\infty(\Omega)} \ \big\|f\big\|_{1/2,\Omega} \\ \big\|\nabla v_\ell(\gamma_1, \cdots, \gamma_\ell; g)\big\|_{L^2(\Omega)} &\le Cc^\ell \big\|\gamma_1\big\|_{L^\infty(\Omega)} \cdots \big\|\gamma_\ell\big\|_{L^\infty(\Omega)} \ \big\|g\big\|_{1/2,\Omega} \end{aligned}$$

By Problem 3.3.4, with $m = 3$, $\mathcal{B} = L^\infty(\Omega; \mathbb{C}^{n^2})$, $\mathcal{B}' = \mathbb{C}$ $\mathcal{B}_1 = L^\infty(\Omega; \mathbb{C}^{n^2})$, $\mathcal{B}_2 = \mathcal{B}_3 = L^2(\Omega; \mathbb{C}^n)$ and

$$M(\gamma', U, V) = \sum_{i,j=1}^{n} \int_\Omega {\gamma'}^{i,j}(x) U^i(x) V^j(x)\ d^n x$$

the functional

$$\gamma' \mapsto \lambda(\gamma'; f, g) = \sum_{i,j=1}^{n} \int_\Omega \big(\gamma^{i,j}(x) + {\gamma'}^{i,j}(x)\big)\big[\tfrac{\partial}{\partial x_i} u(\gamma'; f)\big]\big[\tfrac{\partial}{\partial x_j} v(\gamma'; g)\big]\ d^n x \in \mathbb{C}$$

is analytic in γ' at $\gamma'=0$. The $\ell^{\rm th}$ term in the Taylor expansion of this functional is also linear in its dependence on f and g and obeys the bound

$$\big|\lambda_\ell(\gamma_1,\cdots,\gamma_\ell;f,g)\big| \le Cc^\ell \|\gamma_1\|_{L^\infty(\Omega)}\cdots\|\gamma_\ell\|_{L^\infty(\Omega)}\ \|f\|_{1/2,\Omega}\|g\|_{1/2,\Omega}$$

for new constants C and c. Consequently there exists an ℓ–linear map,

$$(\gamma_1,\cdots,\gamma_\ell)\in L^\infty(\Omega;\mathbb{C}^{n^2})\times\cdots\times L^\infty(\Omega;\mathbb{C}^{n^2})\mapsto \Lambda_\ell(\gamma_1,\cdots,\gamma_\ell)\in\mathcal{BL}_{\frac12,-\frac12}$$

such that

$$\mathcal{L}_{\Lambda_\ell(\gamma_1,\cdots,\gamma_\ell)f}(g)=\lambda_\ell(\gamma_1,\cdots,\gamma_\ell;f,g)$$

Furthermore

$$\big\|\Lambda_\ell(\gamma_1,\cdots,\gamma_\ell)\big\|_{\mathcal{BL}_{\frac12,-\frac12}}\le Cc^\ell\|\gamma_1\|_{L^\infty(\Omega)}\cdots\|\gamma_\ell\|_{L^\infty(\Omega)}$$

When $\gamma+\gamma'$ obeys Hypothesis 3.3.1,

$$\mathcal{L}_{\Lambda_{\gamma+\gamma'}f}(g)=\lambda(\gamma';f,g)$$

for all $f,g\in H^{1/2}(\partial\Omega)$, by Theorem 3.4.1, so that

$$\Lambda_{\gamma+\gamma'}=\sum_{\ell=0}^\infty\Lambda_\ell(\gamma',\cdots,\gamma')$$

(b) By Remark 3.3.6, the constants c and C of the proof of part (a), which actually came from Problem 3.3.5, depend only on Ω and E. In particular they are independent of γ, except through E. Hence the radius of convergence $r(\Omega,E)=\frac1c$ of part (a) is independent of γ in Γ_E. By combining the analytic continuations of part (a), we get an analytic continuation to $\tilde\Gamma_E=\big\{\ \gamma+\gamma'\ \big|\ \gamma\in\Gamma_E,\ \|\gamma'\|_{L^\infty(\Omega)}<r(\Omega,E)\ \big\}$. This set is convex and hence contractible, so the analytic continuation is single–valued.

■

Problem 3.5.1 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q\in L^\infty(\Omega)$.

(a) Prove that there is a unique bounded linear map $L_q: H^1(\Omega)\to H^{-1}(\Omega)$ such that

$$(L_qu)(v)=\int_\Omega v(x)\big[\Delta u(x)+q(x)u(x)\big]\ d^nx$$

for all $u\in C^\infty(\overline{\Omega})$ and all $v\in C_0^\infty(\Omega)$.

(b) Prove that there is a unique continuous sesquilinear form

$$\langle \,\cdot\, , \,\cdot\, \rangle_q : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$$

such that

$$\langle u, v\rangle_q = \sum_{i=1}^n \int_\Omega \tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial v}{\partial x_i}(x)}\ d^nx + \int_{\partial\Omega} u(x)\overline{v(x)}\ d^{n-1}\sigma - \int_\Omega q(x)u(x)\overline{v(x)}\ d^nx$$

for all $u, v \in C^\infty(\overline{\Omega})$. Here $d^{n-1}\sigma$ denotes the surface measure on $\partial\Omega$. Prove that

$$\langle v, u\rangle_q = \overline{\langle u, v\rangle_{\bar q}}$$

for all $u, v \in H^1(\Omega)$ and

$$(L_q u)(v) = -\langle v, \bar u\rangle_q$$

for all $u \in H^1(\Omega)$ and all $v \in H^1_0(\Omega)$.

Solution. (a) It suffices to take $L_q u = L_{\mathbb{1}} u + \mathcal{L}_{qu}$, where $L_{\mathbb{1}}$ is the operator of Lemma 3.3.4, with $\gamma(x)$ being the identity matrix, $q \mapsto qu$ is the operator of multiplication by the function q, which is a bounded operator from $L^2(\Omega) \supset H^1(\Omega)$ to $L^2(\Omega)$ and $\mathcal{L} : L^2(\Omega) \to H^{-1}(\Omega)$ is the bounded operator of Theorem 2.1.13. The uniqueness follows from denseness and boundedness in the usual way.

(b) Linearity in the first argument and conjugate linearity in the second argument are obvious. By hypothesis $q \in L^\infty(\Omega)$ so that

$$\left| \int_\Omega q(x)u(x)\overline{v(x)}\ d^nx \right| \le \|q\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|q\|_{L^\infty(\Omega)}\|u\|_{1,\Omega}\|v\|_{1,\Omega}$$

Since

$$\langle u, v\rangle_q = \langle u, v\rangle_{\gamma=\mathbb{1},\Omega} - \int_\Omega q(x)u(x)\overline{v(x)}\ d^nx$$

and, by Lemma 3.3.3, the norms $\|\ \|_{1,\Omega}$ and $\|\ \|_{\gamma=\mathbb{1},\Omega}$ are equivalent, we have

$$\left| \langle u, v\rangle_q \right| \le \|u\|_{\gamma=\mathbb{1},\Omega}\|v\|_{\gamma=\mathbb{1},\Omega} + \|q\|_{L^\infty(\Omega)}\|u\|_{1,\Omega}\|v\|_{1,\Omega} \le C\|u\|_{1,\Omega}\|v\|_{1,\Omega}$$

The continuity, extendibility and uniqueness are now obvious.

That

$$\langle v, u\rangle_q = \overline{\langle u, v\rangle_{\bar q}}$$

for all $u, v \in C^\infty(\overline{\Omega})$ is obvious. That it is also true for all $u, v \in H^1(\Omega)$ follows by continuity, since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.

By the divergence theorem,

$$(L_q u)(v) = \int_\Omega v(x)\big[\Delta u(x) + q(x)u(x)\big]\ d^nx = -\langle v, \bar u\rangle_q$$

for all $u \in C^\infty(\overline{\Omega})$ and all $v \in C_0^\infty(\Omega)$. By parts (a) and (b), this extends by continuity to all $u \in H^1(\Omega)$ and all $v \in H^1_0(\Omega)$. ■

Problem 3.5.2 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$.

(a) Prove that the map

$$\begin{aligned} H^1(\Omega) &\to H^{-1}(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \\ u &\mapsto \quad \big(L_q u\ ,\ Ru\big) \end{aligned}$$

is bounded.

(b) Assume, in addition, that there is a constant $c > 0$ such that $\big|\left\langle u, u\right\rangle_q\big| \geq c\|u\|_{1,\Omega}^2$ for all $u \in H^1(\Omega)$. Prove that the map of part (a) is also 1–1 and onto with a bounded inverse. In this case, for each $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$ there is a unique $u \in H^1(\Omega)$ such that $\Delta u + qu = F$ in Ω and $u\big|_{\partial\Omega} = f$. Furthermore, there is a constant C such that

$$\|u\|_{1,\Omega}^2 \leq C\big(\|F\|_{-1,\Omega}^2 + \|f\|_{1/2,\partial\Omega}^2\big)$$

for all $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$.

Solution. (a) By part (a) of Lemma 3.5.1, $\|L_q u\|_{-1,\Omega} \leq c_1\|u\|_{1,\Omega}$ and, by part (ii) of Theorem 2.2.2, $\|Ru\|_{\frac{1}{2},\partial\Omega} \leq c_2\|u\|_{1,\Omega}$.

(b) *Injectiveness:* If $Ru = 0$, then $u \in H_0^1(\Omega)$ by part (v) of Theorem 2.2.2. If $u \in H_0^1(\Omega)$ and $L_q u = 0$, then, by part (b) of Lemma 3.5.1,

$$0 = (L_q u)(\bar{u}) = -\left\langle \bar{u}, \bar{u}\right\rangle_q \implies 0 = \big|\left\langle \bar{u}, \bar{u}\right\rangle_q\big| \geq c^2\|\bar{u}\|_{1,\Omega}^2$$

so that $\bar{u} = 0$ and hence $u = 0$.

Surjectiveness: Let $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$. By part (iii) of Theorem 2.2.2, R is onto. So there is a $w \in H^1(\Omega)$ such that $Rw = f$. As $F - L_q w \in H^{-1}(\Omega) = H_0^1(\Omega)^*$ is a bounded linear functional on $H_0^1(\Omega)$, there exists, by the Riesz representation theorem, a $\varphi \in H_0^1(\Omega)$ such that

$$\begin{aligned} (F - L_q w)v &= \left\langle v, \varphi\right\rangle_{1,\Omega} \\ &= \left\langle v, J_q^{-1}\varphi\right\rangle_q && \text{by part (b) of Lemma 3.5.2} \\ &= -\big(L_q \overline{J_q^{-1}\varphi}\big)(v) && \text{by part (b) of Lemma 3.5.1} \end{aligned}$$

for all $v \in H_0^1(\Omega)$. Set $u = w - \overline{J_q^{-1}\varphi}$. Then $L_q u = F$ and, since $\overline{J_q^{-1}\varphi} \in H_0^1(\Omega)$, $Ru = Rw = f$.

Boundedness of the inverse: We use the notation of the surjectiveness proof. By part (iv) of Theorem 2.2.2, we may choose a w obeying $\|w\|_{1,\Omega} \leq C'\|f\|_{1/2,\partial\Omega}$. By the Riesz representation theorem,

$$\begin{aligned} \|\varphi\|_{1,\Omega} &= \|F - L_q w\|_{-1,\Omega} \leq \|F\|_{-1,\Omega} + \|L_q w\|_{-1,\Omega} \\ &\leq \|F\|_{-1,\Omega} + c_1\|w\|_{1,\Omega} && \text{by part (a)} \end{aligned}$$

Hence

$$\begin{aligned}\|u\|_{1,\Omega} &\le \|w\|_{1,\Omega} + \big\|\overline{J_q^{-1}\varphi}\big\|_{1,\Omega} = \|w\|_{1,\Omega} + \big\|J_q^{-1}\varphi\big\|_{1,\Omega} \le \|w\|_{1,\Omega} + \big\|J_q^{-1}\big\|\ \|\varphi\|_{1,\Omega}\\ &\le \big\|J_q^{-1}\big\|\ \|F\|_{-1,\Omega} + \big(1 + c_1\big\|J_q^{-1}\big\|\big)\|w\|_{1,\Omega}\\ &\le \big\|J_q^{-1}\big\|\ \|F\|_{-1,\Omega} + \big(1 + c_1\big\|J_q^{-1}\big\|\big)C'\|f\|_{1/2,\partial\Omega}\end{aligned}$$

■

Problem 3.5.3 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and $\ell \in \mathbb{N}$. Let

$$q \in \begin{cases} L^\infty(\Omega) & \text{if } \ell \le 2 \\ C^{\ell-2}(\overline{\Omega}) & \text{if } \ell > 2 \end{cases}$$

Prove that there is a constant C, depending only on ℓ, q and Ω such that for all $u \in H^1(\Omega)$, $F \in H^{\ell-2}(\Omega)$ and $f \in H^{\ell-\frac{1}{2}}(\partial\Omega)$ obeying

$$L_q u = F \qquad Ru = f$$

we have $u \in H^\ell(\Omega)$ and

$$\|u\|_{\ell,\Omega} \le C\Big(\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega} + \|u\|_{L^2(\Omega)}\Big)$$

Solution. If $\ell \ge 3$, then by part (a) of Lemma 2.1.16, $qu \in H^{\ell-2}(\Omega)$ and $\|qu\|_{\ell-2,\Omega}$ is bounded by a constant times $\|u\|_{\ell-2,\Omega}$. If $\ell = 1, 2$ and $q \in L^\infty(\Omega)$, $qu \in L^2(\Omega) \subset H^{\ell-2}(\Omega)$ and

$$\|qu\|_{\ell-2,\Omega} \le \|qu\|_{L^2(\Omega)} \le \|q\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}$$

Since $L_{\gamma=\mathbb{1}}u = L_q u - qu = F - qu$, Proposition 3.3.10, with $\gamma = \mathbb{1}$ and F replaced by $F - qu$ gives

$$\|u\|_{\ell,\Omega} \le C\Big(\|F - qu\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega}\Big) \le C'\Big(\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega} + \|u\|_{\max\{0,\ell-2\},\Omega}\Big)$$

If $\ell = 1$, this is the bound we want. The bounds for $\ell \in \mathbb{N}\setminus\{1\}$ are achieved by induction on ℓ.

■

Problem 3.5.4 Let $\mathcal{H}$ be any Hilbert space. A linear operator $C : \mathcal{H} \to \mathcal{H}$ is said to be compact if for each bounded sequence $\{v_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$, there is a subsequence of $\{Cv_i\}_{i\in\mathbb{N}}$ that is convergent. See Appendix A for an introduction to compact operators. Let C be a compact operator on $\mathcal{H}$.

(a) Set, for each $\lambda \in \mathbb{C}$, $\mathcal{E}_\lambda = \big\{\ v \in \mathcal{H}\ \big|\ Cv = \lambda v\ \big\}$. If $\mathcal{E}_\lambda \neq \{0\}$, then λ is called an eigenvalue of C and the nonzero elements of $\mathcal{E}_\lambda$ are called the eigenvectors of C with eigenvalue λ. Prove that if $\varepsilon > 0$, then $\#\big\{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| \geq \varepsilon, \mathcal{E}_\lambda \neq \{0\}\ \big\} < \infty$ and furthermore, that if $\lambda \neq 0$, then $\dim \mathcal{E}_\lambda < \infty$.

(b) Prove that if $\lambda \neq 0$ then the range of $C - \lambda\mathbb{1}$ is the orthogonal complement of

$$\mathcal{E}_{*,\bar\lambda} = \big\{\ v \in \mathcal{H}\ \big|\ C^*v = \bar\lambda v\ \big\}$$

Here C^*, the adjoint operator of C, is determined by

$$\langle Cx, y\rangle_{\mathcal{H}} = \langle x, C^*y\rangle_{\mathcal{H}} \qquad \text{for all } x, y \in \mathcal{H}$$

(c) Give an example of a compact operator C whose range is not the orthogonal complement of $\mathcal{E}_{*,0}$.

Solution. (a) This follows immediately from Proposition A.7, but we give an independent proof anyway. Let $\{\lambda_i\}_{i\in\mathbb{N}}$ be any sequence in $\big\{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| \geq \varepsilon\ \big\}$. It suffices for us to exhibit a contradiction to the assumption that there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ of independent vectors such that $Cu_i = \lambda_i u_i$ for all $i \in \mathbb{N}$.

Apply the Gram–Schmidt orthogonalization process to $\{u_i\}_{i\in\mathbb{N}}$ to construct an orthonormal sequence $\{y_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ with each y_i of the form $y_i = \sum\limits_{j=1}^{i} c_{i,j}u_j$. Then $\big\{v_i = \frac{y_i}{\lambda_i}\big\}_{i\in\mathbb{N}}$ is a bounded subset of $\mathcal{H}$. On the other hand, if $\ell < i$,

$$Cv_i - Cv_\ell - y_i = \sum_{j=1}^{i} \tfrac{\lambda_j}{\lambda_i} c_{i,j}u_j - \sum_{j=1}^{\ell} \tfrac{\lambda_j}{\lambda_\ell} c_{\ell,j}u_j - \sum_{j=1}^{i} c_{i,j}u_j = \sum_{j=1}^{i-1} \big(\tfrac{\lambda_j}{\lambda_i} - 1\big) c_{i,j}u_j - \sum_{j=1}^{\ell} \tfrac{\lambda_j}{\lambda_\ell} c_{\ell,j}u_j$$

is in the span of $\big\{u_1, \cdots, u_{i-1}\big\}$ and consequently is orthogonal to y_i. Thus

$$\|Cv_i - Cv_\ell\|_{\mathcal{H}}^2 = \|Cv_i - Cv_\ell - y_i\|_{\mathcal{H}}^2 + \|y_i\|_{\mathcal{H}}^2 \geq \|y_i\|_{\mathcal{H}}^2 = 1$$

and $\{Cv_i\}_{i\in\mathbb{N}}$ may not contain any convergent subsequence, in contradiction to the compactness of C.

(b) Denote by $\mathcal{R}$ the range, $\big\{\ (C - \lambda\mathbb{1})v\ \big|\ v \in \mathcal{H}\ \big\}$, of $C - \lambda\mathbb{1}$. Then

$$\begin{aligned} v \in \mathcal{E}_{*,\bar\lambda} &\iff (C^*v - \bar\lambda)v = 0 \\ &\iff \big\langle u, (C^*v - \bar\lambda)v\big\rangle_{\mathcal{H}} = 0 \text{ for all } u \in \mathcal{H} \\ &\iff \langle (C - \lambda\mathbb{1})u, v\rangle_{\mathcal{H}} = 0 \quad \text{for all } u \in \mathcal{H} \\ &\iff v \perp \mathcal{R} \end{aligned}$$

implies that the orthogonal complement of $\mathcal{E}_{*,\bar\lambda}$ is the orthogonal complement of the orthogonal complement of $\mathcal{R}$, which is the closure of $\mathcal{R}$.

So it suffices to prove that $\mathcal{R}$ is closed. Let $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{R}$ with $\lim\limits_{i\to\infty} u_i = u$. Then, for each $i \in \mathbb{N}$ there is a $v_i \in \mathcal{H}$ such that $(C - \lambda\mathbb{1})v_i = u_i$. Since $(C - \lambda\mathbb{1})w = 0$ for all $w \in \mathcal{E}_\lambda$, we may assume without loss of generality that each $v_i \perp \mathcal{E}_\lambda$.

If $\{v_i\}_{i\in\mathbb{N}}$ is not bounded, we may assume, by restricting to a subsequence, that $\lim\limits_{i\to\infty} \|v_i\|_{\mathcal{H}} = \infty$. Since C is compact, we may also assume, by again restricting to a subsequence, that $\lim\limits_{i\to\infty} C\frac{v_i}{\|v_i\|_{\mathcal{H}}} = w$. But then

$$\lambda \lim_{i\to\infty} \tfrac{v_i}{\|v_i\|_{\mathcal{H}}} = \lim_{i\to\infty} C\tfrac{v_i}{\|v_i\|_{\mathcal{H}}} - \lim_{i\to\infty} \tfrac{u_i}{\|v_i\|_{\mathcal{H}}} = w \implies \|w\|_{\mathcal{H}} = |\lambda| \neq 0,\ w \perp \mathcal{E}_\lambda$$

and

$$Cw = \lambda C \lim_{i\to\infty} \tfrac{v_i}{\|v_i\|_{\mathcal{H}}} = \lambda \lim_{i\to\infty} C\tfrac{v_i}{\|v_i\|_{\mathcal{H}}} = \lambda w \implies w \in \mathcal{E}_\lambda$$

This is a contradiction, since only the zero vector is in both $\mathcal{E}_\lambda$ and its orthogonal complement.

So $\{v_i\}_{i\in\mathbb{N}}$ is bounded. Since C is compact, we may assume, by restricting to a subsequence, that $\{Cv_i\}_{i\in\mathbb{N}}$ converges. Consequently, $\{v_i = \frac{1}{\lambda}(Cv_i - u_i)\}_{i\in\mathbb{N}}$ converges, say to v, and

$$u = \lim_{i\to\infty} \big(Cv_i - \lambda v_i\big) = Cv - \lambda v \in \mathcal{R}$$

This proves that $\mathcal{R}$ is closed.

(c) The problem is that the range of C is often not closed, while the orthogonal complement of $\mathcal{E}_{*,0}$ is always closed. Here is a large family of examples. Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be any sequence of nonzero complex numbers that converges to 0. Then, we claim that

$$C\Big(\sum_{n=1}^\infty \alpha_n e_n\Big) = \sum_{n=1}^\infty \mu_n\alpha_n e_n$$

does the job.

We first prove that C is compact. Set $\nu_n = \sup_{m\ge n}|\mu_m|$. By hypothesis $\lim\limits_{n\to\infty}\nu_n = 0$. Set, for each $N \in \mathbb{N}$,

$$C_N\Big(\sum_{n=1}^\infty \alpha_n e_n\Big) = \sum_{n=1}^N \mu_n\alpha_n e_n$$

Then

$$\begin{aligned}\Big\|(C - C_N)\Big(\sum_{n=1}^\infty \alpha_n e_n\Big)\Big\| = \Big\|\sum_{n=N+1}^\infty \mu_n\alpha_n e_n\Big\| = \sqrt{\sum_{n=N+1}^\infty |\mu_n\alpha_n|^2} \le \nu_{n+1}\sqrt{\sum_{n=N+1}^\infty |\alpha_n|^2}\\ \le \nu_{n+1}\Big\|\sum_{n=1}^\infty \alpha_n e_n\Big\|\end{aligned}$$

so that C_N converges in operator norm to C. As C_N is a nuclear operator (it is Example A.4 with $y_i = e_i$, $x'_i(x) = \langle x, e_i\rangle$ and $c_i = \mu_i$ for $1 \le i \le N$ and zero otherwise), it is compact, by Problem A.4. By part (d) of Proposition A.5, C is also compact.

The adjoint of C is

$$C^*\Big(\sum_{n=1}^{\infty}\alpha_n e_n\Big) = \sum_{n=1}^{\infty}\overline{\mu_n}\alpha_n e_n$$

The eigenvalues of C^* are $\{\overline{\mu_n}\}_{n\in\mathbb{N}}$. In particular, 0 is not an eignevalue so that $\mathcal{E}_{*,0} = \{0\}$ and the orthogonal complement of $\mathcal{E}_{*,0}$ is $\mathcal{H}$. But the range of C is not all of $\mathcal{H}$. In particular, if $(\beta_n)_{n\in\mathbb{N}} \in \ell^2$ but $\big(\frac{\beta_n}{\mu_n}\big)_{n\in\mathbb{N}} \notin \ell^2$ then $\sum_{n=1}^{\infty}\beta_n e_n$ is not in the range of C.

■

Problem 3.5.5 Let $\mathcal{H}$ be a Hilbert space and $C : \mathcal{H} \to \mathcal{H}$ a compact operator that obeys $C = C^*$.

(a) Prove that all eigenvalues of C are real.

(b) Let φ_1 and φ_2 be eigenvectors of C with eigenvalues λ_1 and λ_2, respectively. Prove that if $\lambda_1 \ne \lambda_2$, then $\varphi_1 \perp \varphi_2$.

(c) Prove that if $\mathcal{H} \ne \{0\}$, then there is an eigenvalue λ of C that obeys $|\lambda| = \|C\|$.

(d) Prove that there is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors for C.

Solution. (a) If λ is an eigenvalue of C then there is a unit vector $v \in \mathcal{H}$ such that $Cv = \lambda v$. So

$$\lambda = \lambda\langle v, v\rangle_{\mathcal{H}} = \langle \lambda v, v\rangle_{\mathcal{H}} = \langle Cv, v\rangle_{\mathcal{H}} = \langle v, Cv\rangle_{\mathcal{H}} = \langle v, \lambda v\rangle_{\mathcal{H}} = \bar\lambda\langle v, v\rangle_{\mathcal{H}} = \bar\lambda$$

and $\lambda \in \mathbb{R}$.

(b) Since $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{aligned}(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle_{\mathcal{H}} &= \langle \lambda_1 v_1, v_2\rangle_{\mathcal{H}} - \langle v_1, \lambda_2 v_2\rangle_{\mathcal{H}} = \langle Cv_1, v_2\rangle_{\mathcal{H}} - \langle v_1, Cv_2\rangle_{\mathcal{H}}\\ &= \langle v_1, Cv_2\rangle_{\mathcal{H}} - \langle v_1, Cv_2\rangle_{\mathcal{H}} = 0\end{aligned}$$

But $\lambda_1 \ne \lambda_2$, so it is necessary for $\langle v_1, v_2\rangle_{\mathcal{H}}$ to vanish.

(c) The definition of $\|C\|$ is

$$\|C\| = \sup_{\substack{v\in\mathcal{H}\\ \|v\|_{\mathcal{H}}=1}} \|Cv\|_{\mathcal{H}}$$

So there is a sequence of unit vectors $\{v_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ such that $\lim_{i\to\infty}\|Cv_i\|_{\mathcal{H}} = \|C\|$. Since C is compact, we may assume, by passing to a subsequence, that Cv_i converges to some vector y of length $\|C\|$. If $\|C\| = 0$, then $C = 0$ and all nonzero vectors are eigenvectors of eigenvalue zero, so we may assume that $\|C\| \neq 0$.

Define the operator $A = \|C\|^2 \mathbb{1} - C^2$ and the sesquilinear form $(u, u') = \langle Au, u' \rangle_{\mathcal{H}}$. For all $u \in \mathcal{H}$

$$(u, u) = \langle Au, u \rangle_{\mathcal{H}} = \|C\|^2 \langle u, u \rangle_{\mathcal{H}} - \left\langle C^2 u, u \right\rangle_{\mathcal{H}} = \|C\|^2 \langle u, u \rangle_{\mathcal{H}} - \langle Cu, Cu \rangle_{\mathcal{H}} \geq 0$$

so the form is nonnegative and satisfies the Cauchy–Schwarz inequality. Thus

$$\begin{aligned}
\|Av_i\|_{\mathcal{H}}^2 = (v_i, Av_i) \leq \sqrt{(v_i, v_i)}\sqrt{(Av_i, Av_i)} &= \sqrt{\langle Av_i, v_i \rangle_{\mathcal{H}}}\sqrt{\langle A^2 v_i, Av_i \rangle_{\mathcal{H}}} \\
&\leq \sqrt{\langle Av_i, v_i \rangle_{\mathcal{H}}}\sqrt{\|A^2 v_i\|_{\mathcal{H}} \|Av_i\|_{\mathcal{H}}} \quad \leq \|A\|^{3/2}\sqrt{\langle Av_i, v_i \rangle_{\mathcal{H}}} \\
&= \|A\|^{3/2}\sqrt{\|C\|^2 - \|Cv_i\|_{\mathcal{H}}^2}
\end{aligned}$$

The right hand side converges to zero as i tends to ∞, so that $\lim_{i \to \infty} Av_i = 0$ and

$$Ay = \lim_{i \to \infty} ACv_i = \lim_{i \to \infty} CAv_i = C \lim_{i \to \infty} Av_i = 0$$

Since

$$0 = Ay = (\|C\|\mathbb{1} - C)(\|C\|\mathbb{1} + C)y$$

either $(\|C\|\mathbb{1} + C)y = 0$, in which case y is an eigenvector of C of eigenvalue $-\|C\|$, or $y' = (\|C\|\mathbb{1} + C)y$ is a nonzero vector obeying $(\|C\|\mathbb{1} - C)y' = 0$, , in which case y' is an eigenvector of C of eigenvalue $\|C\|$.

(d) Let $\mathcal{E}$ be the closure of the linear span of all eigenvectors of C. By part (b), there is an orthonormal basis for $\mathcal{E}$ consisting of eigenvectors of C, so it suffices to prove that $\mathcal{E} = \mathcal{H}$.

We claim that C maps $\mathcal{E}^\perp$, the orthogonal complement of $\mathcal{E}$, into itself. To see this, observe that C maps any finite linear combination of eigenvectors into a finite linear combination of eigenvectors. So, by continuity, C maps $\mathcal{E}$ into itself. If $v \perp \mathcal{E}$, then

$$\langle Cv, e \rangle_{\mathcal{H}} = \langle v, Ce \rangle_{\mathcal{H}} = 0 \text{ for all } e \in \mathcal{E}$$

so $Cv \perp \mathcal{E}$.

Restricting C to $\mathcal{E}^\perp$ yields a new compact operator on the new Hilbert space $\mathcal{E}^\perp$. By construction, this new operator may not have any eigenvalues. By part (c), this is only possible if $\mathcal{E}^\perp = \{0\}$. So $\mathcal{E} = \mathcal{H}$, as desired. ■

Problem 3.5.6 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$ be real valued. In the following "eigenvalue" and "eigenvector" always refer to eigenvalues and eigenvectors of $\Delta + q$ with Dirichlet boundary conditions on $\partial\Omega$, as in Definition 3.5.4.

(a) Prove that all eigenvalues are real.

(b) Let φ_1 and φ_2 be eigenvectors of with eigenvalues λ_1 and λ_2, respectively. Prove that if $\lambda_1 \neq \lambda_2$, then $\varphi_1 \perp \varphi_2$.

(c) Prove that there is an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of $\Delta + q$.

Solution. Choose Q as in Lemma 3.5.5. Observe that, for any nonzero $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}(\Delta + q)\varphi = \lambda\varphi &\iff (\Delta + q - Q)\varphi = (\lambda - Q)\varphi \iff \varphi = (\lambda - Q)R_{q,Q}\varphi \\ &\iff R_{q,Q}\varphi = (\lambda - Q)^{-1}\varphi\end{aligned}$$

We have used, in the second equivalence, that $(\Delta + q)\varphi = \lambda\varphi$ is automatically in $L^2(\Omega)$, and, in the last equivalence, that $0 \neq \varphi = (\lambda - Q)R_{q,Q}\varphi$ implies $\lambda - Q \neq 0$. Thus φ is an eigenvector for $\Delta + q$ if and only if it is an eigenvector for $R_{q,Q}$ and λ an eigenvalue for $\Delta + q$ if and only $\frac{1}{\lambda - Q}$ is an eigenvalue for $R_{q,Q}$. As $R_{q,Q}$ is compact, by part (b) of Lemma 3.5.5, all parts of this Problem follow from Problem 3.5.5 with $\mathcal{H} = L^2(\Omega)$ and $C = R_{q,Q}$. ■

Problem 3.5.7 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Let $L_q : H^1(\Omega) \to H^{-1}(\Omega)$ be the Schrödinger operator with potential q defined in part (a) of Lemma 3.5.1. Assume that 0 is not an eigenvalue for L_q with Dirichlet boundary conditions. That is, assume that $\mathcal{K}_q = \left\{\, u \in H_0^1(\Omega) \;\middle|\; L_q u = 0 \,\right\} = \{0\}$.

Let, for each $f \in H^{3/2}(\partial\Omega)$, $u_f \in H^2(\Omega)$ be the solution of

$$\begin{aligned}(L_q u_f)(x) &= 0 && \text{if } x \in \Omega \\ u_f(x) &= f(x) && \text{if } x \in \partial\Omega\end{aligned}$$

provided by part (b) of Theorem 3.5.8 with $\ell = 2$. Prove that there is a unique bounded linear map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ such that

$$\mathcal{L}_{\Lambda_q f}(g) = \sum_{i=1}^{n} \int_{\partial\Omega} \tfrac{\partial u_f}{\partial x_i}(x)\hat{n}^i(x)\ g(x)\ d\sigma(x)$$

for all $f \in H^{3/2}(\partial\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Also prove that, if $f, g \in H^{1/2}(\partial\Omega)$ and $v_g \in H^1(\Omega)$ obeys $Rv_g = g$, then

$$\mathcal{L}_{\Lambda_q f}(g) = \int_\Omega \Big[\sum_{i=1}^{n} \tfrac{\partial u_f}{\partial x_i}\tfrac{\partial v_g}{\partial x_i} - q(x)u_f(x)v_g(x)\Big]\ d^n x$$

Here $\hat{n}^i(x)$ refers to the i^{th} component of $\hat{n}(x)$, the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$, $d\sigma(x)$ refers to the surface measure on $\partial\Omega$ and $\mathcal{L} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)^*$ refers to the isomorphism, from Problem 2.1.24, that is determined by

$$\mathcal{L}_f g = \int_{\partial\Omega} f(x)g(x)\, d\sigma(x) \qquad \text{for all } f, g \in C^\infty(\partial\Omega)$$

Solution. First consider $f \in H^{3/2}(\partial\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ and let $u_f \in H^2(\Omega)$ and $v_g \in H^1(\Omega)$ obey $L_q u_f = 0$, $Ru_f = f$ and $Rv_g = g$. Then, by the part (c) of Problem 3.4.1 (the divergence theorem),

$$\begin{aligned}
\mathcal{L}_{\Lambda_q f}(g) &= \int_{\partial\Omega} (\Lambda_q f)(x)g(x)\, d\sigma(x) = \sum_{i=1}^n \int_{\partial\Omega} \tfrac{\partial u_f}{\partial x_i}(x) v_g(x)\ \hat{n}^i(x)\, d\sigma(x) \\
&= \sum_{i=1}^n \int_\Omega \tfrac{\partial}{\partial x_i}\Big[\tfrac{\partial u_f}{\partial x_i}(x) v_g(x)\Big]\, d^n x \\
&= \sum_{i=1}^n \int_\Omega \tfrac{\partial u_f}{\partial x_i}\tfrac{\partial v_g}{\partial x_i}\, d^n x + \int_\Omega v_g(x)\ \Delta u_f(x)\, d^n x \\
&= \int_\Omega \Big[\sum_{i=1}^n \tfrac{\partial u_f}{\partial x_i}\tfrac{\partial v_g}{\partial x_i} - q(x)u_f(x)v_g(x)\Big]\, d^n x
\end{aligned}$$

By part (iv) of Theorem 2.2.2, we may choose v_g so that $\|v_g\|_{1,\Omega} \le C'\|g\|_{1/2,\partial\Omega}$. Then

$$\begin{aligned}
\big|\mathcal{L}_{\Lambda_q f}(g)\big| &\le \sum_{i=1}^n \Big|\int_\Omega \Big[\sum_{i=1}^n \tfrac{\partial u_f}{\partial x_i}\tfrac{\partial v_g}{\partial x_i} - q(x)u_f(x)v_g(x)\Big]\, d^n x\Big| \\
&\le c_1 \big\|\nabla u_f\big\|_{L^2(\Omega)} \big\|\nabla v_g\big\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)} \|u_f\|_{L^2(\Omega)} \|v_g\|_{L^2(\Omega)} \\
&\le c_2\, \|u_f\|_{1,\Omega}\, \|v_g\|_{1,\Omega} \\
&\le c_3\, \|f\|_{1/2,\partial\Omega}\, \|g\|_{1/2,\partial\Omega}
\end{aligned}$$

by the bound of part (b) of Theorem 3.5.8 with $\ell = 1$. Thus the norm of $\mathcal{L}_{\Lambda_q f}$ is bounded by $c_3\, \|f\|_{1/2,\partial\Omega}$. Since $\mathcal{L}$ is an isomorphism $\|\Lambda_q f\|_{-1/2,\partial\Omega} \le c_4\, \|f\|_{1/2,\partial\Omega}$. Since $H^{3/2}(\partial\Omega)$ is dense in $H^{1/2}(\partial\Omega)$, Λ_q has a unique extension to a bounded linear map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, by the BLT theorem.

If $f, g \in H^{1/2}(\partial\Omega)$ and $u_f, v_g \in H^1(\Omega)$ obey $L_\gamma u_f = 0$, $Ru_f = f$ and $Rv_g = g$, then, by continuity,

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \int_\Omega \Big[\sum_{i=1}^n \tfrac{\partial u_f}{\partial x_i}\tfrac{\partial v_g}{\partial x_i} - q(x)u_f(x)v_g(x)\Big]\, d^n x$$

■

Problem 3.5.8 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Let $L_q : H^1(\Omega) \to H^{-1}(\Omega)$ be the Schrödinger operator with potential q defined in part (a) of Lemma 3.5.1 and $R : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ be the restriction map of Theorem 2.2.2. Do *not* assume that 0 is not an eigenvalue for L_q with Dirichlet boundary conditions. Let $v \in H^1(\Omega)$ obey $L_q v = 0$.

(a) Prove that if $w,\ w' \in H^1(\Omega)$ obey $Rw = Rw'$, then

$$\int_\Omega \big[\nabla w(x) \cdot \nabla v(x) - q(x)w(x)v(x)\big]\ d^n x = \int_\Omega \big[\nabla w'(x) \cdot \nabla v(x) - q(x)w'(x)v(x)\big]\ d^n x$$

(b) Prove that

$$h \in H^{1/2}(\partial\Omega) \mapsto \int_\Omega \big[\nabla w_h(x) \cdot \nabla v(x) - q(x)w_h(x)v(x)\big]\ d^n x \quad \text{with } w_h \in H^1(\Omega),\ Rw_h = h$$

is a well–defined, bounded linear functional on $H^{1/2}(\Omega)$ with norm at most

$$C\big[1 + \|q\|_{L^\infty(\Omega)}\big]\|v\|_{1,\Omega}$$

(c) Prove that if $v \in H^2(\Omega)$, then the linear functional of part (b) is

$$h \in H^{1/2}(\partial\Omega) \mapsto \int_{\partial\Omega} h(x)\tfrac{\partial v}{\partial \nu}(x)\ d\sigma(x)$$

Solution. (a) Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$ there is a sequence $v_i \in C^\infty(\overline{\Omega})$ that converges in $H^1(\Omega)$ to v. Then Δv_i converges in $H^{-1}(\Omega)$ to Δv and qv_i converges in $L^2(\Omega)$ to qv so that $(\Delta + q)v_i = F_i$ converges in $H^{-1}(\Omega)$ to $(\Delta + q)v = 0$. Furthermore ∇v_i converges in $L^2(\Omega)$ to ∇v so that

$$\begin{aligned}
&\int_\Omega \big[\nabla w(x) \cdot \nabla v(x) - q(x)w(x)v(x)\big]\ d^n x - \int_\Omega \big[\nabla w'(x) \cdot \nabla v(x) - q(x)w'(x)v(x)\big]\ d^n x \\
&= \lim_{i\to\infty} \int_\Omega \big[\{\nabla w(x) - \nabla w'(x)\} \cdot \nabla v_i(x) - q(x)\{w(x) - w'(x)\}v_i(x)\big]\ d^n x \\
&= \lim_{i\to\infty} \int_\Omega \big[-\{w(x) - w'(x)\}\Delta v_i(x) - q(x)\{w(x) - w'(x)\}v_i(x)\big]\ d^n x
\end{aligned}$$

by the divergence theorem, since w and w' coincide on $\partial\Omega$. The desired equation now follows from

$$\begin{aligned}
&\lim_{i\to 0}\left| \int_\Omega \big[-\{w(x) - w'(x)\}\Delta v_i(x) - q(x)\{w(x) - w'(x)\}v_i(x)\big]\ d^n x\right| \\
&\qquad = \lim_{i\to 0}\left|\int_\Omega \{w(x) - w'(x)\}F_i(x)\ d^n x\right| \le \|w - w'\|_{1,\Omega} \limsup_{i\to 0} \|F_i\|_{-1,\Omega} = 0
\end{aligned}$$

(b) The well–definedness was proven in part (a). The linearity is obvious, since if $f, f' \in H^{1/2}(\partial\Omega)$ and $\alpha, \alpha' \in \mathbb{C}$ and if $w_f, w_{f'} \in H^1(\Omega)$ obey $Rw_f = f$ and $Rw_{f'} = f'$, we may always choose $w_{\alpha f+\alpha' f'} = \alpha w_f + \alpha' w_{f'}$. By Theorem 2.2.2, there is a constant C such that we may always choose w_h to obey $\|w_h\|_{1,\Omega} \le C\|h\|_{1/2,\partial\Omega}$. With this choice of w_h

$$\begin{aligned}
\Big|\int_\Omega \big[\nabla w_h(x)\cdot\nabla v(x) - q(x)w_h(x)v(x)\big]\ d^n x\Big| \\
&\le \|\nabla w_h\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|w_h\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le \big[\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}\big]\|w_h\|_{1,\Omega} \\
&\le C\big[\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}\big]\|h\|_{1/2,\partial\Omega} \\
&\le C\big[1 + \|q\|_{L^\infty(\Omega)}\big]\|v\|_{1,\Omega}\|h\|_{1/2,\partial\Omega}
\end{aligned}$$

which proves boundedness.

(c) By the divergence theorem (part (c) of Problem Problem 3.4.1)

$$\begin{aligned}
\int_{\partial\Omega} h(x)\tfrac{\partial v}{\partial\nu}(x)\ d\sigma(x) &= \int_\Omega \nabla\cdot w_h(x)\nabla v(x)\ d^n x \\
&= \int_\Omega \big[\nabla w_h(x)\cdot\nabla v(x) + w_h(x)\Delta v(x)\big]\ d^n x \\
&= \int_\Omega \big[\nabla w_h(x)\cdot\nabla v(x) - q(x)w_h(x)v(x)\big]\ d^n x
\end{aligned}$$

■