§3. Elliptic Boundary Value Problems and the Dirichlet to Neumann Map

In this chapter, we give the basic existence, uniqueness and regularity proofs for the second order elliptic boundary value problem

$$\begin{aligned} L_\gamma u(x) = \sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big) &= F(x) \qquad &&\text{if } x\in\Omega \\ u(x) &= f(x) &&\text{if } x\in\partial\Omega \end{aligned} \tag{3.1}$$

Ellipticity is the requirement that, for each $x\in\overline{\Omega}$, the matrix of coefficients, $\gamma(x)$, be a symmetric real, strictly positive definite matrix. That is, there is a constant $C>0$ such that

$$\sum_{i,j=1}^{n}\gamma^{i,j}(x)\xi_i\xi_j \ge C|\xi|^2$$

for all $\xi\in\mathbb{R}^n$. Along the way, we shall develop some basic properties of the Dirichlet to Neumann map.

Problem 3.1 Let γ be a symmetric, real, positive definite matrix. Assume that there are constants $C_1, C_2>0$ such that $C_1|\xi|^2\le\sum_{i,j=1}^{n}\gamma^{i,j}\xi_i\xi_j\le C_2|\xi|^2$ for all $\xi\in\mathbb{R}^n$. Prove that

$$C_1|\xi|^2\le\sum_{i,j=1}^{n}\gamma^{i,j}\xi_i\overline{\xi_j}\le C_2|\xi|^2$$

for all $\xi\in\mathbb{C}^n$.

§3.1. Second Order Elliptic Equations in $\mathbb{R}^n$

To get warmed up, we go through the proof of existence, uniqueness and regularity of solutions to

$$(L_{\gamma,\mu}u)(x) = -\sum_{i,j=1}^{n}\tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big)+\mu(x)u(x)=F(x) \tag{3.1.1}$$

on $\mathbb{R}^n$. The hypotheses on γ and μ are given in

Hypotheses 3.1.1 Let, for each $x\in\mathbb{R}^n$, $\gamma(x)=\big[\gamma^{i,j}(x)\big]_{1\le i,j\le n}$ be a symmetric real matrix that is strictly positive definite. Further assume that there is a symmetric, real, strictly positive definite matrix $\gamma_\infty=\big[\gamma_\infty^{i,j}\big]_{1\le i,j\le n}$ such that $\gamma^{i,j}(x)-\gamma_\infty^{i,j}\in\mathcal{S}(\mathbb{R}^n)$ for each $1\le i,j\le n$. Let $\mu(x)$ be a strictly positive function and assume that there is a $\mu_\infty>0$ such that $\mu-\mu_\infty\in\mathcal{S}(\mathbb{R}^n)$.

Problem 3.1.1 Prove that, under Hypotheses 3.1.1, there are constants $C_1 > 0$ and C_2 such that

$$C_1|\xi|^2 \le \sum_{i,j=1}^{n} \gamma^{i,j}(x)\xi_i\overline{\xi_j} \le C_2|\xi|^2 \tag{3.1.2}$$

for all $\xi \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$.

In the special case that $\gamma(x)$ and $\mu(x)$ are independent of x, the Fourier transform of $L_{\gamma,\mu}u = F$ is

$$\Big(\mu + \sum_{i,j=1}^{n} k_i\gamma^{i,j}k_j\Big)\hat{u}(k) = \hat{F}(k)$$

As μ and the matrix γ are positive definite, the factor $\big(\mu + \sum_{i,j=1}^{n} k_i\gamma^{i,j}k_j\big) \ge \mu > 0$ and we may solve for $\hat{u}(k)$ by simply dividing across.

When $\gamma(x)$ and $\mu(x)$ have a nontrivial dependence on x, Fourier transforming does not help much. By way of motivation for the strategy we shall use for handling the general case, first consider the case in which $\mu = 1$ and γ is the identity matrix. Observe that, if $u, v, F \in \mathcal{S}(\mathbb{R}^n)$ with $(-\Delta + 1)u = F$, then

$$\begin{aligned} \int v(x)F(x)\ d^nx &= \int v(x)(-\Delta+1)u(x)\ d^nx \\ &= \int \big[\nabla v(x)\cdot\nabla u(x) + v(x)u(x)\big]\ d^nx \\ &= \langle v, \bar{u}\rangle_{1,n} \end{aligned} \tag{3.1.3}$$

By the Riesz representation theorem, the linear map $u \in H^1(\mathbb{R}^n) \mapsto \mathcal{R}_u \in H^1(\mathbb{R}^n)^*$ given by

$$\mathcal{R}_u(v) = \langle v, \bar{u}\rangle_{1,n}$$

is a well-defined, norm preserving bijection. Recall from Proposition 2.1.10 that the linear map $F \in H^{-1}(\mathbb{R}^n) \mapsto \mathcal{L}_F \in H^1(\mathbb{R}^n)^*$ given by

$$\mathcal{L}_F(v) = \int \hat{v}(k)\hat{F}(-k)\ \tfrac{d^nk}{(2\pi)^n}$$

is also a well-defined, norm preserving bijection. We chose to express the right hand side in terms of Fourier transforms because it is applicable to all $F \in H^{-1}(\mathbb{R}^n)$ and $v \in H^1(\mathbb{R}^n)$. When $v, F \in \mathcal{S}(\mathbb{R}^n)$,

$$\int \hat{v}(k)\hat{F}(-k)\ \tfrac{d^nk}{(2\pi)^n} = \int v(x)F(x)\ d^nx$$

As $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, (3.1.3) says that

$$(-\Delta+1)u = F \iff \mathcal{R}_u = \mathcal{L}_F$$

at least when $u, F \in \mathcal{S}(\mathbb{R}^n)$. By continuity, the same equivalence applies for all $u \in H^1(\mathbb{R}^n)$ and $F \in H^{-1}(\mathbb{R}^n)$. The existence and uniqueness of a solution $u \in H^1(\mathbb{R}^n)$ for any given $F \in H^{-1}(\mathbb{R}^n)$ is now assured by the Riesz representation theorem.

To implement this strategy for general γ and μ, we construct a variant, $\langle\ ,\ \rangle_{\gamma,\mu}$, of $\langle\ ,\ \rangle_{1,n}$, that is adapted to $L_{\gamma,\mu}$ and whose associated norm is equivalent to $|\!|\!|\ \ |\!|\!|_{1,n}$. Then we prove a generalization of (3.1.3).

Lemma 3.1.2 *There is a unique continuous sesquilinear form*

$$\langle\ \cdot\ ,\ \cdot\ \rangle_{\gamma,\mu} : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{C}$$

such that

$$\langle u, v\rangle_{\gamma,\mu} = \sum_{i,j=1}^{n} \int \gamma^{i,j}(x) \tfrac{\partial u}{\partial x_i}(x) \overline{\tfrac{\partial v}{\partial x_j}(x)}\ d^n x + \int \mu(x) u(x) \overline{v(x)}\ d^n x$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$. It is an inner product. The associated norm, $\|v\|_{\gamma,\mu} = \sqrt{\langle v, v\rangle_{\gamma,\mu}}$ is equivalent to $|\!|\!|\ \cdot\ |\!|\!|_{1,n}$.

Proof: Setting

$$K_1 = \min\big\{C_1, \min_{x\in\mathbb{R}^n} \mu(x)\big\} \qquad K_2 = \max\big\{C_2, \max_{x\in\mathbb{R}^n} \mu(x)\big\}$$

with the C_1, C_2 of Problem 3.1.1, we have

$$\begin{aligned} K_1 |\!|\!|u|\!|\!|_{1,n}^2 &= K_1\big[\|u\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^n)}^2\big] \\ &\le \langle u, u\rangle_{\gamma,\mu} = \int_{\mathbb{R}^n} \mu(x)|u(x)|^2\ d^n x + \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \gamma^{i,j}(x) \tfrac{\partial u}{\partial x_i}(x) \overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^n x \\ &\le K_2\big[\|u\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^n)}^2\big] = K_2 |\!|\!|u|\!|\!|_{1,n}^2 \end{aligned} \tag{3.1.4}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. The map

$$(u, v) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \mapsto \langle u, v\rangle_{\gamma,\mu}$$

is obviously linear in u, conjugate linear in v, symmetric and positive definite. Consequently the Cauchy–Schwartz inequality applies to it and, by (3.1.4), the map is also nondegenerate and continuous in the $H^1(\mathbb{R}^n)$ topology. The existence of a unique continuous extension to $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, that the extension is an inner product and the equivalence to the $|\!|\!|\ \cdot\ |\!|\!|_{1,n}$ norm are now all obvious. ■

Problem 3.1.2 Prove that there is a unique continuous sesquilinear form

$$\langle\ \cdot\ ,\ \cdot\ \rangle_{\gamma,0} : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{C}$$

such that

$$\langle u, v\rangle_{\gamma,0} = \sum_{i,j=1}^{n} \int \gamma^{i,j}(x) \tfrac{\partial u}{\partial x_i}(x) \overline{\tfrac{\partial v}{\partial x_j}(x)}\ d^n x$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$. Is it an inner product? If so, is the associated norm equivalent to $|\ \cdot\ |_{1,n}$?

Lemma 3.1.3 *There is a unique bounded linear map* $L_{\gamma,\mu} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ *such that*

$$(L_{\gamma,\mu} u)(x) = -\sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x) \tfrac{\partial u}{\partial x_j}(x)\Big) + \mu(x) u(x)$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. *Furthermore*

$$\mathcal{L}_{L_{\gamma,\mu} u}(v) = \langle v, \bar{u}\rangle_{\gamma,\mu}$$

for all $u, v \in H^1(\mathbb{R}^n)$.

Proof: We shall show that there is a constant C such that

$$\big|\, \langle v, L_{\gamma,\mu} \bar{u}\rangle_{L^2(\mathbb{R}^n)} \,\big| \le C |u|_1 |v|_1$$

for all $u, v \in C_0^\infty(\mathbb{R}^n)$. Since, by Problem 2.1.8, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, this will imply that $\big|\, \langle v, L_{\gamma,\mu} \bar{u}\rangle_{L^2(\mathbb{R}^n)} \,\big| \le C |u|_1 |v|_1$ for all $u \in C_0^\infty(\mathbb{R}^n)$ and $v \in H^1(\mathbb{R}^n)$ and hence that $|L_{\gamma,\mu} u|_{-1} \le C |u|_1$, by part (b) of Problem 2.1.16. Since $C_0^\infty(\mathbb{R}^n)$ is still dense in $H^1(\mathbb{R}^n)$, this will in turn imply that $L_{\gamma,\mu}$ has a unique continuous extension to all of $H^1(\mathbb{R}^n)$ by the B.L.T. theorem.

By the divergence theorem

$$\begin{aligned} \langle v, L_{\gamma,\mu} \bar{u}\rangle_{L^2(\mathbb{R}^n)} &= -\int_{\mathbb{R}^n} v(x) \Big[\sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x) \tfrac{\partial u}{\partial x_j}(x)\Big)\Big]\ d^n x + \int_{\mathbb{R}^n} v(x)\mu(x)u(x)\ d^n x \\ &= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \gamma^{i,j}(x) \tfrac{\partial v}{\partial x_i}(x) \tfrac{\partial u}{\partial x_j}(x)\ d^n x + \int_{\mathbb{R}^n} \mu(x) v(x) u(x)\ d^n x \\ &= \langle v, \bar{u}\rangle_{\gamma,\mu} \end{aligned} \tag{3.1.5}$$

There are no boundary terms because u and v have compact support. Hence, by Cauchy–Schwarz and the equivalence of the norms $\|\ \cdot\ \|_{\gamma,\mu}$ and $|\ \cdot\ |_{1,n}$ that was proven in Lemma 3.1.2,

$$\big|\, \langle v, L_{\gamma,\mu} \bar{u}\rangle_{L^2(\mathbb{R}^n)} \,\big| \le \|v\|_{\gamma,\mu} \|u\|_{\gamma,\mu} \le C |u|_1 |v|_1$$

By (3.1.5),

$$\mathcal{L}_{L_{\gamma,\mu}u}(v) = \langle v, L_{\gamma,\mu}\bar{u}\rangle_{L^2(\mathbb{R}^n)} = \langle v, \bar{u}\rangle_{\gamma,\mu}$$

for all $u, v \in C_0^\infty(\mathbb{R}^n)$. Since both $\mathcal{L}_{L_{\gamma,\mu}u}(v)$ and $\langle v, \bar{u}\rangle_{\gamma,\mu}$ are continuous in both u and v with respect to the $H^1(\mathbb{R}^n)$ topology and $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, we have that $\mathcal{L}_{L_{\gamma,\mu}u}(v) = \langle v, \bar{u}\rangle_{\gamma,\mu}$ for all $u, v \in H^1(\mathbb{R}^n)$. ■

Problem 3.1.3 Prove that there is a unique bounded linear map $L_{\gamma,0} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ such that

$$(L_{\gamma,0}u)(x) = -\sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big)$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 3.1.4 *For each $F \in H^{-1}(\mathbb{R}^n)$ there is a unique $u \in H^1(\mathbb{R}^n)$ such that $L_{\gamma,\mu}u = F$. Furthermore, there is a constant C such that*

$$F \in H^{-1}(\mathbb{R}^n),\ u \in H^1(\mathbb{R}^n),\ L_{\gamma,\mu}u = F \implies |u|_{1,n} \le C|F|_{-1,n}$$

Thus $L_{\gamma,\mu}$ is an isomorphism (1–1, onto, bounded with bounded inverse).

Proof:

Boundedness was proven in Lemma 3.1.3.

Injectiveness: If $u \in H^1(\mathbb{R}^n)$ and $L_{\gamma,\mu}u = 0$, then, by Lemma 3.1.3,

$$0 = \mathcal{L}_{L_{\gamma,\mu}u}(\bar{u}) = \langle \bar{u}, \bar{u}\rangle_{\gamma,\mu}$$

so that $\bar{u} = 0$ and hence $u = 0$.

Surjectiveness: Let $F \in H^{-1}(\mathbb{R}^n)$. Since the norms $|\ \ |_1$ and $\|\ \ \|_{\gamma,\mu}$ are equivalent, $H^1(\mathbb{R}^n)$ equipped with the inner product $\langle\ ,\ \rangle_{\gamma,\mu}$ is a Hilbert space and, by the Riesz representation theorem, there exists $u \in H^1(\mathbb{R}^n)$ such that

$$\mathcal{L}_F(v) = \langle v, \bar{u}\rangle_{\gamma,\mu}$$

for all $v \in H^1(\mathbb{R}^n)$. By Lemma 3.1.3, $\mathcal{L}_F = \mathcal{L}_{L_{\gamma,\mu}u}$. As $\mathcal{L}$ is injective, $F = L_{\gamma,\mu}u$.

Boundedness of the inverse: We must show there is a constant C such that, if $u \in H^1(\mathbb{R}^n)$ and $F = L_{\gamma,\mu}u$, then

$$|u|_1 \le C|F|_{-1}$$

Since $\mathcal{L}$ is norm preserving

$$|F|_{-1} = \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\neq 0}} \frac{|\mathcal{L}_F(v)|}{|v|_1} = \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\neq 0}} \frac{|\langle v,\bar u\rangle_{\gamma,\mu}|}{|v|_1} = \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\neq 0}} \frac{|\langle \bar v,u\rangle_{\gamma,\mu}|}{|\bar v|_1} = \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\neq 0}} \frac{|\langle v,u\rangle_{\gamma,\mu}|}{|v|_1}$$

By the equivalence of the norms $|\ \ |_1$ and $\|\ \ \|_{\gamma,\mu}$

$$|F|_{-1} \ge C^{-1} \sup_{\substack{v\in H^1(\mathbb{R}^n)\\ v\neq 0}} \frac{|\langle v,u\rangle_{\gamma,\mu}|}{\|v\|_{\gamma,\mu}} = C^{-1}\|u\|_{\gamma,\mu}$$

■

We next prove (Proposition 3.1.7) that the solution, $u \in H^1(\mathbb{R}^n)$, of $L_{\gamma,\mu}u = F$ is more regular than F by two derivatives. That is, if $F \in H^{\ell-2}(\mathbb{R}^n)$ then $u \in H^{\ell}(\mathbb{R}^n)$. This is called elliptic regularity. We shall control $\partial^\alpha u$ through difference quotients $\Delta^\alpha_H u$ that approximate $\partial^\alpha u$.

If $u(x)$ is a function on $\mathbb{R}^n$, $0 \neq h \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$, we define

$$\Delta^\alpha_h u(x) = \tfrac{1}{h}[u(x+h\alpha) - u(x)] \tag{3.1.6}$$

More generally, if $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 1$ and all components of $H = \big(h_{m,j}\big)_{\substack{1\le m\le n\\ 1\le j\le \alpha_m}} \in \mathbb{R}^{|\alpha|}$ are nonzero,

$$\Delta^\alpha_H u(x) = \prod_{m=1}^{n}\prod_{j=1}^{\alpha_m} \Delta^{e_m}_{h_{m,j}} u(x) \tag{3.1.7}$$

where e_m is the multiindex in $\mathbb{N}_0^n$ all of whose components are zero except for the m^{th}, which is one.

Problem 3.1.4 Let $h, h' \in \mathbb{R}$ be nonzero and $\alpha, \alpha' \in \mathbb{N}_0^n$ with $|\alpha| = |\alpha'| = 1$. Prove that the operators Δ^α_h and $\Delta^{\alpha'}_{h'}$ commute.

Lemma 3.1.5 *Let $s \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$ and $u \in H^s(\mathbb{R}^n)$. Then $\partial^\alpha u \in H^s(\mathbb{R}^n)$ if and only if $\limsup_{H\to 0} |\Delta^\alpha_H u|_s$ is finite. In this case $|\Delta^\alpha_H u|_s \le |\partial^\alpha u|_s$ for all H and*

$$|\partial^\alpha u|_s = \lim_{H\to 0} |\Delta^\alpha_H u|_s$$

Proof: If $\alpha = e_m$, then by the translation property (e) of the Fourier transform (2.1.1),

$$\widehat{\Delta_h^\alpha u}(k) = \tfrac{1}{h}[e^{\imath k\cdot he_m} - 1]\hat{u}(k) = \imath e^{\imath hk_m/2}\ \tfrac{\sin(hk_m/2)}{h/2}\ \hat{u}(k)$$

Hence for general α

$$\widehat{\Delta_h^\alpha u}(k) = \hat{u}(k)\prod_{m=1}^{n}\prod_{j=1}^{\alpha_m}\imath\tfrac{\sin(h_{m,j}k_m/2)}{h_{m,j}/2}\ e^{\imath h_{m,j}k_m/2}$$

and

$$\big|\widehat{\Delta_h^\alpha u}(k)\big| = |\hat{u}(k)|\prod_{m=1}^{n}\prod_{j=1}^{\alpha_m}\big|\tfrac{\sin(h_{m,j}k_m/2)}{h_{m,j}/2}\big|$$

Since $|\sin x| \le |x|$ and $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$,

$$\big|\widehat{\Delta_h^\alpha u}(k)\big| \le |k^\alpha\hat{u}(k)| \qquad \text{and} \qquad \lim_{H\to 0}\big|\widehat{\Delta_h^\alpha u}(k)\big| = |k^\alpha\hat{u}(k)|$$

for all $k \in \mathbb{R}^n$.

If $\partial^\alpha u \in H^s(\mathbb{R}^n)$, then $\|\partial^\alpha u\|_s^2 = \int_{\mathbb{R}^n}(1+|k|^2)^s\big|k^\alpha\hat{u}(k)\big|^2\,\frac{d^nk}{(2\pi)^n} < \infty$ and, by the Lebesgue dominated convergence theorem,

$$\|\Delta_H^\alpha u\|_s \le \|\partial^\alpha u\|_s \text{ for all } H \text{ and } \lim_{H\to 0}\|\Delta_H^\alpha u\|_s = \|\partial^\alpha u\|_s$$

On the other hand, even if $\int_{\mathbb{R}^n}(1+|k|^2)^s\big|k^\alpha\hat{u}(k)\big|^2\,\frac{d^nk}{(2\pi)^n} = \infty$, the Lebesgue dominated convergence theorem still implies that, for each $R > 0$,

$$\lim_{H\to 0}\int_{|k|<R}(1+|k|^2)^s\big|\widehat{\Delta_h^\alpha u}(k)\big|^2\,\tfrac{d^nk}{(2\pi)^n} = \int_{|k|<R}(1+|k|^2)^s\big|k^\alpha\hat{u}(k)\big|^2\,\tfrac{d^nk}{(2\pi)^n}$$

Consequently

$$\begin{aligned}\limsup_{H\to 0}\|\Delta_H^\alpha u\|_s^2 &\ge \limsup_{H\to 0}\int_{|k|<R}(1+|k|^2)^s\big|\widehat{\Delta_h^\alpha u}(k)\big|^2\,\tfrac{d^nk}{(2\pi)^n}\\ &= \int_{|k|<R}(1+|k|^2)^s\big|k^\alpha\hat{u}(k)\big|^2\,\tfrac{d^nk}{(2\pi)^n}\end{aligned}$$

As this is true for all $R > 0$,

$$\limsup_{H\to 0}\|\Delta_H^\alpha u\|_s^2 \ge \lim_{R\to\infty}\int_{|k|<R}(1+|k|^2)^s\big|k^\alpha\hat{u}(k)\big|^2\,\tfrac{d^nk}{(2\pi)^n} = \|\partial^\alpha u\|_s^2$$

So $\partial^\alpha u \in H^s(\mathbb{R}^n)$ whenever $\limsup\limits_{H\to 0}\|\Delta_H^\alpha u\|_s$ is finite. ■

Lemma 3.1.6 *Let $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$.*

(a) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Also use φ to denote the operator of multiplication by φ acting on $H^s(\mathbb{R}^n)$. Then there is a $C > 0$ (depending on φ, s, n and α, but independent of H) such that $[\Delta_H^\alpha, \varphi] = \Delta_H^\alpha \varphi - \varphi \Delta_H^\alpha$ is a bounded operator from $H^s(\mathbb{R}^n)$ to $H^{s-|\alpha|+1}$ with norm at most C.

(b) Let $L = \sum_{\substack{\beta \in \mathbb{N}_0 \\ |\beta| \leq k}} a_\beta \partial^\beta$ be a differential operator of order k. Assume that, for each $\beta \in \mathbb{N}_0$ with $|\beta| \leq k$, there is a constant $A_\beta \in \mathbb{C}$ such that the $a_\beta - A_\beta \in \mathcal{S}(\mathbb{R}^n)$. Then there is a $C > 0$, independent of H, such that $[\Delta_H^\alpha, L]$ is a bounded operator from $H^s(\mathbb{R}^n)$ to $H^{s-k-|\alpha|+1}(\mathbb{R}^n)$ with norm at most C.

Proof: (a) We first consider $|\alpha| = 1$. Then

$$\begin{aligned}
([\Delta_h^\alpha, \varphi]u)(x) &= (\Delta_h^\alpha \varphi u)(x) - (\varphi \Delta_h^\alpha u)(x) \\
&= \tfrac{1}{h}\big[\varphi(x + h\alpha)u(x + h\alpha) - \varphi(x)u(x) - \varphi(x)u(x + h\alpha) + \varphi(x)u(x)\big] \\
&= \tfrac{1}{h}\big[\varphi(x + h\alpha)u(x + h\alpha) - \varphi(x)u(x + h\alpha)\big] \\
&= \Phi_h(x)u_h(x)
\end{aligned}$$

where $\Phi_h = \Delta_h^\alpha \varphi$ and $u_h(x) = u(x + h\alpha)$ is a translate of u. We have already seen, in the proof of Lemma 3.1.5, that the Fourier transform $|\hat{\Phi}_h(k)| \leq |k^\alpha \hat{\varphi}(k)| \leq (1 + |k|^2)^{\frac{1}{2}} |\hat{\varphi}(k)|$ for all $h \neq 0$. Hence, by Lemma 2.3.5, the norm of Φ_h as a multiplication operator on $H^s(\mathbb{R}^n) = H^{s-|\alpha|+1}(\mathbb{R}^n)$ is at most $C_s = 2^{\frac{|s|}{2}} \int (1 + |p|^2)^{\frac{|s|+1}{2}} |\hat{\varphi}(p)| \, \frac{d^n p}{(2\pi)^n}$ for all $h \neq 0$. The proof, in the case $|\alpha| = 1$, is completed by observing that $\big|u_h\big|_s = \big|u\big|_s$, since $\hat{u}_h(k) = e^{ihk\cdot\alpha}\hat{u}(k)$.

For general α, we use the telescoping sum

$$\begin{aligned}
[A_1 A_2 \cdots A_\ell, B] &= A_1 A_2 \cdots A_\ell B - B A_1 A_2 \cdots A_\ell \\
&= \big(A_1 A_2 \cdots A_\ell B - A_1 A_2 \cdots B A_\ell\big) + \big(A_1 A_2 \cdots B A_\ell - \cdots \\
&\quad + \cdots \qquad - A_1 B A_2 \cdots A_\ell\big) + \big(A_1 B A_2 \cdots A_\ell - B A_1 A_2 \cdots A_\ell\big) \\
&= \sum_{i=1}^{\ell} A_1 \cdots A_{i-1} [A_i, B] A_{i+1} \cdots A_\ell
\end{aligned}$$

to express $[\Delta_h^\alpha, \varphi]$ as a sum of $|\alpha|$ terms, each of the form $\Delta_{H'}^{\alpha'} [\Delta_h^\beta, \varphi] \Delta_{H''}^{\alpha''}$ with $\alpha' + \beta + \alpha'' = \alpha$ and $|\beta| = 1$. By Lemma 3.1.5 and the just proven bound for the special case $|\alpha| = 1$,

$$\begin{aligned}
\big|\Delta_{H'}^{\alpha'} [\Delta_h^\beta, \varphi] \Delta_{H''}^{\alpha''} u\big|_{s-|\alpha|+1} &\leq \big|[\Delta_h^\beta, \varphi] \Delta_{H''}^{\alpha''} u\big|_{s-|\alpha|+|\alpha'|+1} \\
&\leq C_{s-|\alpha|+|\alpha'|+1} \big|\Delta_{H''}^{\alpha''} u\big|_{s-|\alpha|+|\alpha'|+1} \\
&\leq C_{s-|\alpha|+|\alpha'|+1} \big|u\big|_{s-|\alpha|+|\alpha'|+|\alpha''|+1} \\
&= C_{s-|\alpha''|} \big|u\big|_s
\end{aligned}$$

So it suffices to take $C = |\alpha| \max\limits_{0\le j\le|\alpha|} C_{s-j}$.

(b) As the differential operator ∂^β commutes with translation, it also commutes with Δ_H^α. Consequently,

$$[\Delta_H^\alpha, L] = \sum_{\substack{\beta\in\mathbb{N}_0 \\ |\beta|\le k}} [\Delta_H^\alpha, a_\beta]\partial^\beta$$

As multiplication by the constant A_β also commutes with Δ_H^α, we may assume, without loss of generality, that all of the A_β's are zero. So, by part (a) of this lemma

$$\begin{aligned}\big|[\Delta_H^\alpha, L]u\big|_{s-k-|\alpha|+1} &\le \sum_{\substack{\beta\in\mathbb{N}_0 \\ |\beta|\le k}} \big|[\Delta_H^\alpha, a_\beta]\partial^\beta u\big|_{s-k-|\alpha|+1} \le \sum_{\substack{\beta\in\mathbb{N}_0 \\ |\beta|\le k}} C\,\big|\partial^\beta u\big|_{s-k} \le \sum_{\substack{\beta\in\mathbb{N}_0 \\ |\beta|\le k}} C\,|u|_{s-k+|\beta|} \\ &\le C n^{k+1} |u|_s\end{aligned}$$

since $|\beta| \le k$ and there are at most n^{k+1} terms in the sum. ■

Proposition 3.1.7 (Elliptic Regularity in $\mathbb{R}^n$) *Let $\gamma(x)$ and μ obey Hypotheses 3.1.1. Let $\ell \in \mathbb{N}$. There is a constant C, depending only on ℓ, γ and μ such that, for all $u \in H^1(\mathbb{R}^n)$ and $F \in H^{\ell-2}(\mathbb{R}^n)$ obeying $L_{\gamma,\mu}u = F$ we have $u \in H^\ell(\mathbb{R}^n)$ and*

$$|u|_\ell \le C\,|F|_{\ell-2}$$

Proof: We use induction. We proved the result for $\ell = 1$ in Theorem 3.1.4. Assume that we have proven the desired result for some $\ell \in \mathbb{N}$. We now prove it for $\ell + 1$. That is, we assume that for all $u \in H^1(\mathbb{R}^n)$ and $F \in H^{\ell-2}(\mathbb{R}^n)$ obeying $L_{\gamma,\mu}u = F$ we have $u \in H^\ell(\mathbb{R}^n)$ and $|u|_\ell \le C\,|F|_{\ell-2}$. We must prove that if $u \in H^1(\mathbb{R}^n)$, $F \in H^{\ell-1}(\mathbb{R}^n)$ and $L_{\gamma,\mu}u = F$, then $u \in H^{\ell+1}(\mathbb{R}^n)$ and $|u|_{\ell+1} \le C'\,|F|_{\ell-1}$.

So let $u \in H^1(\mathbb{R}^n)$ and $F \in H^{\ell-1}(\mathbb{R}^n)$ with $L_{\gamma,\mu}u = F$. For each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and each $h \in \mathbb{R}\setminus\{0\}$, set $u_{\alpha,h} = \Delta_h^\alpha u$. Then

$$L_{\gamma,\mu}u_{\alpha,h} = F_{\alpha,h} \text{ where } F_{\alpha,h} = \Delta_h^\alpha F + [L_{\gamma,\mu}, \Delta_h^\alpha]u$$

By the inductive hypothesis, $u \in H^\ell(\mathbb{R}^n)$ with $|u|_\ell \le C\,|F|_{\ell-2}$. Hence by Lemma 3.1.6.b with $s = \ell$ and $k = 2$, $[L_{\gamma,\mu}, \Delta_h^\alpha]u \in H^{\ell-2}(\mathbb{R}^n)$ with

$$\big|[L_{\gamma,\mu}, \Delta_h^\alpha]u\big|_{\ell-2} \le C_1|u|_\ell \le C_1C\,|F|_{\ell-2} \le C_1C\,|F|_{\ell-1}$$

By Lemma 3.1.5, with $s = \ell - 2$,

$$\big|\Delta_h^\alpha F\big|_{\ell-2} \le \big|\partial^\alpha F\big|_{\ell-2} \le |F|_{\ell-1}$$

Hence $F_{\alpha,h} \in H^{\ell-2}(\mathbb{R}^n)$ with $|F_{\alpha,h}|_{\ell-2} \le (1+C_1C)|F|_{\ell-1}$ and, by the inductive hypothesis, $u_{\alpha,h} = \Delta_h^\alpha u \in H^\ell(\mathbb{R}^n)$ with

$$|\Delta_h^\alpha u|_\ell \le C\,|F_{\alpha,h}|_{\ell-2} \le C(1+C_1C)|F|_{\ell-1}$$

By Lemma 3.1.5, with $s=\ell$, $\partial^\alpha u \in H^\ell(\mathbb{R}^n)$ with $|\partial^\alpha u|_\ell \le C(1+C_1C)|F|_{\ell-1}$. Hence $u \in H^{\ell+1}(\mathbb{R}^n)$ and

$$|u|_{\ell+1}^2 = |u|_\ell^2 + \sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|=1}} |\partial^\alpha u|_\ell^2 \le C^2|F|_{\ell-2}^2 + nC^2(1+C_1C)^2|F|_{\ell-1}^2 \le C'^2|F|_{\ell-1}^2$$

with $C'^2 = C^2 + nC^2(1+C_1C)^2$. ■

Problem 3.1.5 Let $\gamma(x)$ obey Hypotheses 3.1.1 and let $\ell \in \mathbb{N}$. Prove that there is a constant C, depending only on ℓ and γ such that, for all $u \in H^1(\mathbb{R}^n)$ and $F \in H^{\ell-2}(\mathbb{R}^n)$ obeying $L_{\gamma,0}u = F$ (setting $\mu = 0$ eliminates the $\mu(x)u(x)$ term from the differential equation) we have $u \in H^\ell(\mathbb{R}^n)$ and

$$|u|_\ell \le C\Big(|F|_{\ell-2} + |u|_{-1}\Big) \le C\Big(|F|_{\ell-2} + |u|_1\Big)$$

Problem 3.1.6 In this problem we explore the need for the $\mu(x)u(x)$ term in $L_{\gamma,\mu}u = F$.

(a) Find a $\gamma(x)$ obeying Hypotheses 3.1.1 and an $F \in H^{-1}(\mathbb{R}^n)$ such that **no** $u \in H^1(\mathbb{R}^n)$ obeys $L_{\gamma,0}u = F$.

(b) Find a $\gamma(x)$ obeying Hypotheses 3.1.1 such that there does **not** exist a constant C obeying $|u|_{1,n} \le C|F|_{-1,n}$ for all $F \in H^{-1}(\mathbb{R}^n)$ and $u \in H^1(\mathbb{R}^n)$ with $L_{\gamma,0}u = F$.

(c) Let $\gamma(x)$ obey Hypotheses 3.1.1. Prove that the map $L_{\gamma,0} : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ is bounded and injective, but that it is NOT surjective. Prove that the inverse map $L_{\gamma,0}^{-1}$ (defined on the range of $L_{\gamma,0}$) is NOT bounded.

(d) Let $\gamma(x)$ obey Hypotheses 3.1.1, let $\ell \in \mathbb{N}$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Prove that there is a constant C, depending only on ℓ, γ and Ω such that, for all $F \in H^{\ell-2}(\mathbb{R}^n)$ and all $u \in H^1(\mathbb{R}^n)$ that vanish outside of Ω and obey $L_{\gamma,0}u = F$ we have $u \in H^\ell(\mathbb{R}^n)$ and

$$|u|_\ell \le C|F|_{\ell-2}$$

§3.2. Second Order Elliptic Equations in $\mathbb{R}^n_+$

We now introduce a simple boundary and study the boundary value problem

$$\begin{aligned} -\sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big) + \mu(x)u(x) = F(x) &\qquad \text{if } x \in \mathbb{R}^n_+ \\ u(x) = f(x) &\qquad \text{if } x \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \end{aligned} \tag{3.2.1}$$

Throughout this section we assume that $\gamma(x)$ and $\mu(x)$ are restrictions to $\mathbb{R}^n_+$ of functions satisfying Hypotheses 3.1.1.

Lemma 3.2.1 *There is a unique continuous sesquilinear form*

$$\langle\ \cdot\ ,\ \cdot\ \rangle_{\gamma,\mu,+} : H^1(\mathbb{R}^n_+) \times H^1(\mathbb{R}^n_+) \to \mathbb{C}$$

such that

$$\langle u, v\rangle_{\gamma,\mu,+} = \sum_{i,j=1}^n \int_{\mathbb{R}^n_+} \gamma^{i,j}(x) \tfrac{\partial u}{\partial x_i}(x) \overline{\tfrac{\partial v}{\partial x_j}(x)}\ d^n x + \int_{\mathbb{R}^n_+} \mu(x) u(x) \overline{v(x)}\ d^n x$$

for all u, v that are restrictions to $\mathbb{R}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$. It is an inner product. The associated norm, $\|v\|_{\gamma,\mu,+} = \sqrt{\langle v, v\rangle_{\gamma,\mu,+}}$ is equivalent to $\|\ \cdot\ \|_{1,\mathbb{R}^n_+}$.

Proof: The proof is virtually identical to that of Lemma 3.1.2. ■

Lemma 3.2.2 *There is a unique bounded linear map $L_{\gamma,\mu,+} : H^1(\mathbb{R}^n_+) \to H^{-1}(\mathbb{R}^n_+) = H^1_0(\mathbb{R}^n_+)^*$ such that*

$$(L_{\gamma,\mu,+} u)(v) = -\sum_{i,j=1}^n \int_{\mathbb{R}^n_+} v(x) \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x) \tfrac{\partial u}{\partial x_j}(x)\Big)\ d^n x + \int_{\mathbb{R}^n_+} \mu(x) u(x) v(x)\ d^n x$$

for all u that are restrictions to $\mathbb{R}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$ and all $v \in C_0^\infty(\mathbb{R}^n_+)$. Furthermore

$$(L_{\gamma,\mu,+} u)(v) = \langle v, \bar{u}\rangle_{\gamma,\mu,+} \tag{3.2.2}$$

for all $u \in H^1(\mathbb{R}^n_+)$ and all $v \in H^1_0(\mathbb{R}^n_+)$.

Proof: The proof is virtually identical to that of Lemma 3.1.3. ■

Theorem 3.2.3 *Let $L_{\gamma,\mu,+} : H^1(\mathbb{R}^n_+) \to H^{-1}(\mathbb{R}^n_+)$ be as in Lemma 3.2.2 and let $R : H^1(\mathbb{R}^n_+) \to H^{1/2}(\mathbb{R}^{n-1})$ be the restriction map of Proposition 2.2.10. Then the map*

$$\begin{aligned} H^1(\mathbb{R}^n_+) &\to H^{-1}(\mathbb{R}^n_+) \oplus H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \\ u &\mapsto \quad \big(L_{\gamma,\mu,+} u\ ,\ Ru\big) \end{aligned}$$

is an isomorphism (1–1, onto, bounded with bounded inverse). That is, for each $F \in H^{-1}(\mathbb{R}^n_+)$ and $f \in H^{1/2}(\mathbb{R}^{n-1})$ there is a unique $u \in H^1(\mathbb{R}^n_+)$ such that

$$\begin{aligned} -\sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x) \tfrac{\partial u}{\partial x_j}(x)\Big) + \mu(x) u(x) &= F \qquad \text{in } \mathbb{R}^n_+ \\ u\big|_{\mathbb{R}^{n-1}} &= f \end{aligned}$$

Furthermore, there is a constant C such that

$$\|u\|^2_{1,\mathbb{R}^n_+} \leq C\big(\|F\|^2_{-1,\mathbb{R}^n_+} + |f|^2_{1/2,n-1}\big)$$

for all $F \in H^{-1}(\mathbb{R}^n_+)$ and $f \in H^{1/2}(\mathbb{R}^{n-1})$.

Proof:
Boundedness: $\|L_{\gamma,\mu,+}u\|_{-1,\mathbb{R}^n_+} \leq c_1\|u\|_{1,\mathbb{R}^n_+}$ by Lemma 3.2.2 and $|Ru|_{\frac{1}{2},n-1} \leq c_2\|u\|_{1,\mathbb{R}^n_+}$ by part (ii) of Proposition 2.2.10.
Injectiveness: If $Ru = 0$, then $u \in H^1_0(\mathbb{R}^n_+)$ by Proposition 2.2.11. If $u \in H^1_0(\mathbb{R}^n_+)$ and $L_{\gamma,\mu,+}u = 0$, then, by Lemma 3.2.2,

$$0 = (L_{\gamma,\mu,+}u)(\bar{u}) = \langle \bar{u}, \bar{u}\rangle_{\gamma,\mu,+}$$

so that $\bar{u} = 0$ and hence $u = 0$.
Surjectiveness: Let $F \in H^{-1}(\mathbb{R}^n_+) = H^1_0(\mathbb{R}^n_+)^*$ and $f \in H^{1/2}(\mathbb{R}^{n-1})$. Since the norms $\|\ \|_{1,\mathbb{R}^n_+}$ and $\|\ \|_{\gamma,\mu,+}$ are equivalent, $H^1_0(\mathbb{R}^n_+)$, equipped with the inner product $\langle\ ,\ \rangle_{\gamma,\mu,+}$, is still a Hilbert space and F is still a bounded linear functional on this Hilbert space. By the Riesz representation theorem, there exists $w_1 \in H^1_0(\mathbb{R}^n_+)$ such that

$$Fv = \langle v, w_1\rangle_{\gamma,\mu,+}$$

for all $v \in H^1_0(\mathbb{R}^n_+)$. By Lemma 3.2.2, $F = L_{\gamma,\mu,+}u_1$ for $u_1 = \bar{w}_1$.

By part (iii) of Proposition 2.2.10, R is onto. So there is a $w_2 \in H^1(\mathbb{R}^n_+)$ such that $Rw_2 = f$. Since $H^1(\mathbb{R}^n_+)$ is still a Hilbert space when equipped with the inner product $\langle\ ,\ \rangle_{\gamma,\mu,+}$ and $H^1_0(\mathbb{R}^n_+)$ is still a closed linear subspace of this Hilbert space, there exist unique $w_0 \in H^1_0(\mathbb{R}^n_+)$ and $u_2 \in H^1_0(\mathbb{R}^n_+)^{\gamma,\mu\perp}$ such that $w_2 = u_2 + w_0$. Here $H^1_0(\mathbb{R}^n_+)^{\gamma,\mu,\perp}$ is the orthogonal complement of $H^1_0(\mathbb{R}^n_+)$ in $H^1(\mathbb{R}^n_+)$ with respect to the inner product $\langle\ ,\ \rangle_{\gamma,\mu,+}$.

Set $u = u_1 + u_2$. By (3.2.2), $L_{\gamma,\mu,+}$ annihilates $H^1_0(\mathbb{R}^n_+)^{\gamma,\mu,\perp}$ so that $L_{\gamma,\mu,+}u = L_{\gamma,\mu,+}u_1 = F$. As R annihilates $H^1_0(\mathbb{R}^n_+)$,

$$Ru = Ru_1 + Ru_2 = Ru_1 + R(w_2 - w_0) = Rw_2 = f$$

Boundedness of the inverse: We must prove the existence of a constant C such that if $u \in H^1(\mathbb{R}^n_+)$, $F = L_{\gamma,\mu,+}u$ and $f = Ru$ then

$$\|u\|^2_{1,\mathbb{R}^n_+} \leq C\Big(\|F\|^2_{-1,\mathbb{R}^n_+} + |f|^2_{\frac{1}{2},n-1}\Big)$$

Write $u = u_1 + u_2$ with $u_1 \in H^1_0(\mathbb{R}^n_+)$ and $u_2 \in H^1_0(\mathbb{R}^n_+)^{\gamma,\mu,\perp}$. Again, $H^1_0(\mathbb{R}^n_+)^{\gamma,\mu,\perp}$ is the orthogonal complement of $H^1_0(\mathbb{R}^n_+)$ in $H^1(\mathbb{R}^n_+)$ with respect to the inner product $\langle\ ,\ \rangle_{\gamma,\mu,+}$. By (3.2.2),

$$\begin{aligned}
\|F\|_{-1,\mathbb{R}^n_+} &= \sup_{\substack{v\in H^1_0(\mathbb{R}^n_+)\\ v\neq 0}} \frac{|Fv|}{\|v\|_{1,\mathbb{R}^n_+}} = \sup_{\substack{v\in H^1_0(\mathbb{R}^n_+)\\ v\neq 0}} \frac{|\langle v,\bar{u}\rangle_{\gamma,\mu,+}|}{\|v\|_{1,\mathbb{R}^n_+}} = \sup_{\substack{v\in H^1_0(\mathbb{R}^n_+)\\ v\neq 0}} \frac{|\langle \bar{v},u\rangle_{\gamma,\mu,+}|}{\|\bar{v}\|_{1,\mathbb{R}^n_+}}\\
&= \sup_{\substack{v\in H^1_0(\mathbb{R}^n_+)\\ v\neq 0}} \frac{|\langle v,u\rangle_{\gamma,\mu,+}|}{\|v\|_{1,\mathbb{R}^n_+}}
\end{aligned}$$

By the equivalence of the norms $\|\ \|_{1,\mathbb{R}^n_+}$ and $\|\ \|_{\gamma,\mu,+}$

$$\|F\|_{-1,\mathbb{R}^n_+} \geq c_3 \sup_{\substack{v\in H^1_0(\mathbb{R}^n_+)\\ v\neq 0}} \frac{|\langle v,u\rangle_{\gamma,\mu,+}|}{\|v\|_{\gamma,\mu,+}} = c_3 \sup_{\substack{v\in H^1_0(\mathbb{R}^n_+)\\ v\neq 0}} \frac{|\langle v,u_1\rangle_{\gamma,\mu,+}|}{\|v\|_{\gamma,\mu,+}} = c_3\|u_1\|_{\gamma,\mu,+}$$

By part (iv) of Proposition 2.2.10, which gives the existence of a bounded right inverse for R, there is a $w \in H^1(\mathbb{R}^n_+)$ such that $Rw = f$ and

$$\|w\|_{\gamma,\mu,+} \leq c_4\|w\|_{1,\mathbb{R}^n_+} \leq c_5 |f|_{\frac{1}{2},n-1}$$

As $Ru_2 = Ru = f$ and $Rw = f$, $w - u_2$ is in the kernel of R, which is $H^1_0(\mathbb{R}^n_+)$. As $w - u_2$ and u_2 are perpendicular to each other with respect to the inner product $\langle\ ,\ \rangle_{\gamma,\mu,+}$,

$$\|w\|^2_{\gamma,mu,+} = \|u_2\|^2_{\gamma,\mu,+} + \|w-u_2\|^2_{\gamma,\mu,+} \geq \|u_2\|^2_{\gamma,\mu,+}$$

so that $\|u_2\|^2_{\gamma,\mu,+} \leq c_5^2 |f|_{\frac{1}{2},n-1}$. All together

$$\begin{aligned}
\|F\|^2_{-1,\mathbb{R}^n_+} + |f|^2_{\frac{1}{2},n-1} &\geq c_3^2\|u_1\|^2_{\gamma,\mu,+} + \tfrac{1}{c_5^2}\|u_2\|^2_{\gamma,\mu,+} \geq \min\left\{c_3^2, \tfrac{1}{c_5^2}\right\}\left[\|u_1\|^2_{\gamma,\mu,+} + \|u_2\|^2_{\gamma,\mu,+}\right]\\
&= \min\left\{c_3^2, \tfrac{1}{c_5^2}\right\}\|u\|^2_{\gamma,\mu,+} \qquad \geq c_6\|u\|^2_{1,\mathbb{R}^n_+}
\end{aligned}$$

■

Remark 3.2.4 By (3.2.2), the kernel of $L_{\gamma,\mu,+}$ is $H^1_0(\mathbb{R}^n_+)^{\gamma,\mu,\perp}$, the orthogonal complement of $H^1_0(\mathbb{R}^n_+)$ in $H^1(\mathbb{R}^n_+)$ with respect to the inner product $\langle\ ,\ \rangle_{\gamma,\mu,+}$. So, if $f \in H^{1/2}(\mathbb{R}^{n-1})$, then there exists a unique $u \in H^1_0(\mathbb{R}^n_+)^\perp$ such that

$$L_{\gamma,\mu,+}u = 0 \qquad Ru = f$$

The kernel of R is $H^1_0(\mathbb{R}^n_+)$. So if $F \in H^{-1}(\mathbb{R}^n_+)$, then there exists a unique $u \in H^1_0(\mathbb{R}^n_+)$ such that

$$L_{\gamma,\mu,+}u = F \qquad Ru = 0$$

In preparation for proving elliptic regularity in $\mathbb{R}^n_+$, we prove analogs of Lemmas 3.1.5 and 3.1.6.

Lemma 3.2.5 *Let* $\ell \in \mathbb{N}_0$*,* $u \in H^\ell(\mathbb{R}^n_+)$ *and* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \geq 1$ *and* $\alpha_n = 0$*. Then* $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$ *if and only if* $\limsup\limits_{H\to 0} \|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+}$ *is finite. There is a constant* C*, depending only on* ℓ, α *and* n *such that, in this case,* $\|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+} \leq C\|\partial^\alpha u\|_{\ell,\mathbb{R}^n_+}$ *for all* H *and*

$$\|\partial^\alpha u\|_{\ell,\mathbb{R}^n_+} \leq C \limsup_{H\to 0} \|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+}$$

Proof: Let $E : H^\ell(\mathbb{R}^n_+) \to H^\ell(\mathbb{R}^n)$ be the extension operator of Lemma 2.2.8. First suppose that $\limsup\limits_{H\to 0} \|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+}$ is finite. Then $\limsup\limits_{H\to 0} |E\Delta_H^\alpha u|_\ell$ is also finite. Since $\alpha_n = 0$, the construction of (2.2.1) gives $E\Delta_H^\alpha u = \Delta_H^\alpha Eu$. Thus $\limsup\limits_{H\to 0} |\Delta_H^\alpha Eu|_\ell$ is finite too. By Lemma 3.1.5, $\partial^\alpha Eu \in H^\ell(\mathbb{R}^n)$ and

$$|\partial^\alpha Eu|_\ell = \lim_{H\to 0} |\Delta_H^\alpha Eu|_\ell = \lim_{H\to 0} |E\Delta_H^\alpha u|_\ell \leq C \limsup_{H\to 0} \|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+}$$

by Lemma 2.2.8. By Problem 2.2.7, $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|\partial^\alpha u\|_{\ell,\mathbb{R}^n_+} \leq |E\partial^\alpha u|_\ell = |\partial^\alpha Eu|_\ell \leq C \limsup_{H\to 0} \|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+}$$

Conversely, suppose that $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$. By Problem 2.2.7, $\partial^\alpha Eu \in H^\ell(\mathbb{R}^n)$ and $E\partial^\alpha u = \partial^\alpha Eu$. Then, since $\alpha_n = 0$,

$$\begin{aligned} \|\Delta_H^\alpha u\|_{\ell,\mathbb{R}^n_+} &\leq |E\Delta_H^\alpha u|_\ell = |\Delta_H^\alpha Eu|_\ell \leq |\partial^\alpha Eu|_\ell \qquad \text{(by Lemma 3.1.5)} \\ &= |E\partial^\alpha u|_\ell \ \leq C\|\partial^\alpha u\|_{\ell,\mathbb{R}^n_+} \end{aligned}$$

■

Lemma 3.2.6 *Let* $\ell \in \mathbb{N}_0$*,* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \geq 1$ *and* $\alpha_n = 0$*.*

(a) Let $\varphi \in C(\mathbb{R}^n_+)$ *and all of its derivatives of order at most* $\ell + |\alpha|$ *be bounded and continuous. Also use* φ *to denote the operator of multiplication by* φ *acting on* $H^\ell(\mathbb{R}^n_+)$*. Then there is a* $C > 0$ *(depending on* φ*,* ℓ*,* n *and* α*, but independent of* H*) such that*

$$\big\|[\Delta_H^\alpha, \varphi]u\big\|_{\ell,\mathbb{R}^n_+} \leq \sum_{\substack{\beta\leq\alpha \\ |\beta|<|\alpha|}} C\|\partial^\beta u\|_{\ell,\mathbb{R}^n_+}$$

on $\left\{\ u \in H^\ell(\mathbb{R}^n_+)\ \middle|\ \partial^\beta u \in H^\ell(\mathbb{R}^n_+) \text{ for all } \beta \leq \alpha \text{ with } |\beta| < |\alpha|\ \right\}$*.*

(b) Let $L = \sum_{\beta\in B} a_\beta \partial^\beta$*, with* $B \subset \left\{\ \beta \in \mathbb{N}_0^n\ \middle|\ |\beta| \leq k\ \right\}$*, be a differential operator of order* k*. Assume that, for each* $\beta \in B$*,* $a_\beta \in C(\mathbb{R}^n_+)$ *with all of its derivatives of order at most* $\ell + |\alpha|$ *bounded and continuous. Then there is a* $C > 0$*, independent of* H*, such that*

$$\big\|[\Delta_H^\alpha, L]u\big\|_{\ell,\mathbb{R}^n_+} \leq \sum_{\beta\in B} \sum_{\substack{\alpha'\leq\alpha \\ |\alpha'|<|\alpha|}} C\|\partial^{\alpha'+\beta} u\|_{\ell,\mathbb{R}^n_+}$$

on $\left\{\ u \in H^\ell(\mathbb{R}^n_+)\ \middle|\ \partial^{\alpha'+\beta} u \in H^\ell(\mathbb{R}^n_+) \text{ for all } \alpha' \leq \alpha \text{ with } |\alpha'| < |\alpha| \text{ and all } \beta \in B\ \right\}$*.*

Proof: (a) We first consider $|\alpha| = 1$. We have already seen, in the proof of Lemma 3.1.6, that

$$([\Delta_h^\alpha, \varphi]u)(x) = \Phi_h^\alpha(x) u_h(x) \tag{3.2.3}$$

where $\Phi_h^\alpha = \Delta_h^\alpha \varphi$ and $u_h(x) = u(x + h\alpha)$ is a translate of u. In general, the norm of a multiplication operator ψ acting on $H^\ell(\mathbb{R}^n_+)$ is bounded by a constant, depending on n and ℓ, times $\sup\{\ |\partial^\beta \psi(x)|\ \big|\ x \in \mathbb{R}^n_+,\ |\beta| \le \ell\ \}$. See the solution to part (b) of Problem 2.1.3. By the mean value theorem,

$$\sup\{\ |\partial^\beta \Delta_h^\alpha \varphi(x)|\ \big|\ x \in \mathbb{R}^n_+,\ |\beta| \le \ell\ \} \le \sup\{\ |\partial^\beta \varphi(x)|\ \big|\ x \in \mathbb{R}^n_+,\ |\beta| \le \ell + |\alpha|\ \} \tag{3.2.4}$$

So the proof, for $|\alpha| = 1$, is completed by observing that

$$\|u_h\|_{\ell,\mathbb{R}^n_+} = \|u\|_{\ell,\mathbb{R}^n_+} \tag{3.2.5}$$

when $\alpha_n = 0$.

For general α, we use, as in the proof of Lemma 3.1.6,

$$[A_1 A_2 \cdots A_\ell, B] = \sum_{j=1}^{\ell} A_1 \cdots A_{i-1} [A_i, B] A_{i+1} \cdots A_\ell$$

to express $[\Delta_h^\alpha, \varphi]$ as a sum of $|\alpha|$ terms, each of the form $\Delta_{H''}^\gamma [\Delta_h^{\alpha'}, \varphi] \Delta_{H'}^{\beta'}$ with $\gamma + \alpha' + \beta' = \alpha$ and $|\alpha'| = 1$. Denote $v = \Delta_{H'}^{\beta'} u$. The desired bound is provided by

$$\begin{aligned}
\big\|\Delta_{H''}^\gamma [\Delta_h^{\alpha'}, \varphi] \Delta_{H'}^{\beta'} u\big\|_{\ell,\mathbb{R}^n_+} &\le C_1 \big\|\partial^\gamma [\Delta_h^{\alpha'}, \varphi] \Delta_{H'}^{\beta'} u\big\|_{\ell,\mathbb{R}^n_+} && \text{by Lemma 3.2.5} \\
&\le C_2 \sum_{\beta'' \le \gamma} \big\|(\partial^{\gamma - \beta''} \Phi_h^{\alpha'}) \partial^{\beta''} v_h\big\|_{\ell,\mathbb{R}^n_+} && \text{by (3.2.3), the product rule} \\
&\le C_3 \sum_{\beta'' \le \gamma} \big\|\partial^{\beta''} \Delta_{H'}^{\beta'} u\big\|_{\ell,\mathbb{R}^n_+} && \text{by (3.2.4), (3.2.5)} \\
&= C_3 \sum_{\beta'' \le \gamma} \big\|\Delta_{H'}^{\beta'} \partial^{\beta''} u\big\|_{\ell,\mathbb{R}^n_+} \\
&\le C_4 \sum_{\beta'' \le \gamma} \big\|\partial^{\beta' + \beta''} u\big\|_{\ell,\mathbb{R}^n_+} && \text{by Lemma 3.2.5}
\end{aligned}$$

(b) As the differential operator ∂^β commutes with translation, it also commutes with Δ_H^α. Consequently,

$$[\Delta_H^\alpha, L] = \sum_{\beta \in B} [\Delta_H^\alpha, a_\beta] \partial^\beta$$

So just apply part (a) of this lemma. ■

Proposition 3.2.7 (Elliptic Regularity in $\mathbb{R}^n_+$) *Let $\ell \in \mathbb{N}$. There is a constant C, depending only on ℓ, γ and μ such that, for all $u \in H^1(\mathbb{R}^n_+)$, $F \in H^{\ell-2}(\mathbb{R}^n_+)$ and $f \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$ obeying*

$$L_{\gamma,\mu,+}u = F \qquad Ru = f$$

we have $u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|u\|_{\ell,\mathbb{R}^n_+} \le C\Big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1}\Big)$$

Proof: We proceed by induction. We proved the result for $\ell = 1$ in Theorem 3.2.3. Assume that we have proven the desired result for some $\ell \in \mathbb{N}$. We now prove it for $\ell+1$. That is, we assume that for all $u \in H^1(\mathbb{R}^n_+)$, $F \in H^{\ell-2}(\mathbb{R}^n_+)$, $f \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$ obeying $L_{\gamma,\mu,+}u = F$, $Ru = f$ we have $u \in H^\ell(\mathbb{R}^n_+)$ and $\|u\|_{\ell,\mathbb{R}^n_+} \le C\big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1}\big)$. We must prove that if $u \in H^1(\mathbb{R}^n_+)$, $F \in H^{\ell-1}(\mathbb{R}^n_+)$, $f \in H^{\ell+\frac{1}{2}}(\mathbb{R}^{n-1})$, $L_{\gamma,\mu,+}u = F$ and $Ru = f$, then $u \in H^{\ell+1}(\mathbb{R}^n_+)$ and $\|u\|_{\ell+1,\mathbb{R}^n_+} \le C'\big(\|F\|_{\ell-1,\mathbb{R}^n_+} + |f|_{\ell+\frac{1}{2},n-1}\big)$.

So let $u \in H^1(\mathbb{R}^n_+)$, $F \in H^{\ell-1}(\mathbb{R}^n_+)$ and $f \in H^{\ell+\frac{1}{2}}(\mathbb{R}^{n-1})$ with $L_{\gamma,\mu,+}u = F$ and $Ru = f$. For each $\alpha \in \mathbb{N}^n_0$ with $|\alpha| = 1$, $\alpha_n = 0$ and each $h \in \mathbb{R} \setminus \{0\}$, set $u_{\alpha,h} = \Delta^\alpha_h u$ and $f_{\alpha,h} = \Delta^\alpha_h f$. Then

$$L_{\gamma,\mu,+}u_{\alpha,h} = F_{\alpha,h} \qquad Ru_{\alpha,h} = f_{\alpha,h} \qquad \text{where } F_{\alpha,h} = \Delta^\alpha_h F + [L_{\gamma,\mu,+}, \Delta^\alpha_h]u$$

By the inductive hypothesis, $u \in H^\ell(\mathbb{R}^n_+)$ with $\|u\|_{\ell,\mathbb{R}^n_+} \le C\big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1}\big)$. Hence, by part (b) of Lemma 3.2.6 with $\ell \to \ell-2$ and $k = 2$, $[L_{\gamma,\mu}, \Delta^\alpha_h]u \in H^{\ell-2}(\mathbb{R}^n)$ with

$$\begin{aligned}\big\|[L_{\gamma,\mu,+}, \Delta^\alpha_h]u\big\|_{\ell-2,\mathbb{R}^n_+} &\le C_1\|u\|_{\ell,\mathbb{R}^n_+} \le C_1C\left(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1}\right)\\ &\le C_1C\left(\|F\|_{\ell-1,\mathbb{R}^n_+} + |f|_{\ell+\frac{1}{2},n-1}\right)\end{aligned}$$

By Lemma 3.1.5, with $s = \ell - \frac{1}{2}$,

$$|f_{\alpha,h}|_{\ell-\frac{1}{2},n-1} = \big|\Delta^\alpha_h f\big|_{\ell-\frac{1}{2},n-1} \le \big|\partial^\alpha f\big|_{\ell-\frac{1}{2},n-1} \le |f|_{\ell+\frac{1}{2},n-1}$$

By Lemma 3.2.5, with $\ell \to \ell-2$,

$$\big\|\Delta^\alpha_h F\big\|_{\ell-2,\mathbb{R}^n_+} \le C_2\big\|\partial^\alpha F\big\|_{\ell-2,\mathbb{R}^n_+} \le C_2\|F\|_{\ell-1,\mathbb{R}^n_+}$$

Hence $F_{\alpha,h} \in H^{\ell-2}(\mathbb{R}^n_+)$ with $\|F_{\alpha,h}\|_{\ell-2,\mathbb{R}^n_+} \le (C_2 + C_1C)\big(\|F\|_{\ell-1,\mathbb{R}^n_+} + |f|_{\ell+\frac{1}{2},n-1}\big)$ and, by the inductive hypothesis, $u_{\alpha,h} = \Delta^\alpha_h u \in H^\ell(\mathbb{R}^n_+)$ with

$$\begin{aligned}\|\Delta^\alpha_H u\|_{\ell,\mathbb{R}^n_+} &\le C\left(\|F_{\alpha,h}\|_{\ell-2,\mathbb{R}^n_+} + |f_{\alpha,h}|_{\ell-\frac{1}{2},n-1}\right)\\ &\le C(C_2 + C_1C + 1)\big(\|F\|_{\ell-1,\mathbb{R}^n_+} + |f|_{\ell+\frac{1}{2},n-1}\big)\end{aligned}$$

By Lemma 3.2.5, with C renamed to C_3, $\partial^\alpha u \in H^\ell(\mathbb{R}^n_+)$ with

$$\|\partial^\alpha u\|_{\ell,\mathbb{R}^n_+} \le C_3 C(C_2 + C_1 C + 1)\big(\|F\|_{\ell-1,\mathbb{R}^n_+} + |f|_{\ell+\frac{1}{2},n-1}\big) \tag{3.2.6}$$

So far, we have assumed that $\alpha_n = 0$. So we now know that $\partial^\beta u \in L^2(\mathbb{R}^n_+)$ for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \ell + 1$ and $\beta_n = 0$ and that $\|\partial^\beta u\|_{L^2(\mathbb{R}^n_+)}$ is bounded by the right hand side of (3.2.6). By part (a) of Proposition 2.2.9, it remains only to prove the same bound for $\partial^\gamma u$ with $\gamma = (0, \cdots, 0, \ell + 1)$. From the differential equation

$$L_{\gamma,\mu,+} u = F$$

we have

$$\tfrac{\partial}{\partial x_n}\Big(\gamma^{n,n}(x)\tfrac{\partial u}{\partial x_n}(x)\Big) = - \sum_{\substack{1\le i,j\le n\\ (i,j)\ne(n,n)}} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big) + \mu(x)u(x) - F(x)$$

and hence

$$\tfrac{\partial^2 u}{\partial x_n^2}(x) = \tfrac{1}{\gamma^{n,n}(x)}\Big\{ - \sum_{\substack{1\le i,j\le n\\ (i,j)\ne(n,n)}} \tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big) + \mu(x)u(x) - F(x) - \tfrac{\partial \gamma^{n,n}}{\partial x_n}(x)\tfrac{\partial u}{\partial x_n}(x)\Big\}$$

By (3.2.6) and part (a) of Problem 2.1.3 (multiplication by a nice function is a bounded operator), the $\|\ \cdot\ \|_{\ell-1,\mathbb{R}^n_+}$ norm of each term on the right hand side is bounded by a constant times the right hand side of (3.2.6). This is all we need. ■

Problem 3.2.1 Let $\gamma(x)$ obey Hypotheses 3.1.1 and let $\ell \in \mathbb{N}$.

(a) Prove that there is a constant C, depending only on ℓ and γ such that for all $u \in H^1(\mathbb{R}^n_+)$, $F \in H^{\ell-2}(\mathbb{R}^n_+)$ and $f \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$ obeying

$$L_{\gamma,0,+} u = F \qquad Ru = f$$

we have $u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|u\|_{\ell,\mathbb{R}^n_+} \le C\Big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1} + \|u\|_{-1,\mathbb{R}^n_+}\Big)$$

(b) Let Ω be a bounded open subset of $\mathbb{R}^n$. Prove that there is a constant C, depending only on ℓ, γ and Ω such that, for all $F \in H^{\ell-2}(\mathbb{R}^n_+)$, all $f \in H^{\ell-\frac{1}{2}}(\mathbb{R}^{n-1}_+)$ and all $u \in H^1(\mathbb{R}^n_+)$ that vanish outside of Ω and obey

$$L_{\gamma,0,+} u = F \qquad Ru = f$$

we have $u \in H^\ell(\mathbb{R}^n_+)$ and

$$\|u\|_{\ell,\mathbb{R}^n_+} \le C\Big(\|F\|_{\ell-2,\mathbb{R}^n_+} + |f|_{\ell-\frac{1}{2},n-1}\Big)$$

§3.3. The Dirichlet Problem for $\sum\limits_{i,j=1}^n \frac{\partial}{\partial x_i}\gamma^{i,j}(x)\frac{\partial}{\partial x_j}$

We now mimic the arguments of §3.2 to prove the existence of a unique solution to the Dirichlet problem (3.1). Throughout this section we assume that Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary and that γ obeys

Hypothesis 3.3.1 *Let, for each $x \in \overline{\Omega}$, $\gamma(x) = \big[\gamma^{i,j}(x)\big]_{1\le i,j\le n}$ be a symmetric real matrix that is strictly positive definite. Further assume*[1] *that, for each $1 \le i,j \le n$, $\gamma^{i,j} \in C^\infty(\overline{\Omega})$.*

We start by constructing a new inner product $\langle\ ,\ \rangle_{\gamma,\Omega}$ that

- is adapted to the boundary value problem (3.1) the way that $\langle\ ,\ \rangle_{\gamma,\mu,+}$ is adapted to the boundary value problem (3.2.1) and
- whose associated norm is equivalent to $\|\ \|_{1,\Omega}$

Definition 3.3.2 $(H_\gamma(\Omega),\ H_{\gamma,0}(\Omega))$ Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Define, for all $u, v \in C^\infty(\overline{\Omega})$,

$$\langle u, v\rangle_{\gamma,\Omega} = \sum_{i,j=1}^n \int_\Omega \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial v}{\partial x_j}(x)}\ d^n x + \int_{\partial\Omega} u(x)\overline{v(x)}\ d^{n-1}\sigma$$

where $d^{n-1}\sigma$ denotes the surface measure on $\partial\Omega$. This is an inner product. In particular, the proof of nondegeneracy is Problem 3.3.1. The completion of $C^\infty(\overline{\Omega})$ under the associated norm

$$\|u\|_{\gamma,\Omega} = \sqrt{\langle u, u\rangle_{\gamma,\Omega}}$$

is called $H_\gamma(\Omega)$. Similarly, $H_{\gamma,0}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under $\|\ \|_{\gamma,\Omega}$.

Problem 3.3.1 Prove that $\langle u, u\rangle_{\gamma,\Omega} = 0$ if and only if $u = 0$.

Lemma 3.3.3 *The norms $\|\ \|_{1,\Omega}$ and $\|\ \|_{\gamma,\Omega}$ are equivalent.*

---

[1] This assumption has been made stronger than necessary for simplicity of notation. It is easy to generalize the results of this section to $\gamma \in C^\ell(\overline{\Omega})$ for suitable ℓ.

Proof: By Problem 3.1,

$$C_1\|\nabla u\|^2_{L^2(\Omega)} \le \sum_{i,j=1}^n \int_\Omega \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^nx \le C_2\|\nabla u\|^2_{L^2(\Omega)}$$

By Poincaré's inequality (Proposition 2.3.7)

$$\begin{aligned}\|u\|^2_{1,\Omega} &= \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \le (C+1)\|\nabla u\|^2_{L^2(\Omega)} + C\|Ru\|^2_{L^2(\partial\Omega)}\\ &\le \tfrac{C+1}{C_1}\sum_{i,j=1}^n \int_\Omega \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^nx + C\|Ru\|^2_{L^2(\partial\Omega)}\\ &\le \max\left\{\tfrac{C+1}{C_1}, C\right\}\|u\|^2_{\gamma,\Omega}\end{aligned}$$

By Problem 2.1.22, the norm $\|\ \cdot\ \|_{0,\partial\Omega}$ is equivalent to the norm $\|\ \cdot\ \|_{L^2(\partial\Omega)}$. As $\|Ru\|_{s,\partial\Omega}$ increases with s,

$$\|Ru\|_{L^2(\partial\Omega)} \le C_3\|Ru\|_{0,\partial\Omega} \le C_3\|Ru\|_{\frac{1}{2},\partial\Omega} \le C_4\|u\|_{1,\Omega}$$

by Theorem 2.2.2.ii, the boundedness of the operator R. Hence

$$\begin{aligned}\|u\|^2_{\gamma,\Omega} &= \sum_{i,j=1}^n \int_\Omega \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial u}{\partial x_j}(x)}\ d^nx + \|Ru\|^2_{L^2(\partial\Omega)}\\ &\le C_2\|\nabla u\|^2_{L^2(\Omega)} + C_4^2\|u\|^2_{1,\Omega}\\ &\le (C_2 + C_4^2)\|u\|^2_{1,\Omega}\end{aligned}$$

■

Lemma 3.3.4 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. There is a unique bounded linear map $L_\gamma : H^1(\Omega) \to H^{-1}(\Omega)$ such that*

$$(L_\gamma u)(v) = \sum_{i,j=1}^n \int_\Omega v(x)\tfrac{\partial}{\partial x_i}\Big(\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_j}(x)\Big)\ d^nx$$

for all $u \in C^2(\overline{\Omega})$ and all $v \in C_0^\infty(\Omega)$. Furthermore

$$(L_\gamma u)(v) = -\left\langle v, \bar{u}\right\rangle_{\gamma,\Omega}$$

for all $u \in H^1(\Omega)$ and all $v \in H_0^1(\Omega)$.

Proof: The proof is virtually identical to that of Lemma 3.1.3 and Lemma 3.2.2. Just note that the boundary term $\int_{\partial\Omega} v(x)u(x)\ d^{n-1}\sigma$, in the definition of $\left\langle v, \bar{u}\right\rangle_{\gamma,\Omega}$ vanishes for all $v \in C_0^\infty(\Omega)$. ■

Problem 3.3.2 Prove that, for each $\ell \in \mathbb{N}$, L_γ is bounded as a map from $H^\ell(\Omega)$ to $H^{\ell-2}(\Omega)$.

Theorem 3.3.5 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Let $L_\gamma : H^1(\Omega) \to H^{-1}(\Omega)$ be as in Lemma 3.3.4 and let $R : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ be the restriction map of Theorem 2.2.2. Then the map*

$$\begin{aligned} H^1(\Omega) &\to H^{-1}(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \\ u &\mapsto \quad \big(L_\gamma u\ ,\ Ru\big) \end{aligned}$$

is an isomorphism (1–1, onto, bounded with bounded inverse). That is, for each $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$ there is a unique $u \in H^1(\Omega)$ such that $\nabla \cdot \gamma \nabla u = F$ in Ω and $u\big|_{\partial\Omega} = f$. Furthermore, there is a constant C such that

$$\|u\|_{1,\Omega}^2 \le C\big(\|F\|_{-1,\Omega}^2 + \|f\|_{1/2,\partial\Omega}^2\big)$$

for all $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$.

Proof:

Boundedness: $\|L_\gamma u\|_{-1,\Omega} \le c_1 \|u\|_{1,\Omega}$ by Lemma 3.3.4 and $\|Ru\|_{\frac{1}{2},\partial\Omega} \le c_2 \|u\|_{1,\Omega}$ by part (ii) of Theorem 2.2.2.

Injectiveness: If $Ru = 0$, then $u \in H_0^1(\Omega)$ by part (v) of Theorem 2.2.2. If $u \in H_0^1(\Omega)$ and $L_\gamma u = 0$, then, by Lemma 3.3.4,

$$0 = (L_\gamma u)(\bar{u}) = -\left\langle \bar{u}, \bar{u} \right\rangle_{\gamma,\Omega}$$

so that $\bar{u} = 0$ and hence $u = 0$.

Surjectiveness: Let $F \in H^{-1}(\Omega) = H_0^1(\Omega)^*$ and $f \in H^{1/2}(\partial\Omega)$. Since the norms $\|\ \|_{1,\Omega}$ and $\|\ \|_{\gamma,\Omega}$ are equivalent, $H_0^1(\Omega) = H_{\gamma,0}(\Omega)$, as sets, and F may be viewed as a bounded linear functional on $H_{\gamma,0}(\Omega)$. Hence, by the Riesz representation theorem, there exists $w_1 \in H_{\gamma,0}(\Omega)$ such that

$$Fv = \left\langle v, w_1 \right\rangle_{\gamma,\Omega}$$

for all $v \in H_{\gamma,0}(\Omega) = H_0^1(\Omega)$. By Lemma 3.3.4, $F = L_\gamma u_1$ for $u_1 = -\bar{w}_1$.

By part (iii) of Theorem 2.2.2, R is onto. So there is a $w \in H^1(\Omega) = H_\gamma(\Omega)$ such that $Rw = f$. Write $w = u_2 + w_0$, where $w_0 \in H_{\gamma,0}(\Omega)$ and $u_2 \in H_{\gamma,0}(\Omega)^\perp$. Here, the orthogonal complement is with respect to the $\langle\ ,\ \rangle_{\gamma,\Omega}$ inner product. Set $u = u_1 + u_2$. By the last part of Lemma 3.3.4, L_γ annihilates $H_{\gamma,0}(\Omega)^\perp$, so that $L_\gamma u = L_\gamma u_1 = F$. As R annihilates $H_{\gamma,0}(\Omega)$,

$$Ru = Ru_1 + Ru_2 = Ru_1 + R(w - w_0) = Rw = f$$

Boundedness of the inverse: We must show that if $u \in H^1(\Omega)$, $F = L_\gamma u$ and $f = Ru$ then there is a constant C such that

$$\|u\|^2_{1,\Omega} \le C\Big(\|F\|^2_{-1,\Omega} + \|f\|^2_{\frac{1}{2},\Omega}\Big)$$

Write $u = u_1 + u_2$ with $u_1 \in H_{\gamma,0}(\Omega)$ and $u_2 \in H_{\gamma,0}(\Omega)^\perp$. By Lemma 3.3.4,

$$\|F\|_{-1,\Omega} = \sup_{\substack{v\in H^1_0(\Omega)\\ v\ne 0}} \frac{|Fv|}{\|v\|_{1,\Omega}} = \sup_{\substack{v\in H^1_0(\Omega)\\ v\ne 0}} \frac{|\langle v, \bar{u}\rangle_{\gamma,\Omega}|}{\|v\|_{1,\Omega}} = \sup_{\substack{v\in H^1_0(\Omega)\\ v\ne 0}} \frac{|\langle \bar{v}, u\rangle_{\gamma,\Omega}|}{\|\bar{v}\|_{1,\Omega}} = \sup_{\substack{v\in H^1_0(\Omega)\\ v\ne 0}} \frac{|\langle v, u\rangle_{\gamma,\Omega}|}{\|v\|_{1,\Omega}}$$

By the equivalence of the norms $\|\ \|_{1,\Omega}$ and $\|\ \|_{\gamma,\Omega}$

$$\|F\|_{-1,\Omega} \ge c_3 \sup_{\substack{v\in H_{\gamma,0}(\Omega)\\ v\ne 0}} \frac{|\langle v, u\rangle_{\gamma,\Omega}|}{\|v\|_{\gamma,\Omega}} = c_3 \sup_{\substack{v\in H_{\gamma,0}(\Omega)\\ v\ne 0}} \frac{|\langle v, u_1\rangle_{\gamma,\Omega}|}{\|v\|_{\gamma,\Omega}} = c_3\|u_1\|_{\gamma,\Omega}$$

By part (iv) of Theorem 2.2.2, which gives the existence of a bounded right inverse for R, there is a $w \in H^1(\Omega)$ such that $Rw = f$ and

$$\|w\|_{\gamma,\Omega} \le c_4\|w\|_{1,\Omega} \le c_5\|f\|_{\frac{1}{2},\Omega}$$

As $Ru_2 = Ru = f$ and $Rw = f$, $w - u_2$ is in the kernel of R, which is $H_{\gamma,0}(\Omega)$. Hence $w - u_2$ and u_2 are perpendicular to each other with respect to the $\langle\ ,\ \rangle_{\gamma,\Omega}$ inner product and

$$\|w\|^2_{\gamma,\Omega} = \|u_2\|^2_{\gamma,\Omega} + \|w - u_2\|^2_{\gamma,\Omega} \ge \|u_2\|^2_{\gamma,\Omega}$$

so that $\|u_2\|^2_{\gamma,\Omega} \le c_5^2\|f\|_{\frac{1}{2},\Omega}$. All together

$$\begin{aligned}\|F\|^2_{-1,\Omega} + \|f\|^2_{\frac{1}{2},\Omega} &\ge c_3^2\|u_1\|^2_{\gamma,\Omega} + \tfrac{1}{c_5^2}\|u_2\|^2_{\gamma,\Omega} \ge \min\big\{c_3^2, \tfrac{1}{c_5^2}\big\}\big[\|u_1\|^2_{\gamma,\Omega} + \|u_2\|^2_{\gamma,\Omega}\big]\\ &= \min\big\{c_3^2, \tfrac{1}{c_5^2}\big\}\|u\|^2_{\gamma,\Omega} \qquad \ge c_6\|u\|^2_{1,\Omega}\end{aligned}$$

■

Remark 3.3.6 Apply Problem 3.1 to each $\gamma(x)$, $x \in \Omega$. Let E_1 denote the infimum over $x \in \Omega$ of the resulting constants C_1 and E_2 denote the supremum over $x \in \Omega$ of the resulting constants C_2. A brief review of the proofs of Lemmas 3.3.3 and 3.3.4 and Theorem 3.3.5 shows that

- there are constants c_1 and c_2 depending only on E_1, E_2 and Ω such that

$$\|u\|_{1,\Omega} \le c_1\|u\|_{\gamma,\Omega} \qquad \|u\|_{\gamma,\Omega} \le c_2\|u\|_{1,\Omega}$$

for all $U \in H^1(\Omega)$,

- there is a constant c depending only on E_2 such that the operator norm of L_γ is bounded by c and
- the constant C of Theorem 3.3.5 depends only on E_1, E_2 and Ω.

Denote by $u\big(\gamma, (F, f)\big)$ the unique solution to

$$(L_\gamma u, Ru) = (F, f)$$

provided by Theorem 3.3.5. Let $\alpha, \alpha' \in \mathbb{C}$. Adding together

$$\begin{aligned} &\alpha\Big(L_\gamma u\big(\gamma, (F, f)\big), Ru\big(\gamma, (F, f)\big)\Big) = \alpha(F, f) \\ \text{and} \qquad &\alpha'\Big(L_\gamma u\big(\gamma, (F', f')\big), Ru\big(\gamma, (F', f')\big)\Big) = \alpha'(F', f') \end{aligned}$$

and invoking the uniqueness conclusion of Theorem 3.3.5 yields

$$u\big(\gamma, (\alpha F + \alpha' F', \alpha f + \alpha' f')\big) = \alpha u\big(\gamma, (F, f)\big) + \alpha' u\big(\gamma, (F', f')\big)$$

That is, $u\big(\gamma, (F, f)\big)$ is linear in $(F, f) \in H^{-1}(\Omega) \oplus H^{1/2}(\Omega)$. We shall see, in Problem 3.3.5, below, that $u\big(\gamma, (F, f)\big)$ is analytic in $\gamma \in L^\infty(\Omega)$. In general,

Definition 3.3.7 Let $\mathcal{B}$ and $\mathcal{B}'$ be Banach spaces. A map

$$f : \mathcal{D} \subset \mathcal{B} \to \mathcal{B}'$$

is said to be analytic[2] at $x \in \mathcal{D}$ if there is a family $\{F_\ell\}_{\ell \in \mathbb{N}_0}$ of bounded multilinear functions

$$F_\ell : \overbrace{\mathcal{B} \times \cdots \times \mathcal{B}}^{\ell \text{ times}} \to \mathcal{B}'$$

and constants C and $c > 0$ such that

$$\|F_\ell(z_1, \cdots, z_\ell)\|_{\mathcal{B}'} \le C c^\ell \|z_1\|_{\mathcal{B}} \cdots \|z_\ell\|_{\mathcal{B}}$$

for all $z_1, \cdots, z_\ell \in \mathcal{B}$ and

$$f(x + z) = \sum_{\ell=0}^{\infty} F_\ell(z, \cdots, z)$$

for all $z \in \mathcal{B}$ obeying $\|z\|_{\mathcal{B}} < \frac{1}{c}$ and $x + z \in D$.

---

[2] Actually, there are several equivalent characterizations of analyticity for maps between Banach spaces. We have just picked the characterization in terms of convergent Taylor expansions because it is convenient for our purposes. For a more thorough, but still short and elementary, treatment of analyticity for maps between Banach spaces, see [PT].

Problem 3.3.3 Let $\mathcal{B}$ and $\mathcal{B}'$ be Banach spaces and let the map

$$f : \mathcal{D} \subset \mathcal{B} \to \mathcal{B}'$$

be analytic at $x \in \mathcal{D}$.

(a) Prove that there are constants C and δ such that

$$\|f(x') - f(x)\|_{\mathcal{B}'} \le C\|x' - x\|_{\mathcal{B}}$$

for all $x' \in \mathcal{D}$ with $\|x' - x\|_{\mathcal{B}} < \delta$. In particular, f is continuous at x.

(b) Prove that there exists a bounded linear map $D : \mathcal{B} \to \mathcal{B}'$ such that

- ∘ for any curve $x(t)$ that is defined for t in a neighbourhood of 0, takes values in $\mathcal{D}$, obeys $x(0) = x$ and is differentiable in the sense that there is an $X \in \mathcal{B}$ such that
$$\lim_{t\to 0}\big\| \tfrac{1}{t}[x(t) - x] - X \big\|_{\mathcal{B}} = 0$$

we have

$$\lim_{t\to 0}\big\| \tfrac{1}{t}[f\big(x(t)\big) - f(x)] - DX \big\|_{\mathcal{B}'} = 0$$

The linear map D is called the Fréchet derivative of f at x.

Problem 3.3.4 Let $m \in \mathbb{N}$, $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}_1, \cdots, \mathcal{B}_m$ be Banach spaces and, for each $1 \le i \le m$, let the map

$$f_i : \mathcal{D} \subset \mathcal{B} \to \mathcal{B}_i$$

be analytic at $x \in \mathcal{D}$. Let $M : \mathcal{B}_1 \times \cdots \mathcal{B}_m \to \mathcal{B}'$ be a bounded m–linear map. Prove that

$$\begin{aligned} f : \mathcal{D} \subset \mathcal{B} &\to \mathcal{B}' \\ x' &\mapsto M\big(f_1(x'), \cdots f_m(x')\big) \end{aligned}$$

is analytic at x.

Now back to the question of the analyticity of $u\big(\gamma, (F, f)\big)$ with respect to γ. We shall also include the (F, f) dependence. For each fixed γ obeying Hypothesis 3.3.1, we shall construct a family $\{U_\ell\}_{\ell\in\mathbb{N}_0}$ of bounded multilinear functions

$$\begin{aligned} U_\ell : \overbrace{L^\infty(\Omega) \times \cdots \times L^\infty(\Omega)}^{\ell \text{ times}} \times \big(H^{-1}(\Omega) \oplus H^{1/2}(\Omega)\big) &\to H^1(\Omega) \\ \big(\gamma_1, \cdots, \gamma_\ell, (F, f)\big) &\mapsto U_\ell\big(\gamma_1, \cdots, \gamma_\ell, (F, f)\big) \end{aligned}$$

such that

$$u\big(\gamma + \gamma', (F, f)\big) = \sum_{\ell=0}^{\infty} U_\ell\big(\gamma', \cdots, \gamma', (F, f)\big)$$

for all $\|\gamma'\|_{L^\infty(\Omega)} < \frac{1}{c}$. The function U_ℓ will be linear in each of $\gamma_1, \cdots, \gamma_\ell$ and (F, f) and will be bounded in the sense that

$$\big\|U_\ell\big(\gamma_1, \cdots, \gamma_\ell, (F, f)\big)\big\|_{1,\Omega} \le Cc^\ell \big\|\gamma_1\big\|_{L^\infty(\Omega)} \cdots \big\|\gamma_\ell\big\|_{L^\infty(\Omega)} \ \big\|(F, f)\big\|_{H^{-1}(\Omega)\oplus H^{1/2}(\Omega)}$$

It will also depend on γ, but this dependence does not appear in the notation. In particular $U_0\big((F, f)\big)$ will be exactly $u\big(\gamma, (F, f)\big)$.

By way of motivation for the construction of Problem 3.3.5, shorten $u\big(\gamma, (F, f)\big)$ to u and $u\big(\gamma + \gamma', (F, f)\big)$ to u' and observe that

$$L_\gamma u' = L_{\gamma+\gamma'} u' - \nabla \cdot \gamma' \nabla u' = F - \nabla \cdot \gamma' \nabla u' \qquad Ru' = f \tag{3.3.1}$$

We shall construct a solution to (3.3.1) by an iterative procedure reminiscent of the contraction mapping theorem. We start with $u_0 = u\big(\gamma, (F, f)\big)$ and then iteratively define u_ℓ, $\ell \ge 1$ as the solution to

$$L_\gamma u_\ell = F - \nabla \cdot \gamma' \nabla u_{\ell-1} \qquad Ru_\ell = f \tag{3.3.2}$$

If we can prove that the u_ℓ's converge in $H^1(\Omega)$, then $u' = \lim\limits_{\ell\to\infty} u_\ell$ will obey (3.3.1). To prove convergence, we get a bound on $u_\ell - u_{\ell-1}$ by applying Theorem 3.3.5 to

$$L_\gamma(u_\ell - u_{\ell-1}) = -\nabla \cdot \gamma' \nabla (u_{\ell-1} - u_{\ell-2}) \qquad R(u_\ell - u_{\ell-1}) = 0$$

with $u_{-1} = 0$.

Problem 3.3.5 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let γ satisfy Hypothesis 3.3.1, $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\Omega)$. Define, for any sequence $\{\gamma_m\}_{m\in\mathbb{N}} \subset L^\infty(\Omega)$, the sequence $\{U_\ell(\gamma_1, \cdots, \gamma_\ell)\}_{\ell\in\mathbb{N}_0} \subset H^1(\Omega)$ inductively by

$$\begin{aligned} U_0 &= u\big(\gamma, (F, f)\big) \\ L_\gamma U_\ell(\gamma_1, \cdots, \gamma_\ell) &= -\nabla \cdot \gamma_\ell \nabla U_{\ell-1}(\gamma_1, \cdots, \gamma_{\ell-1}) \qquad RU_\ell(\gamma_1, \cdots, \gamma_\ell) = 0 \end{aligned}$$

(a) Prove that $U_\ell(\gamma_1, \cdots, \gamma_\ell)$ is linear in in each of $\gamma_1, \cdots, \gamma_\ell$ and that there are constants C and c that depend only on n, Ω and γ such that

$$\big\|U_\ell\big(\gamma_1, \cdots, \gamma_\ell\big)\big\|_{1,\Omega} \le Cc^\ell \big\|\gamma_1\big\|_{L^\infty(\Omega)} \cdots \big\|\gamma_\ell\big\|_{L^\infty(\Omega)} \ \big\|(F, f)\big\|_{H^{-1}(\Omega)\oplus H^{1/2}(\Omega)}$$

(b) Prove that if $\gamma' \in L^\infty(\Omega)$ obeys $\|\gamma'\|_{L^\infty(\Omega)} < \frac{1}{c}$, then

$$u' = \sum_{\ell=0}^{\infty} U_\ell\big(\gamma', \cdots, \gamma'\big)$$

converges in $H^1(\Omega)$ and obeys (3.3.1).

Problem 3.3.6 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let γ satisfy Hypothesis 3.3.1. Prove that there is a constant $\delta > 0$ such that, for each $F \in H^{-1}(\Omega)$, $f \in H^{1/2}(\Omega)$ and $\gamma' \in L^\infty(\Omega)$ with $\|\gamma'\|_{L^\infty(\Omega)} < \delta$, there is a unique $u \in H^1(\Omega)$ obeying

$$L_\gamma u = F - \nabla \cdot \gamma' \nabla u \qquad Ru = f$$

Problem 3.3.7 Let, for $i = 1, 2$, γ_i obey Hypothesis 3.3.1, $F_i \in H^{-1}(\Omega)$, $f_i \in H^{1/2}(\partial\Omega)$ and u_i be the solution of

$$L_{\gamma_i} u_i = F_i \qquad u_i\big|_{\partial\Omega} = f_i$$

Prove that there are constants C and $\varepsilon > 0$, depending only on n, Ω and γ_1, such that, if $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} < \varepsilon$, then

$$\|u_1 - u_2\|_{1,\Omega} \le C\Big[\|F_1 - F_2\|_{-1,\Omega} + \|f_1 - f_2\|_{\frac{1}{2},\partial\Omega} + \big(\|F_1\|_{-1,\Omega} + \|f_1\|_{\frac{1}{2},\Omega}\big)\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}\Big]$$

Problem 3.3.8 Let, for each t in a neighbourhood of 0, γ_t obey Hypothesis 3.3.1, $F_t \in H^{-1}(\Omega)$ and $f_t \in H^{1/2}(\partial\Omega)$. Assume that γ_t is differentiable with respect to t at $t = 0$ in $L^\infty(\Omega)$. In other words, assume that there is a $\overset{\bullet}{\gamma}(x) = \big[\overset{\bullet}{\gamma}{}^{i,j}(x)\big]_{1 \le i,j \le n} \in L^\infty(\Omega)$ such that

$$\lim_{t \to 0} \big\|\overset{\bullet}{\gamma} - \tfrac{1}{t}(\gamma_t - \gamma_0)\big\|_{L^\infty(\Omega)} = 0$$

Similarly, assume that F_t and f_t are differentiable with respect to t at $t = 0$ in $H^{-1}(\Omega)$ and $H^{1/2}(\partial\Omega)$, respectively. Let u_t be the solution of

$$L_{\gamma_t} u_t = F_t \qquad u_t\big|_{\partial\Omega} = f_t$$

Prove that u_t differentiable with respect to t at $t = 0$ in $H^1(\Omega)$ and that

$$L_{\gamma_0} \overset{\bullet}{u} = \overset{\bullet}{F} - \nabla \cdot \overset{\bullet}{\gamma} \nabla u_0 \qquad \overset{\bullet}{u}\big|_{\partial\Omega} = \overset{\bullet}{f}$$

Here $\overset{\bullet}{X} = \lim_{t \to 0} \frac{1}{t}(X_t - X_0)$ in $H^1(\Omega)$ when $X = u$, in $H^{-1}(\Omega)$ when $X = F$ and in $H^{1/2}(\partial\Omega)$ when $X = f$.

We will find it useful (see the discussion at the beginning of §4.1) to characterise the unique solution of the boundary value problem $L_\gamma u = F$, $u\big|_{\partial\Omega} = f$, provided by Theorem 3.3.5, as the unique solution to a variational problem.

Definition 3.3.8 Let $F \in H^{-1}(\Omega)$ and denote by P the orthogonal projection from $H^1(\Omega)$ onto $H^1_0(\Omega)$ (using the $H^1(\Omega)$ inner product). The Dirichlet integral is the map $D_F : H^1(\Omega) \to \mathbb{C}$ defined by

$$D_F(w) = \tfrac{1}{2} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j}(x) \tfrac{\partial w}{\partial x_i}(x) \overline{\tfrac{\partial w}{\partial x_j}(x)}\ d^n x + \operatorname{Re} F(P\bar{w})$$

Theorem 3.3.9 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Let*

- ∘ $L_\gamma : H^1(\Omega) \to H^{-1}(\Omega)$ *be as in Lemma 3.3.4,*
- ∘ $R : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ *be the restriction map of Theorem 2.2.2*
- ∘ $F \in H^{-1}(\Omega)$ *and* $f \in H^{1/2}(\partial\Omega)$.

Then $u \in H^1(\Omega)$ obeys $L_\gamma u = F$, $Ru = f$ if and only if u is the unique minimizer to $D_F(w)$ in the set $\big\{\ w \in H^1(\Omega)\ \big|\ Rw = f\ \big\}$.

Proof: Let $u \in H^1(\Omega)$ be the unique solution to $L_\gamma u = F$, $Ru = f$ provided by Theorem 3.3.5 and let w be any element of $H^1(\Omega)$ that obeys $Rw = f$. Note that $w' = w - u \in H^1_0(\Omega)$, by part (v) of Theorem 2.2.2. Then

$$\begin{aligned} D_F(w) - D_F(u) &= D_F(u + w') - D_F(u) \\ &= \tfrac{1}{2} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial w'}{\partial x_i} \overline{\tfrac{\partial w'}{\partial x_j}}\ d^n x + \operatorname{Re} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u}{\partial x_i} \overline{\tfrac{\partial w'}{\partial x_j}}\ d^n x + \operatorname{Re} F(\overline{w'}) \end{aligned}$$

By the last statement of Lemma 3.3.4,

$$F(\overline{w'}\,) = (L_\gamma u)(\overline{w'}\,) = -\big\langle \overline{w'}, \bar{u} \big\rangle_{\gamma,\Omega} = -\sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u}{\partial x_i} \overline{\tfrac{\partial w'}{\partial x_j}}\ d^n x$$

so

$$D_F(w) - D_F(u) = \tfrac{1}{2} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial w'}{\partial x_i} \overline{\tfrac{\partial w'}{\partial x_j}}\ d^n x = \|w'\|^2_{\gamma,\Omega} = \|w - u\|^2_{\gamma,\Omega}$$

The right hand side is manifestly positive and vanishes only for $w = u$. So $D_F(w)$ has a unique minimizer on the set $\big\{\ w \in H^1(\Omega)\ \big|\ Rw = f\ \big\}$ and that minimizer is u. ■

Proposition 3.3.10 (Elliptic Regularity) *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary, $\ell \in \mathbb{N}$ and γ obey Hypothesis 3.3.1. There is a constant C, depending only on ℓ, Ω and γ such that, for all $u \in H^1(\Omega)$, $F \in H^{\ell-2}(\Omega)$ and $f \in H^{\ell-\frac{1}{2}}(\partial\Omega)$ obeying*

$$L_\gamma u = F \qquad Ru = f$$

we have $u \in H^\ell(\Omega)$ and

$$\|u\|_{\ell,\Omega} \le C\Big(\ \|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega}\Big)$$

Proof: We proceed by induction. We already know the result for $\ell = 1$. Assume that we have proven the desired bound for some $\ell \in \mathbb{N}$. We now prove it for $\ell + 1$. That is, we assume that $F \in H^{\ell-1}(\Omega)$ and $f \in H^{\ell+\frac{1}{2}}(\Omega)$ and that we already know $u \in H^{\ell}(\Omega)$ and

$$\|u\|_{\ell,\Omega} \le C\Big(\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega}\Big) \tag{3.3.3}$$

We are to prove that $u \in H^{\ell+1}(\Omega)$ and

$$\|u\|_{\ell+1,\Omega} \le C'\Big(\|F\|_{\ell-1,\Omega} + \|f\|_{\ell+\frac{1}{2},\partial\Omega}\Big)$$

Let N and, for $1 \le i \le N$, $\mathcal{U}(p_i)$, χ_i, ψ_{p_i} be as in Definition 2.1.20. In particular, each $\chi_i \in C_0^\infty\big(\mathcal{U}(p_i)\big)$. Let $\mathcal{V}$ be a neighbourhood of $\partial\Omega$ on which $\sum_{i=1}^N \chi_i = 1$. Select a function $\varphi \in C_0^\infty(\Omega)$ that is identically one on $\Omega \setminus \mathcal{V}$. We prove that each of the $N+1$ terms on the right hand side of

$$u = \varphi u + \sum_{i=1}^N \chi_i(1-\varphi)u$$

is in $H^{\ell+1}(\Omega)$ and obeys the desired bound.

The term φu:

Let $v = \varphi u$. We find a $G \in H^{\ell-1}(\mathbb{R}^n)$ obeying $|G|_{\ell-1,n} \le c\big(\|F\|_{\ell-1,\Omega} + \|f\|_{\ell+\frac{1}{2},\partial\Omega}\big)$ and

$$L_\gamma v = G$$

We also find, in Problem 3.3.9, a γ' that satisfies Hypotheses 3.1.1 and coincides with γ on the support of φ. It will then suffice to apply either Problem 3.1.5 or part (d) of Problem 3.1.6 with $F = G$, u replaced by v, γ replaced by γ' and ℓ replaced by $\ell+1$. To find G, we just have to compute

$$\begin{aligned} L_\gamma v &= \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\big[\gamma^{i,j}(x)\tfrac{\partial}{\partial x_j}(\varphi u)(x)\big] \\ &= \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\big[\gamma^{i,j}(x)\varphi(x)\tfrac{\partial u}{\partial x_j}(x)\big] + \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\big[\gamma^{i,j}(x)u(x)\tfrac{\partial \varphi}{\partial x_j}(x)\big] \\ &= \varphi L_\gamma u + \sum_{i,j=1}^n \gamma^{i,j}(x)\tfrac{\partial\varphi}{\partial x_i}(x)\tfrac{\partial u}{\partial x_j}(x) + \sum_{i,j=1}^n \gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\tfrac{\partial \varphi}{\partial x_j}(x) + uL_\gamma\varphi \\ &= \varphi F + 2\sum_{i,j=1}^n \gamma^{i,j}\tfrac{\partial\varphi}{\partial x_i}\tfrac{\partial u}{\partial x_j} + uL_\gamma\varphi \\ &\equiv G \end{aligned} \tag{3.3.4}$$

Recalling that $\varphi \in C_0^\infty(\Omega)$ and that, by Lemma 2.1.16,

- multiplication by any function in $C^\infty(\overline{\Omega})$ is a bounded operator on $H^p(\Omega)$ for all $p \in \mathbb{Z}$
- $\frac{\partial}{\partial x_j}$ is a bounded operator from $H^p(\Omega)$ to $H^{p-1}(\Omega)$ for all $p \in \mathbb{Z}$ and $1 \le j \le n$

we have $G \in H^{\ell-1}(\mathbb{R}^n)$ with

$$\begin{aligned}|G|_{\ell-1,n} = \|G\|_{\ell-1,\Omega} &\le c_1\Big(\|F\|_{\ell-1,\Omega} + \|u\|_{\ell,\Omega} + \|u\|_{\ell-1,\Omega}\Big)\\ &\le 2c_1C\Big(\|F\|_{\ell-1,\Omega} + \|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega}\Big) \text{ by (3.3.3)}\\ &\le 4c_1C\Big(\|F\|_{\ell-1,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega}\Big)\end{aligned}$$

Here we have assumed, without loss of generality, that $C \ge \frac{1}{2}$.

The term $\chi_{i_0}(1-\varphi)u$ *– algebra:*

Fix $1 \le i_0 \le N$ and write $\chi_{i_0} = \chi$ and $\psi_{p_{i_0}} = \psi$. The strategy for treating this term is analogous to that we applied to φu, but this time we construct a boundary value problem on $\mathbb{R}^n_+$ and apply Problem 3.2.1. Recall that ψ is a diffeomorphism from $\mathcal{U}(p_{i_0})$ to $\mathbb{R}^n$ that maps $\mathcal{U}(p_{i_0}) \cap \Omega$ to $\mathbb{R}^n_+$. We use x to denote coordinates in $\mathcal{U}(p_{i_0})$ and y to denote coordinates in $\mathbb{R}^n$. Set $v = \chi(1-\varphi)u$, $w = v \circ \psi^{-1}$ and

$$\tilde\gamma^{i,j}(y) = \sum_{k,\ell=1}^n \gamma^{k,\ell}\big(\psi^{-1}(y)\big)\tfrac{\partial\psi_i}{\partial x_k}\big(\psi^{-1}(y)\big)\tfrac{\partial\psi_j}{\partial x_\ell}\big(\psi^{-1}(y)\big)$$

By the chain rule

$$\psi_m^{-1}\big(\psi(x)\big) = x_m \implies \sum_{j=1}^n \tfrac{\partial\psi_j}{\partial x_\ell}(x)\tfrac{\partial\psi_m^{-1}}{\partial y_j}\big(\psi(x)\big) = \begin{cases} 1 & \text{if } \ell = m\\ 0 & \text{if } \ell \ne m\end{cases}$$

and

$$\begin{aligned}\sum_{j=1}^n \tfrac{\partial\psi_j}{\partial x_\ell}\big(\psi^{-1}(y)\big)\tfrac{\partial}{\partial y_j}v\big(\psi^{-1}(y)\big) &= \sum_{j,m=1}^n \tfrac{\partial\psi_j}{\partial x_\ell}\big(\psi^{-1}(y)\big)\tfrac{\partial\psi_m^{-1}}{\partial y_j}(y)\tfrac{\partial\, v}{\partial x_m}\big(\psi^{-1}(y)\big)\\ &= \tfrac{\partial\, v}{\partial x_\ell}\big(\psi^{-1}(y)\big)\end{aligned} \tag{3.3.5}$$

Hence

$$\begin{aligned}L_{\tilde\gamma,0,+}w &= \sum_{i,j=1}^n \tfrac{\partial}{\partial y_i}\Big(\tilde\gamma^{i,j}(y)\tfrac{\partial}{\partial y_j}v\circ\psi^{-1}(y)\Big)\\ &= \sum_{i,k,\ell=1}^n \tfrac{\partial}{\partial y_i}\Big(\gamma^{k,\ell}\big(\psi^{-1}(y)\big)\tfrac{\partial\psi_i}{\partial x_k}\big(\psi^{-1}(y)\big)\tfrac{\partial\, v}{\partial x_\ell}\big(\psi^{-1}(y)\big)\Big)\\ &= \sum_{k,\ell=1}^n \big[\tfrac{\partial}{\partial x_k}\gamma^{k,\ell}(x)\tfrac{\partial\, v}{\partial x_\ell}(x)\big]_{x=\psi^{-1}(y)}\\ &\qquad + \sum_{i,k,\ell=1}^n \gamma^{k,\ell}\big(\psi^{-1}(y)\big)\tfrac{\partial\, v}{\partial x_\ell}\big(\psi^{-1}(y)\big)\tfrac{\partial}{\partial y_i}\tfrac{\partial\psi_i}{\partial x_k}\big(\psi^{-1}(y)\big)\end{aligned}$$

For the final equality, we used the product rule (using $\gamma^{k,\ell}\big(\psi^{-1}(y)\big)\frac{\partial v}{\partial x_\ell}\big(\psi^{-1}(y)\big)$ and $\frac{\partial \psi_i}{\partial x_k}\big(\psi^{-1}(y)\big)$ as the two factors) followed by a second application of (3.3.5) with $v(x)$ replaced by $\gamma^{k,\ell}(x)\frac{\partial v}{\partial x_\ell}(x)$, j replaced by i and ℓ replaced by k. Substitute in

$$\begin{aligned}
\sum_{k,\ell=1}^{n} \tfrac{\partial}{\partial x_k}\gamma^{k,\ell}(x)\tfrac{\partial v}{\partial x_\ell}(x) &= \sum_{k,\ell=1}^{n} \tfrac{\partial}{\partial x_k}\gamma^{k,\ell}\chi(1-\varphi)\tfrac{\partial u}{\partial x_\ell} + \sum_{k,\ell=1}^{n} \tfrac{\partial}{\partial x_k}\gamma^{k,\ell}u\tfrac{\partial}{\partial x_\ell}\chi(1-\varphi) \\
&= \chi(1-\varphi)L_\gamma u + \sum_{k,\ell=1}^{n}\gamma^{k,\ell}\tfrac{\partial u}{\partial x_\ell}\tfrac{\partial}{\partial x_k}\chi(1-\varphi) + \sum_{k,\ell=1}^{n}\gamma^{k,\ell}\tfrac{\partial u}{\partial x_k}\tfrac{\partial}{\partial x_\ell}\chi(1-\varphi) + uL_\gamma\chi(1-\varphi) \\
&= \chi(1-\varphi)F + 2\sum_{i,j=1}^{n}\gamma^{i,j}\tfrac{\partial u}{\partial x_i}\tfrac{\partial}{\partial x_j}\chi(1-\varphi) + uL_\gamma\chi(1-\varphi)
\end{aligned}$$

to give

$$L_{\tilde\gamma,0,+}w = G \qquad w\restriction_{\mathbb{R}^{n-1}} = \big(\chi(1-\varphi)f\big)\circ\psi^{-1} = (\chi f)\circ\psi^{-1}$$

(recall that φ is supported in the interior of Ω) where

$$\begin{aligned}
G(y) = \chi(1-\varphi)F\circ\psi^{-1} &+ [uL_\gamma\chi(1-\varphi)]\circ\psi^{-1} \\
&+ 2\sum_{i,j=1}^{n}\Big[\gamma^{i,j}(x)\tfrac{\partial u}{\partial x_i}(x)\tfrac{\partial}{\partial x_j}\chi(x)(1-\varphi(x))\Big]_{x=\psi^{-1}(y)} \\
&+ \sum_{i,j,k=1}^{n}\gamma^{k,j}\big(\psi^{-1}(y)\big)\tfrac{\partial v}{\partial x_j}\big(\psi^{-1}(y)\big)\tfrac{\partial}{\partial y_i}\tfrac{\partial \psi_i}{\partial x_k}\big(\psi^{-1}(y)\big)
\end{aligned}$$

The term $\chi_{i_0}(1-\varphi)u$ – bounds:

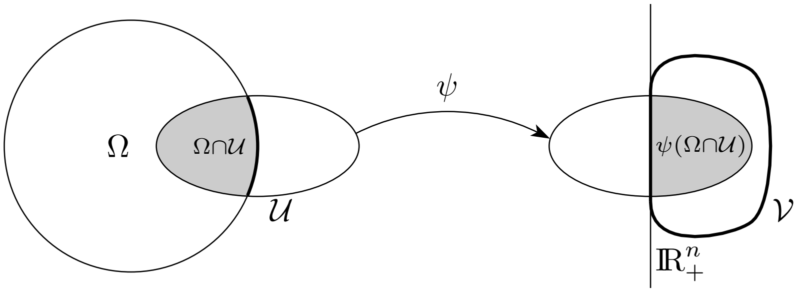
By part (c) of Lemma 2.1.23 and (3.3.3), we get the following bounds on the various parts of G.

$$\begin{aligned}
\|\chi(1-\varphi)F\circ\psi^{-1}\|_{\ell-1,\mathbb{R}^n_+} &\le c_2\|F\|_{\ell-1,\Omega} \\
\big\|[uL_\gamma\chi(1-\varphi)]\circ\psi^{-1}\big\|_{\ell-1,\mathbb{R}^n_+} &\le c_3\|u\|_{\ell-1,\Omega} \le c_3\|u\|_{\ell,\Omega} \\
&\le c_3C\Big(\,\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac12,\partial\Omega}\Big) \\
\big\|\big[\tfrac{\partial u}{\partial x_i}(x)\tfrac{\partial}{\partial x_j}\chi(x)(1-\varphi(x))\big]\circ\psi^{-1}\big\|_{\ell-1,\mathbb{R}^n_+} &\le c_3\big\|\tfrac{\partial u}{\partial x_i}\big\|_{\ell-1,\Omega} \le c_3\|u\|_{\ell,\Omega} \\
&\le c_3C\Big(\,\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac12,\partial\Omega}\Big) \\
\big\|\tfrac{\partial v}{\partial x_j}\circ\psi^{-1}\big\|_{\ell-1,\mathbb{R}^n_+} &\le c_3\Big[\big\|\tfrac{\partial u}{\partial x_j}\big\|_{\ell-1,\Omega} + \|u\|_{\ell-1,\Omega}\Big] \le 2c_3\|u\|_{\ell,\Omega} \\
&\le 2c_3C\Big(\,\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac12,\partial\Omega}\Big)
\end{aligned}$$

Consequently, by part (a) of Lemma 2.1.16, $G \in H^{\ell-1}(\mathbb{R}^n_+)$ with

$$\|G\|_{\ell-1,\mathbb{R}^n_+} \le c_4\Big(\,\|F\|_{\ell-1,\Omega} + \|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac12,\partial\Omega}\Big) \le 2c_4\Big(\,\|F\|_{\ell-1,\Omega} + \|f\|_{\ell-\frac12,\partial\Omega}\Big)$$

Let $\mathcal{U}$ be an open neighbourhood of the support of χ whose closure is contained in $\mathcal{U}(p_{i_0})$. Then $\psi(\mathcal{U}) \cap \mathbb{R}^n_+ = \psi(\mathcal{U} \cap \Omega)$ is a bounded open subset of $\mathbb{R}^n$ and w vanishes outside of $\psi(\mathcal{U} \cap \Omega)$. Enlarge $\psi(\mathcal{U}) \cap \mathbb{R}^n_+$ to $\mathcal{V}$, a bounded open subset of $\mathbb{R}^n_+$ with smooth boundary.



In Problem 3.3.9, we construct a γ' that satisfies Hypotheses 3.1.1 and coincides with $\tilde{\gamma}$ on $\mathcal{V}$. Applying part (b) of Problem 3.2.1 with $F = G$, $u = w$, $\gamma = \gamma'$, f replaced by $(\chi f) \circ \psi^{-1}$ and ℓ replaced by $\ell + 1$ gives

$$\begin{aligned} \|w\|_{\ell+1,\mathbb{R}^n_+} &\le c_5\big(\|G\|_{\ell-1,\mathbb{R}^n_+} + |\chi f \circ \psi^{-1}|_{\ell+\frac{1}{2},n-1}\big) \\ &\le c_6\big(\|F\|_{\ell-1,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega} + |\chi f \circ \psi^{-1}|_{\ell+\frac{1}{2},n-1}\big) \end{aligned}$$

By Definition 2.1.20,

$$\|w\|_{\ell+1,\mathbb{R}^n_+} \le 2c_6\big(\|F\|_{\ell-1,\Omega} + \|f\|_{\ell+\frac{1}{2},\partial\Omega}\big)$$

Finally, by part (a) of Lemma 2.1.23, with $\mathcal{U}$ replaced by $\mathcal{V}$, $\mathcal{V}$ replaced by $\psi^{-1}(\mathcal{V})$, ψ replaced by ψ^{-1} and ℓ replaced by $\ell + 1$,

$$\begin{aligned} \|\chi_{i_0}(1-\varphi)u\|_{\ell+1,\Omega} = \|w \circ \psi\|_{\ell+1,\psi^{-1}(\mathcal{V})} &\le \|w\|_{\ell+1,\mathcal{V}} = \|w\|_{\ell+1,\mathbb{R}^n_+} \\ &\le 2c_6\big(\|F\|_{\ell-1,\Omega} + \|f\|_{\ell+\frac{1}{2},\partial\Omega}\big) \end{aligned}$$

■

Problem 3.3.9 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let γ obey Hypothesis 3.3.1. Prove that there is a γ' obeying Hypotheses 3.1.1 that coincides with γ on $\overline{\Omega}$.

The next proposition shows that the regularity of u in some region, in the interior of its domain of definition, is really just determined by the regularity of F in essentially the same region.

Proposition 3.3.11 (Interior Regularity) *Let Ω and Ω' be bounded open subsets of $\mathbb{R}^n$ with smooth boundaries and $\overline{\Omega} \subset \Omega'$. Assume that γ obeys Hypothesis 3.3.1 in Ω'.*

Let $\ell \in \mathbb{N}$. There is a constant C, depending only on ℓ, Ω, Ω' and γ such that, for all $u \in H^1(\Omega')$ and $F \in H^{\ell-2}(\Omega')$ obeying

$$L_\gamma u = F$$

we have $u \in H^\ell(\Omega)$ and

$$\|u\|_{\ell,\Omega} \le C\Big(\|F\|_{\ell-2,\Omega'} + \|u\|_{L^2(\Omega')}\Big)$$

Proof: Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \chi \le 1$ be such that $\chi \equiv 1$ on Ω and $\operatorname{supp}\chi \subset \Omega'$. Set $v = \chi u$. As in the proof of Proposition 3.3.10, we shall shortly find a $G \in H^{\ell-2}(\mathbb{R}^n)$ obeying $|G|_{\ell-2,n} \le c\big(\|F\|_{\ell-2,\Omega'} + \|u\|_{\ell-1,\Omega'}\big)$ and

$$L_\gamma v = G$$

We have already found, in Problem 3.3.9, a γ' that satisfies Hypotheses 3.1.1 and coincides with γ on the support of χ. Then by Problem 3.1.5 with $F = G$, u replaced by v and γ replaced by γ'

$$\begin{aligned} \|u\|_{\ell,\Omega} \le |v|_{\ell,n} &\le C\Big(|G|_{\ell-2,n} + |v|_{-1,n}\Big) \le C\Big(|G|_{\ell-2,n} + \|v\|_{L^2(\mathbb{R}^n)}\Big) \\ &\le C\Big(\|F\|_{\ell-2,\Omega'} + \|u\|_{\ell-1,\Omega'} + \|u\|_{L^2(\Omega')}\Big) \end{aligned} \tag{3.3.6}$$

It will then suffice to apply induction on ℓ to (3.3.6), using an increasing family

$$\Omega \subset \Omega_1 \subset \ \cdots\ \subset \Omega_{\ell-1} \subset \Omega'$$

of bounded open subsets of $\mathbb{R}^n$ with smooth boundary.

To find G, we just have to replace φ by χ in (3.3.4). This gives

$$\begin{aligned} L_\gamma v &= \chi F + 2 \sum_{i,j=1}^n \gamma^{i,j} \tfrac{\partial \chi}{\partial x_i} \tfrac{\partial u}{\partial x_j} + u L_\gamma \chi \\ &\equiv G \end{aligned}$$

Recalling that $\chi \in C_0^\infty(\Omega')$ and that, by Lemma 2.1.16,

- multiplication by any function in $C^\infty(\overline{\Omega'})$ is a bounded operator on $H^p(\Omega')$ for all $p \in \mathbb{Z}$
- $\frac{\partial}{\partial x_j}$ is a bounded operator from $H^p(\Omega')$ to $H^{p-1}(\Omega')$ for all $p \in \mathbb{Z}$ and $1 \le j \le n$

we have $G \in H^{\ell-2}(\mathbb{R}^n)$ with

$$|G|_{\ell-2,n} = \|G\|_{\ell-2,\Omega'} \le C\Big(\|F\|_{\ell-2,\Omega'} + \|u\|_{\ell-1,\Omega'}\Big)$$

■

Now we prove, as a corollary to Proposition 3.3.10, regularity up to the boundary, in the classical sense, of solutions of the Dirichlet problem

Theorem 3.3.12 (Regularity up to the boundary) *Let $F \in C^\infty(\overline{\Omega})$ and $f \in C^\infty(\partial\Omega)$. Then the solution u to*

$$L_\gamma u = F \qquad Ru = f$$

is in $C^\infty(\overline{\Omega})$.

Proof: The conclusion follows immediately from Lemma 2.2.14, since we already know, from Proposition 3.3.10, that $u \in H^\ell(\Omega)$ for all $\ell \in \mathbb{N}$. ■

§3.4. The Dirichlet to Neumann Map

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Denote by $\hat{n}(x)$ the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$. Denote by $d\sigma(x)$ the surface measure on $\partial\Omega$ and recall, from Problem 2.1.22, that the norm $\|\ \cdot\ \|_{0,\partial\Omega}$ on $H^0(\partial\Omega)$ is equivalent to

$$\|f\|_{L^2(\partial\Omega)} = \sqrt{\int_{\partial\Omega} |f(x)|^2 \, d\sigma(x)}$$

Also recall, from Problem 2.1.24, that there is an isomorphism $\mathcal{L} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)^*$ such that

$$\mathcal{L}_f g = \int_{\partial\Omega} f(x)g(x)\, d\sigma(x)$$

for all $f, g \in C^\infty(\partial\Omega)$.

We assume throughout this subsection, as we did in the last, that γ obeys Hypothesis 3.3.1.

Theorem 3.4.1 *Let, for each $f \in C^\infty(\partial\Omega)$, $u_f \in C^\infty(\overline{\Omega})$ be the solution to*

$$\begin{aligned} (L_\gamma u_f)(x) &= 0 \qquad && \text{if } x \in \Omega \\ u_f(x) &= f(x) && \text{if } x \in \partial\Omega \end{aligned} \tag{3.4.1}$$

of Theorem 3.3.12. There is a unique bounded linear map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ such that

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \sum_{i,j=1}^n \int_{\partial\Omega} \gamma^{i,j}(x) \tfrac{\partial u_f}{\partial x_i}(x) \hat{n}^j(x)\ g(x)\, d\sigma(x)$$

for all $f, g \in C^\infty(\partial\Omega)$. Here $\hat{n}^j(x)$ refers to the j^{th} component of $\hat{n}(x)$. We think of $\Lambda_\gamma f$ as $\hat{n} \cdot \gamma \nabla u_f \big|_{\partial\Omega}$. Furthermore, if $f, g \in H^{1/2}(\partial\Omega)$ and $v_g \in H^1(\Omega)$ obeys $Rv_g = g$, then

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \sum_{i,j=1}^n \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial v_g}{\partial x_j}\, d^n x$$

Proof: First consider $f, g \in C^\infty(\partial\Omega)$ and let $u_f, v_g \in C^\infty(\overline{\Omega})$ obey $L_\gamma u_f = 0$, $Ru_f = f$ and $Rv_g = g$. Then, by the divergence theorem (for the divergence theorem in general dimension n, see Problem 3.4.1, below),

$$\begin{aligned}
\mathcal{L}_{\Lambda_\gamma f}(g) &= \sum_{i,j=1}^n \int_{\partial\Omega} \gamma^{i,j}(x) \tfrac{\partial u_f}{\partial x_i}(x) v_g(x)\ \hat{n}^j(x)\ d\sigma(x) \\
&= \sum_{i,j=1}^n \int_{\Omega} \tfrac{\partial}{\partial x_j}\Big[\gamma^{i,j}(x) \tfrac{\partial u_f}{\partial x_i}(x) v_g(x)\Big]\ d^n x \\
&= \sum_{i,j=1}^n \int_{\Omega} \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial v_g}{\partial x_j}\ d^n x + \sum_{i,j=1}^n \int_{\Omega} v_g(x) \tfrac{\partial}{\partial x_j}\Big[\gamma^{i,j}(x) \tfrac{\partial u_f}{\partial x_i}(x)\Big]\ d^n x \\
&= \sum_{i,j=1}^n \int_{\Omega} \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial v_g}{\partial x_j}\ d^n x
\end{aligned}$$

The second sum vanished because $L_\gamma u_f = 0$. By part (iv) of Theorem 2.2.2, we may choose v_g so that $\|v_g\|_{1,\Omega} \le C' \|g\|_{1/2,\partial\Omega}$. Then

$$\begin{aligned}
\big|\mathcal{L}_{\Lambda_\gamma f}(g)\big| &\le \sum_{i,j=1}^n \left| \int_{\Omega} \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial v_g}{\partial x_j}\ d^n x \right| \\
&\le c_1 \big\|\nabla u_f\big\|_{L^2(\Omega)} \big\|\nabla v_g\big\|_{L^2(\Omega)} \\
&\le c_1\ \|u_f\|_{1,\Omega}\ \|v_g\|_{1,\Omega} \\
&\le c_2\ \|f\|_{1/2,\partial\Omega}\ \|g\|_{1/2,\partial\Omega}
\end{aligned}$$

by the bound of Theorem 3.3.5. Since $C^\infty(\partial\Omega)$ is dense in $H^{1/2}(\partial\Omega)$, the norm of $\mathcal{L}_{\Lambda_\gamma f}$ is bounded by $c_2\ \|f\|_{1/2,\partial\Omega}$. Since $\mathcal{L}$ is an isomorphism $\|\Lambda_\gamma f\|_{-1/2,\partial\Omega} \le c_2\ \|f\|_{1/2,\partial\Omega}$. So, by the BLT theorem, Λ_γ has a unique extension to a bounded linear map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

If $f, g \in H^{1/2}(\partial\Omega)$ and $u_f, v_g \in H^1(\Omega)$ obey $L_\gamma u_f = 0$, $Ru_f = f$ and $Rv_g = g$, then, by continuity,

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \sum_{i,j=1}^n \int_{\Omega} \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial v_g}{\partial x_j}\ d^n x$$

■

Remark 3.4.2 Apply Problem 3.1 to each $\gamma(x)$, $x \in \Omega$. Let E_1 denote the infimum over $x \in \Omega$ of the resulting constants C_1 and E_2 denote the supremum over $x \in \Omega$ of the resulting constants C_2. By Remark 3.3.6, the operator norm of Λ_γ is bounded by a constant that depends only on E_1, E_2 and Ω.

Problem 3.4.1 Let $2 \le n \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Denote by $\hat{n}(x)$ the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$ and by $d\sigma(x)$ the surface measure on $\partial\Omega$. Let $\vec{f}(x) = \big(f_j(x)\big)_{1\le j\le n}$ be a smooth vector field on $\overline{\Omega}$.

(a) Prove that

$$\sum_{j=1}^n \int_{\partial\Omega} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = (-1)^{j-1} \sum_{j=1}^n \int_{\partial\Omega} f_j(x)\ dx_1 \wedge \cdots \wedge \cancel{dx_j} \wedge \cdots \wedge dx_n$$

On the right hand side, the orientation of $\partial\Omega$ is such that, for an open neighbourhood $\mathcal{O} \subset \mathbb{R}^n$ in which $\partial\Omega \cap \mathcal{O}$ is given by the equation

$$x_\ell = \varphi(x_1, \cdots, \cancel{x_\ell}, \cdots, x_n), \qquad (x_1, \cdots, \cancel{x_\ell}, \cdots, x_n) \in \mathcal{R}$$

we have

$$\int_{\partial\Omega\cap\mathcal{O}} g(x)\ dx_1 \wedge \cdots \wedge \cancel{dx_\ell} \wedge \cdots \wedge dx_n = (-1)^\ell \int_{\mathcal{R}} g(x)\big|_{x_\ell=\varphi(x)} \prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i$$

when Ω is given by $x_\ell > \varphi(x_1, \cdots, \cancel{x_\ell}, \cdots, x_n)$ and

$$\int_{\partial\Omega\cap\mathcal{O}} g(x)\ dx_1 \wedge \cdots \wedge \cancel{dx_\ell} \wedge \cdots \wedge dx_n = (-1)^{\ell-1} \int_{\mathcal{R}} g(x)\big|_{x_\ell=\varphi(x)} \prod_{\substack{1\le i\le n\\ i\ne\ell}} dx_i$$

when Ω is given by $x_\ell < \varphi(x_1, \cdots, \cancel{x_\ell}, \cdots, x_n)$. For a brief introduction to wedge products and differential forms, see Chapter 7 of [Ar].

(b) Prove that

$$\sum_{j=1}^n \int_{\partial\Omega} f_j(x)\ \hat{n}^j(x)\ d\sigma(x) = \sum_{j=1}^n \int_{\Omega} \tfrac{\partial f_j}{\partial x^j}(x)\ d^n x$$

(c) Let $v \in H^1(\Omega)$ and, for each $1 \le j \le n$, $u_j \in H^1(\Omega)$. Prove that

$$\sum_{j=1}^n \int_{\Omega} v(x)\tfrac{\partial u_j}{\partial x^j}(x)\ d^n x = \sum_{j=1}^n \int_{\partial\Omega} v(x)u_j(x)\ \hat{n}^j(x)\ d\sigma(x) - \sum_{j=1}^n \int_{\Omega} \tfrac{\partial v}{\partial x^j}(x)u_j(x)\ d^n x$$

Remark 3.4.3 An alternate strategy for proving that Λ_γ is a bounded operator is to express it in terms of four operators that we already know are bounded.

Two of these operators are related to the restriction operator of Theorem 2.2.2. By that theorem,

$$R : H^1_0(\Omega)^{\perp_\gamma} \to H^{1/2}(\partial\Omega)$$

is 1–1, onto and bounded and has a bounded inverse. Here $\perp_\gamma$ refers to the orthogonal complement in $H^1_\gamma(\Omega)$ with respect to the $\langle\ ,\ \rangle_{\gamma,\Omega}$ inner product. Let

$$R^{-1} : H^{1/2}(\partial\Omega) \to H^1_0(\Omega)^{\perp_\gamma}$$

be its inverse operator and

$$(R^{-1})^* : H_0^1(\Omega)^{\perp_\gamma} \to H^{1/2}(\partial\Omega)$$

be the adjoint of the inverse. That is

$$\big\langle (R^{-1})^* f, g\big\rangle_{\frac{1}{2},\partial\Omega} = \big\langle f, R^{-1}g\big\rangle_{\gamma,\Omega} \qquad \text{for all } f \in H_0^1(\Omega)^{\perp_\gamma},\ g \in H^{1/2}(\partial\Omega)$$

Note that if f and u_f are as in (3.4.1), then $Ru_f = f$ and, by Lemma 3.3.4,

$$\langle u_f, v\rangle_{\gamma,\Omega} = -\overline{(L_\gamma u)(\bar v)} = 0$$

for all $v \in H_0^1(\Omega)$, so that $u_f \in H_0^1(\Omega)^{\perp_\gamma}$ and $R^{-1}f = u_f$.

The remaining two operators provide isomorphisms from $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, respectively, to $H^{1/2}(\partial\Omega)^*$. By Problem 2.1.23 or Problem 2.1.24, if $f \in C^\infty(\partial\Omega)$, then

$$\mathcal{L}_f : g \in C^\infty(\partial\Omega) \mapsto \mathcal{L}_f(g) = \langle f, \bar g\rangle_{L^2(\partial\Omega)}$$

determines a unique $\mathcal{L}_f \in H^{1/2}(\partial\Omega)^*$ and furthermore $c\|f\|_{-\frac{1}{2},\partial\Omega} \le \|\mathcal{L}_f\| \le C\|f\|_{-\frac{1}{2},\partial\Omega}$. By the Riesz representation theorem

$$(\mathfrak{L}g)(f) = \langle f, \bar g\rangle_{1/2,\partial\Omega}$$

defines an isomorphism

$$\mathfrak{L} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)^*$$

In Lemma 3.4.4, below, we show that $\mathcal{L}_{\Lambda_\gamma f} = \mathfrak{L}(R^{-1})^* R^{-1} f - \mathcal{L}_f$. As

$$\begin{aligned}\|\Lambda_\gamma f\|_{-\frac{1}{2},\partial\Omega} &\le \tfrac{1}{c}\|\mathcal{L}_{\Lambda_\gamma f}\| \le \tfrac{1}{c}\Big\{\|\mathfrak{L}\|\,\|(R^{-1})^*\|\,\|R^{-1}\|\,\|f\|_{\frac{1}{2},\partial\Omega} + C\|f\|_{-\frac{1}{2},\partial\Omega}\Big\}\\ &\le \tfrac{1}{c}\Big\{\|R^{-1}\|^2 + C\Big\}\|f\|_{\frac{1}{2},\partial\Omega}\end{aligned}$$

boundedness is proven again.

Lemma 3.4.4 *For all $f \in C^\infty(\partial\Omega)$*

$$\mathcal{L}_{\Lambda_\gamma f} = \mathfrak{L}(R^{-1})^* R^{-1} f - \mathcal{L}_f$$

Proof: Let $g \in C^\infty(\partial\Omega)$. Then

$$\begin{aligned}
\mathfrak{L}\Big(\big(R^{-1}\big)^* R^{-1} f\Big)(g) &= \big\langle g, \overline{\big(R^{-1}\big)^* R^{-1} f}\big\rangle_{\frac{1}{2},\partial\Omega} \\
&= \big\langle \big(R^{-1}\big)^* R^{-1} f, \bar g\big\rangle_{\frac{1}{2},\partial\Omega} \\
&= \big\langle R^{-1} f, R^{-1}\bar g\big\rangle_{\gamma,\Omega} \\
&= \sum_{i,j=1}^n \int_\Omega \gamma^{i,j}(x) \tfrac{\partial u}{\partial x_i}(x) \tfrac{\partial v}{\partial x_j}(x)\ d^n x + \int_{\partial\Omega} f(x) g(x)\ d^{n-1}\sigma \\
&\qquad\qquad\qquad\qquad\qquad \text{with } u = R^{-1} f,\ v = R^{-1} g \\
&= -\sum_{i,j=1}^n \int_\Omega \Big(\tfrac{\partial}{\partial x_j} \gamma^{i,j} \tfrac{\partial u}{\partial x_i}\Big) v\ d^n x + \sum_{i,j=1}^n \int_{\partial\Omega} \gamma^{i,j} \tfrac{\partial u}{\partial x_i} \hat n^j\ g\ d^{n-1}\sigma \\
&\qquad\qquad + \int_{\partial\Omega} f g\ d^{n-1}\sigma
\end{aligned}$$

By integration by parts. Since $L_\gamma u = 0$ we conclude that

$$\mathfrak{L}\Big(\big(R^{-1}\big)^* R^{-1} f\Big)(g) = \langle \Lambda_\gamma f + f, \bar g\rangle_{L^2(\partial\Omega)}$$

as desired. ■

Problem 3.4.2

(a) Prove that Λ_γ is self–adjoint in the sense that

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \mathcal{L}_{\Lambda_\gamma g}(f)$$

for all $f, g \in H^{1/2}(\partial\Omega)$.

(b) Let A be any bounded linear operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. Define the operator norm

$$\|A\|_{1/2,-1/2} = \sup\Big\{\ \big|\mathcal{L}_{Af}(g)\big|\ \Big|\ f, g \in H^{1/2}(\partial\Omega),\ \|f\|_{1/2,\partial\Omega} = \|g\|_{1/2,\partial\Omega} = 1\ \Big\}$$

Define the quadratic form $Q_A : H^{1/2}(\partial\Omega) \to \mathbb{C}$ associated with A by

$$Q_A(f) = \mathcal{L}_{Af}(\bar f)$$

Prove that if A is self–adjoint, in the sense that $\mathcal{L}_{Af}(g) = \mathcal{L}_{Ag}(f)$ for all $f, g \in H^{1/2}(\partial\Omega)$, then

$$\|A\|_{1/2,-1/2} = \sup\Big\{\ \big|Q_A(f)\big|\ \Big|\ f \in H^{1/2}(\partial\Omega),\ \|f\|_{1/2,\partial\Omega} = 1\ \Big\}$$

Problem 3.4.3 Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary. Denote by $\mathcal{BL}_{\frac{1}{2},-\frac{1}{2}}$ the Banach space of bounded linear maps from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ endowed with the operator norm.

(a) Let γ satisfy Hypothesis 3.3.1. Prove that the map $\gamma' \mapsto \Lambda_{\gamma+\gamma'}$ has an analytic continuation, as a map from a neighbourhood of the origin in $L^\infty(\Omega)$ to $\mathcal{BL}_{\frac{1}{2},-\frac{1}{2}}$.

(b) Let $E > 0$. Denote by Γ_E the set of γ's that obey Hypothesis 3.3.1 and also obey

$$\tfrac{1}{E}|\xi|^2 < \sum_{i,j=1}^n \gamma_1(x)^{i,j}\xi_i\xi_j < E|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Prove that the map $\gamma \mapsto \Lambda_\gamma$ has an analytic continuation, as a map from a neighbourhood of Γ_E in $L^\infty(\Omega)$ to $\mathcal{BL}_{\frac{1}{2},-\frac{1}{2}}$.

It is immediate from Problems 3.4.3 and 3.3.3 that the map $\gamma \mapsto \Lambda_\gamma$ is continuous and indeed has a Fréchet derivative at each $\gamma \in L^\infty(\Omega)$ that satisfies Hypothesis 3.3.1. We next provide a direct proof that $\gamma \mapsto \Lambda_\gamma$ is continuous and in fact is uniformly continuous on a large class of γ's.

Theorem 3.4.5 *Let γ_1, γ_2 obey Hypothesis 3.3.1. In particular, let $E > 0$ be such that*

$$\tfrac{1}{E}|\xi|^2 \le \sum_{i,j=1}^n \gamma_1(x)^{i,j}\xi_i\xi_j \le E|\xi|^2 \qquad \tfrac{1}{E}|\xi|^2 \le \sum_{i,j=1}^n \gamma_2(x)^{i,j}\xi_i\xi_j \le E|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then there is a constant $C(\Omega, E)$ such that

$$\big\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\big\|_{1/2,-1/2} \le C(\Omega, E)\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}$$

We put the bulk of the proof into the following two lemmas.

Lemma 3.4.6 *Let γ obey Hypothesis 3.3.1 and let, for each $1 \le i, j \le n$, $\beta^{i,j}(x) \in C^\infty(\overline{\Omega})$. In particular, let $m_\gamma, M_\beta > 0$ be such that*

$$\sum_{i,j=1}^n \gamma(x)^{i,j}\xi_i\overline{\xi_j} \ge m_\gamma|\xi|^2 \qquad \Big|\sum_{i,j=1}^n \beta(x)^{i,j}\xi_i\overline{\xi'_j}\Big| \le M_\beta|\xi|\,|\xi'|$$

for all $x \in \Omega$ and $\xi, \xi' \in \mathbb{C}^n$. If $\sigma \in H^1(\Omega)$ and $w \in H^1(\Omega)$ is the solution of

$$\begin{aligned} (L_\gamma w)(x) &= \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\Big(\beta^{i,j}(x)\tfrac{\partial\sigma}{\partial x_j}(x)\Big) && \textit{if } x \in \Omega \\ w(x) &= 0 && \textit{if } x \in \partial\Omega \end{aligned}$$

Then

$$\big\|\nabla w\big\|_{L^2(\Omega)} \le \tfrac{M_\beta}{m_\gamma}\big\|\nabla\sigma\big\|_{L^2(\Omega)}$$

Proof: For $\sigma \in C^\infty(\overline{\Omega})$, we also have $w \in C^\infty(\overline{\Omega})$, by Theorem 3.3.12. Then, by the divergence theorem and the assumptions on m_γ and M_β,

$$\begin{aligned} m_\gamma \big\|\nabla w\big\|_{L^2(\Omega)}^2 &\le \sum_{i,j=1}^n \int_\Omega \gamma^{i,j}(x) \tfrac{\partial w}{\partial x_i}(x) \overline{\tfrac{\partial w}{\partial x_j}(x)}\ d^n x = -\int_\Omega w(x) \overline{(L_\gamma w)(x)}\ d^n x \\ &= -\sum_{i,j=1}^n \int_\Omega w(x) \overline{\tfrac{\partial}{\partial x_i}\big(\beta^{i,j}(x) \tfrac{\partial \sigma}{\partial x_j}(x)\big)}\ d^n x \\ &= \sum_{i,j=1}^n \int_\Omega \tfrac{\partial w}{\partial x_i}(x) \overline{\beta^{i,j}(x) \tfrac{\partial \sigma}{\partial x_j}(x)}\ d^n x \\ &\le M_\beta \int_\Omega \big|\nabla w(x)\big|\, \big|\nabla \sigma(x)\big|\ d^n x \\ &\le M_\beta \big\|\nabla w\big\|_{L^2(\Omega)} \big\|\nabla \sigma\big\|_{L^2(\Omega)} \end{aligned}$$

by Cauchy–Schwarz. So

$$m_\gamma \big\|\nabla w\big\|_{L^2(\Omega)} \le M_\beta \big\|\nabla \sigma\big\|_{L^2(\Omega)}$$

for all $\sigma \in C^\infty(\overline{\Omega})$. This bound extends to all $\sigma \in H^1(\Omega)$, since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, by Lemma 2.2.13, and w depends continuously on σ, by Lemma 2.1.16 and Theorem 3.3.5.

■

Lemma 3.4.7 *There is a constant $C(\Omega)$, depending only on Ω, such that the following holds. Let $f \in H^{1/2}(\partial\Omega)$ and $F \in H^1(\Omega)$ be the solution of*

$$\begin{aligned} \Delta F &= 0 && \text{if } x \in \Omega \\ F(x) &= f(x) && \text{if } x \in \partial\Omega \end{aligned}$$

Let, for $i = 1, 2$, γ_i obey Hypothesis 3.3.1, u_i be the solution of

$$\begin{aligned} (L_{\gamma_i} u_i)(x) &= 0 && \text{if } x \in \Omega \\ u_i(x) &= f(x) && \text{if } x \in \partial\Omega \end{aligned}$$

and $v_i = u_i - F$. If

$$m_1 |\xi|^2 \le \sum_{i,j=1}^n \gamma_1(x)^{i,j} \xi_i \xi_j \le M_1 |\xi|^2 \qquad m_2 |\xi|^2 \le \sum_{i,j=1}^n \gamma_2(x)^{i,j} \xi_i \xi_j \le M_2 |\xi|^2$$

and

$$\Big| \sum_{i,j=1}^n \big[\gamma_1(x)^{i,j} - \gamma_2(x)^{i,j}\big] \xi_i \xi_j \Big| \le M_{1,2} |\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, then, for $i = 1, 2$

$$\big\|\nabla v_i\big\|_{L^2(\Omega)} \le C(\Omega) \tfrac{M_i}{m_i} \big\|f\big\|_{1/2,\Omega} \tag{3.4.2}$$

$$\big\|\nabla u_i\big\|_{L^2(\Omega)} \le C(\Omega) \big[1 + \tfrac{M_i}{m_i}\big] \big\|f\big\|_{1/2,\Omega} \tag{3.4.3}$$

and

$$\big\|\nabla (u_1 - u_2)\big\|_{L^2(\Omega)} \le C(\Omega) \tfrac{M_{1,2}}{m_1} \big[1 + \tfrac{M_2}{m_2}\big] \big\|f\big\|_{1/2,\Omega} \tag{3.4.4}$$

Remark. The hypotheses $m_\ell|\xi|^2 \le \sum_{i,j=1}^n \gamma_\ell(x)^{i,j}\xi_i\xi_j \le M_\ell|\xi|^2$, $\ell = 1, 2$, immediately imply that $\left|\sum_{i,j=1}^n \left[\gamma_1(x)^{i,j} - \gamma_2(x)^{i,j}\right]\xi_i\xi_j\right| \le (M_1 + M_2)|\xi|^2$. But if γ_1 and γ_2 are close together, we can get $\left|\sum_{i,j=1}^n \left[\gamma_1(x)^{i,j} - \gamma_2(x)^{i,j}\right]\xi_i\xi_j\right| \le M_{1,2}|\xi|^2$ with $M_{1,2} \ll M_1 + M_2$.

Proof: We have that, for $i = 1, 2$,

$$\begin{aligned} (L_{\gamma_i} v_i)(x) &= -(L_{\gamma_i} F)(x) && \text{if } x \in \Omega \\ v_i(x) &= 0 && \text{if } x \in \partial\Omega \end{aligned}$$

So, by Lemma 3.4.6 and the triangle inequality,

$$\big\|\nabla v_i\big\|_{L^2(\Omega)} \le \tfrac{M_i}{m_i}\big\|\nabla F\big\|_{L^2(\Omega)} \qquad \big\|\nabla u_i\big\|_{L^2(\Omega)} \le \big[1 + \tfrac{M_i}{m_i}\big]\big\|\nabla F\big\|_{L^2(\Omega)}$$

By Theorem 3.3.5, with γ replaced by the identity matrix,

$$\big\|\nabla F\big\|_{L^2(\Omega)} \le \|F\|_{1,\Omega} \le C(\Omega)\|f\|_{1/2,\Omega}$$

which proves (3.4.2) and (3.4.3).

Now

$$\begin{aligned} (L_{\gamma_1}(v_1 - v_2))(x) &= L_{\gamma_1} v_1(x) - L_{\gamma_2} v_2(x) + \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\Big(\big[\gamma_2^{i,j}(x) - \gamma_1^{i,j}(x)\big]\tfrac{\partial v_2}{\partial x_j}(x)\Big) \\ &= -L_{\gamma_1} F(x) + L_{\gamma_2} F(x) + \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\Big(\big[\gamma_2^{i,j}(x) - \gamma_1^{i,j}(x)\big]\tfrac{\partial v_2}{\partial x_j}(x)\Big) \\ &= \sum_{i,j=1}^n \tfrac{\partial}{\partial x_i}\Big(\big[\gamma_2^{i,j}(x) - \gamma_1^{i,j}(x)\big]\tfrac{\partial}{\partial x_j}\big[v_2(x) + F(x)\big]\Big) \qquad \text{if } x \in \Omega \\ v_1(x) - v_2(x) &= 0 \qquad \text{if } x \in \partial\Omega \end{aligned}$$

By Lemma 3.4.6 and (3.4.3),

$$\begin{aligned} \big\|\nabla(v_1 - v_2)\big\|_{L^2(\Omega)} &\le \tfrac{M_{1,2}}{m_1}\big\|\nabla(v_2 + F)\big\|_{L^2(\Omega)} = \tfrac{M_{1,2}}{m_1}\big\|\nabla u_2\big\|_{L^2(\Omega)} \\ &\le C(\Omega)\tfrac{M_{1,2}}{m_1}\big[1 + \tfrac{M_2}{m_2}\big]\|f\|_{1/2,\Omega} \end{aligned}$$

This proves (3.4.4), since $u_1 - u_2 = v_1 - v_2$, ■

Proof of Theorem 3.4.5: Let $f \in H^{1/2}(\partial\Omega)$. By Theorem 3.4.1,

$$\mathcal{L}_{\Lambda_{\gamma_1} f}(f) - \mathcal{L}_{\Lambda_{\gamma_2} f}(f) = \sum_{i,j=1}^n \int_\Omega \gamma_1^{i,j}\tfrac{\partial u_1}{\partial x_i}\tfrac{\partial u_1}{\partial x_j}\ d^n x - \sum_{i,j=1}^n \int_\Omega \gamma_2^{i,j}\tfrac{\partial u_2}{\partial x_i}\tfrac{\partial u_2}{\partial x_j}\ d^n x$$

where, for $i = 1, 2$, u_i solve

$$\begin{aligned} (L_{\gamma_i} u_i)(x) &= 0 \qquad && \text{if } x \in \Omega \\ u_i(x) &= f(x) && \text{if } x \in \partial\Omega \end{aligned}$$

Thus

$$\begin{aligned} &\left|\mathcal{L}_{\Lambda_{\gamma_1} f}(f) - \mathcal{L}_{\Lambda_{\gamma_2} f}(f)\right| \\ &\qquad = \left| \sum_{i,j=1}^{n} \int_\Omega \left(\gamma_1^{i,j} - \gamma_2^{i,j}\right) \tfrac{\partial u_1}{\partial x_i} \tfrac{\partial u_1}{\partial x_j}\ d^n x + \sum_{i,j=1}^{n} \int_\Omega \gamma_2^{i,j} \tfrac{\partial (u_1+u_2)}{\partial x_i} \tfrac{\partial (u_1-u_2)}{\partial x_j}\ d^n x \right| \\ &\qquad \le n^2 \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \|\nabla u_1\|^2_{L^2(\Omega)} + E\|\nabla(u_1+u_2)\|_{L^2(\Omega)} \|\nabla(u_1-u_2)\|_{L^2(\Omega)} \\ &\qquad \le C(\Omega, E) \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \|f\|^2_{1/2,\partial\Omega} \end{aligned}$$

by (3.4.3) and (3.4.4) with $M_{1,2} = n^2 \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}$. The theorem now follows by part (b) of Problem 3.4.2. ■

We shall prove, in Theorem 4.3.1 and Corollary 4.4.21,

Theorem 3.4.8 *Let*

$$\gamma_1^{i,j}(x) = \alpha_1(x)\delta_{i,j} \qquad \gamma_2^{i,j}(x) = \alpha_2(x)\delta_{i,j}$$

with $\alpha_1, \alpha_2 \in C^\infty(\overline{\Omega})$ bounded below by a strictly positive constant and

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.

The proof of Theorem 3.4.8 uses a characterization of Λ_γ in terms of the variational problem in Theorem 3.3.9, which we give below, and also uses a reformulation of Λ_γ in terms of Schrödinger operators, that we give in the next subsection.

Definition 3.4.9 We define the quadratic functional $Q_\gamma : H^{1/2}(\partial\Omega) \to \mathbb{R}$ by

$$Q_\gamma(f) = \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \overline{\tfrac{\partial u_f}{\partial x_j}}\ d^n x = \min_{\substack{w \in H^1(\Omega) \\ Rw = f}} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial w}{\partial x_i} \overline{\tfrac{\partial w}{\partial x_j}}\ d^n x$$

by Theorem 3.3.9. Here $u_f \in H^1(\Omega)$ is the solution of $L_\gamma u_f = 0$, $Ru_f = f$.

Proposition 3.4.10 *Assume that* γ_1 *and* γ_2 *obey Hypothesis 3.3.1. Then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \iff Q_{\gamma_1}(f) = Q_{\gamma_2}(f) \text{ for all } f \in H^{1/2}(\partial\Omega)$$

Proof: The proof is very much like the argument that an inner product is determined by its associated norm via the polarization identity. Since $\sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \frac{\partial u_f}{\partial x_i} \overline{\frac{\partial u_g}{\partial x_j}}\, d^n x$ is linear in f and conjugate linear in g

$$\begin{aligned} Q_\gamma(f+g) - Q_\gamma(f-g) &= 4\mathrm{Re} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \overline{\tfrac{\partial u_g}{\partial x_j}}\, d^n x \\ Q_\gamma(f+ig) - Q_\gamma(f-ig) &= 4\mathrm{Im} \sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \overline{\tfrac{\partial u_g}{\partial x_j}}\, d^n x \end{aligned}$$

Hence

$$\sum_{i,j=1}^{n} \int_\Omega \gamma^{i,j} \tfrac{\partial u_f}{\partial x_i} \overline{\tfrac{\partial u_g}{\partial x_j}}\, d^n x = \tfrac{1}{4}\big[Q_\gamma(f+g) - Q_\gamma(f-g) + iQ_\gamma(f+ig) - iQ_\gamma(f-ig)\big]$$

and, by Theorem 3.4.1 and the observation that $u_{\bar f} = \overline{u_f}$,

$$\mathcal{L}_{\Lambda_\gamma f}(g) = \tfrac{1}{4}\big[Q_\gamma(f+\bar g) - Q_\gamma(f-\bar g) + iQ_\gamma(f+i\bar g) - iQ_\gamma(f-i\bar g)\big] \tag{3.4.5}$$

Since $\mathcal{L}$ is an isomorphism

$$\begin{aligned} \Lambda_{\gamma_1} = \Lambda_{\gamma_2} &\iff \mathcal{L}_{\Lambda_{\gamma_1} f}(g) = \mathcal{L}_{\Lambda_{\gamma_2} f}(g) \text{ for all } f, g \in H^{1/2}(\partial\Omega) \\ &\iff Q_{\gamma_1}(f) = Q_{\gamma_2}(f) \text{ for all } f \in H^{1/2}(\partial\Omega) \end{aligned}$$

■

§3.5. Schrödinger Operators

It is often easier to deal with differential operators whose highest order derivatives have constant coefficients. So we shall use Theorem 3.5.9, below, to relate the Dirichlet to Neumann map for the isotropic conductivity equation $\nabla\cdot\gamma\nabla u = 0$ to the Dirichlet to Neumann map for the Schrödinger equation $\Delta u + qu = 0$ with $q = -\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$. The Schrödinger operator $-\frac{1}{2m}\Delta + q$ is the analog in quantum mechanics of the classical Hamiltonian $\frac{1}{2m}p^2 + q$, which represents the sum of the kinetic and potential energies of a particle of mass m. So it is not surprising that Schrödinger operators have been extensively studied. See, for example, [CF, RS4]. In this subsection, we just provide a bare introduction.

When $q \le 0$, the methods of the previous sections lead to similar existence, uniqueness and regularity results for the boundary value problem

$$\Delta u + qu = 0 \text{ in } \Omega \qquad u = f \text{ on } \partial\Omega$$

See Problems 3.5.2 and 3.5.3. We shall be interested in q's that are not necessarily negative. This leads to one new phenomenon that we shall get to shortly. To try and reduce the degree of repetition a bit, we shall even allow q to be complex, and, for some results, not smooth.

Lemma 3.5.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$.*

(a) There is a unique bounded linear map $L_q : H^1(\Omega) \to H^{-1}(\Omega)$ such that

$$(L_q u)(v) = \int_\Omega v(x)\big[\Delta u(x) + q(x)u(x)\big]\ d^n x$$

for all $u \in C^\infty(\overline{\Omega})$ and all $v \in C_0^\infty(\Omega)$.

(b) There is a unique continuous sesquilinear[3] form

$$\langle\ \cdot\ ,\ \cdot\ \rangle_q : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$$

such that

$$\langle u, v\rangle_q = \sum_{i=1}^n \int_\Omega \tfrac{\partial u}{\partial x_i}(x)\overline{\tfrac{\partial v}{\partial x_i}(x)}\ d^n x + \int_{\partial\Omega} u(x)\overline{v(x)}\ d^{n-1}\sigma - \int_\Omega q(x)u(x)\overline{v(x)}\ d^n x$$

for all $u, v \in C^\infty(\overline{\Omega})$. Here $d^{n-1}\sigma$ denotes the surface measure on $\partial\Omega$. Furthermore

$$\langle v, u\rangle_q = \overline{\langle u, v\rangle_{\bar q}}$$

for all $u, v \in H^1(\Omega)$ and

$$(L_q u)(v) = -\langle v, \bar u\rangle_q$$

for all $u \in H^1(\Omega)$ and all $v \in H_0^1(\Omega)$.

Problem 3.5.1 Prove Lemma 3.5.1.

---

[3] "Sesquilinear" just means linear in the first argument and conjugate linear in the second. That is, $\langle \alpha u + \beta v, w\rangle_q = \alpha\langle u, w\rangle_q + \beta\langle v, w\rangle_q$ and $\langle u, \alpha v + \beta w\rangle_q = \bar\alpha\langle u, v\rangle_q + \bar\beta\langle u, w\rangle_q$ for all $u, v, w \in H^1(\Omega)$ and $\alpha, \beta \in \mathbb{C}$.

Lemma 3.5.2 *Let Ω be a bounded open subset of $\mathrm{I\!R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$.*

(a) If $\operatorname{Re} q(x) \le 0$ for almost all $x \in \Omega$, then there is a $c > 0$ such that

$$\left| \langle u, u \rangle_q \right| \ge c\|u\|_{1,\Omega}^2$$

for all $u \in H^1(\Omega)$.

(b) If there is a $c > 0$ such that $\left| \langle u, u \rangle_q \right| \ge c\|u\|_{1,\Omega}^2$ for all $u \in H^1(\Omega)$, then there are isomorphisms $I_q : H^1(\Omega) \to H^1(\Omega)$ and $J_q : H_0^1(\Omega) \to H_0^1(\Omega)$ such that

$$\begin{aligned} \langle u, v \rangle_q &= \langle u, I_q v \rangle_{1,\Omega} \qquad \text{for all } u, v \in H^1(\Omega) \\ \langle u, v \rangle_q &= \langle u, J_q v \rangle_{1,\Omega} \qquad \text{for all } u, v \in H_0^1(\Omega) \end{aligned}$$

(c) If $q(x) \le 0$ for almost all $x \in \Omega$, then $\langle\ \cdot\ ,\ \cdot\ \rangle_q$ is an inner product and the associated norm, $\|u\|_q = \sqrt{\langle u, u \rangle_q}$ is equivalent to $\|\ \cdot\ \|_{1,\Omega}$.

Proof: (a) Assume that $\operatorname{Re} q(x) \le 0$ for almost all $x \in \Omega$. Then

$$\left| \langle u, u \rangle_q \right| \ge \operatorname{Re}\, \langle u, u \rangle_q = \|u\|_{\gamma=\mathbb{1},\Omega}^2 - \operatorname{Re} \int_\Omega q(x)|u(x)|^2 \, d^n x \ge \|u\|_{\gamma=\mathbb{1},\Omega}^2 \ge c\|u\|_{1,\Omega}^2$$

since $\|\ \cdot\ \|_{\gamma=\mathbb{1},\Omega}$ and $\|\ \cdot\ \|_{1,\Omega}$ are equivalent.

(b) The arguments for I_q and J_q are virtually identical, so we only give the former. For the latter just replace $H^1(\Omega)$ with $H_0^1(\Omega)$ and I_q with J_q.

There is a constant C such that $\left| \langle u, v \rangle_q \right| \le C\|u\|_{1,\Omega}\|v\|_{1,\Omega}$ for all $u, v \in H^1(\Omega)$. So the map $u \mapsto \langle u, v \rangle_q$ is a bounded linear functional on $H^1(\Omega)$ with norm at most $C\|v\|_{1,\Omega}$. By the Riesz representation theorem, there is a unique $w_v \in H^1(\Omega)$ such that $\langle u, v \rangle_q = \langle u, w_v \rangle_{1,\Omega}$ for all $u \in H^1(\Omega)$. Set $I_q v = w_v$. This map is obviously linear and has operator norm at most C. As

$$c\|v\|_{1,\Omega}^2 \le \left| \langle v, v \rangle_q \right| = \left| \langle v, I_q v \rangle_{1,\Omega} \right| \le \|v\|_{1,\Omega}\|I_q v\|_{1,\Omega} \implies \|v\|_{1,\Omega} \le \tfrac{1}{c}\|I_q v\|_{1,\Omega}$$

I_q is 1–1 and the operator norm of the inverse is at most $\frac{1}{c}$.

It remains only to prove that I_q is onto. We first show that the range of I_q is closed. If $w = \lim_{\ell\to\infty} I_q v_\ell$, then $\{v_\ell\}$ is a Cauchy sequence because the inverse of I_q is bounded. Because $H^1(\Omega)$ is complete $v = \lim_{\ell\to\infty} v_\ell$ exists. As I_q is continuous $w = I_q v$ and the range of I_q is closed. If the range is not all of $H^1(\Omega)$, there is a non zero $u \in H^1(\Omega)$ that is orthogonal to the range. But then

$$c\|u\|_{1,\Omega}^2 \le \left| \langle u, u \rangle_q \right| = \left| \langle u, I_q u \rangle_{1,\Omega} \right| = 0$$

provides a contradiction.

(c) The claims are obvious from part (b) of Lemma 3.5.1 and part (a) of this Lemma. ■

Problem 3.5.2 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$.

(a) Prove that the map

$$\begin{aligned} H^1(\Omega) &\to H^{-1}(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \\ u &\mapsto \quad \big(L_q u \ , \ Ru\big) \end{aligned}$$

is bounded.

(b) Assume, in addition, that there is a constant $c > 0$ such that $\big|\langle u, u\rangle_q\big| \ge c\|u\|_{1,\Omega}^2$ for all $u \in H^1(\Omega)$. Prove that the map of part (a) is also 1–1 and onto with a bounded inverse. In this case, for each $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$ there is a unique $u \in H^1(\Omega)$ such that $\Delta u + qu = F$ in Ω and $u\big|_{\partial\Omega} = f$. Furthermore, there is a constant C such that

$$\|u\|_{1,\Omega}^2 \le C\big(\|F\|_{-1,\Omega}^2 + \|f\|_{1/2,\partial\Omega}^2\big)$$

for all $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$.

Problem 3.5.3 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and $\ell \in \mathbb{N}$. Let

$$q \in \begin{cases} L^\infty(\Omega) & \text{if } \ell \le 2 \\ C^{\ell-2}(\overline{\Omega}) & \text{if } \ell > 2 \end{cases}$$

Prove that there is a constant C, depending only on ℓ, q and Ω such that for all $u \in H^1(\Omega)$, $F \in H^{\ell-2}(\Omega)$ and $f \in H^{\ell-\frac{1}{2}}(\partial\Omega)$ obeying

$$L_q u = F \qquad Ru = f$$

we have $u \in H^\ell(\Omega)$ and

$$\|u\|_{\ell,\Omega} \le C\Big(\ \|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega} + \|u\|_{L^2(\Omega)}\Big)$$

The following example shows that the conclusions of part (b) of Problem 3.5.2 cannot hold for all $q \in L^\infty(\Omega)$ or even for all $q \in C^\infty(\overline{\Omega})$.

Example 3.5.3 Let $\Omega = (0, \pi) \subset \mathbb{R}$. We compute the kernel of the map

$$\begin{aligned} H^1(\Omega) &\to H^{-1}(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \\ u &\mapsto \quad \big(L_q u \ , \ Ru\big) \end{aligned}$$

in the special case that the function q happens to be a constant, that we call $-\lambda$. In other words, we find all $u \in H^1(\Omega)$ obeying

$$u''(x) = \lambda u(x) \text{ for } 0 < x < \pi \qquad u(0) = u(\pi) = 0$$

By Lemma 2.2.14 and Problem 3.5.3, with $F = f = 0$, $u \in C^\infty([0,\pi])$ so that u'' is the conventional second derivative of u. The general solution to the ordinary differential equation $u''(x) = \lambda u(x)$ is

$$u(x) = A \sin\left(\sqrt{-\lambda}\, x\right) + B \cos\left(\sqrt{-\lambda}\, x\right)$$

The boundary condition $u(0) = 0$ is satisfied if and only if $B = 0$. With $B = 0$, the boundary condition $u(\pi) = 0$ is satisfied if and only either $A = 0$ or $\sin\left(\sqrt{-\lambda}\, \pi\right) = 0$. The latter condition is equivalent to

$$\sqrt{-\lambda} \in \mathbb{Z} \iff -\lambda \in \mathbb{N}$$

So if λ is not strictly negative integer, the kernel is trivial. If λ is a strictly negative integer, the kernel has dimension one.

Definition 3.5.4 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. The number $\lambda \in \mathbb{C}$ is said to be an eigenvalue of the differential operator $\Delta + q$ with Dirichlet boundary conditions on $\partial\Omega$ if there is a nonzero $u \in H_0^1(\Omega)$ such that $(\Delta + q)u = \lambda u$. The vector u is said to be an eigenfunction of $\Delta + q$ with eigenvalue λ. In particular, zero is an eigenvalue of the differential operator $\Delta + q$ with Dirichlet boundary conditions on $\partial\Omega$ if and only if the map $u \in H^1(\Omega) \mapsto \left(L_q u\ ,\ Ru\right)$ has a nontrivial kernel.

We have just seen, in Example 3.5.3, that the eigenvalues of Δ with Dirichlet boundary conditions on the boundary of $(0, \pi)$ are precisely the strictly negative integers. We shall see below, in Lemma 3.5.6, that, for any $q \in L^\infty(\Omega)$, with Ω a bounded open set with smooth boundary, the set of eigenvalues of L_q with Dirichlet boundary conditions on $\partial\Omega$ is a discrete subset of $\mathbb{C}$. Furthermore if zero is not such an eigenvalue, the conclusions of part (b) of Problem 3.5.2 hold. This will be part of Theorem 3.5.8.

Lemma 3.5.5 (The Resolvent of L_q) *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Let*

$$Q \geq \|q\|_{L^\infty(\Omega)}$$

(a) There is a linear transformation $R_{q,Q} : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$, called the resolvent of $\Delta + q$, and a constant $C = C(\Omega, q, Q)$ such that

$$\begin{aligned} \|R_{q,Q}v\|_{2,\Omega} &\leq C\|v\|_{L^2(\Omega)} && \text{for all } v \in L^2(\Omega) \\ (\Delta + q - Q)R_{q,Q}v &= v && \text{for all } v \in L^2(\Omega) \\ R_{q,Q}(\Delta + q - Q)u &= u && \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega) \end{aligned}$$

(b) The map $R_{q,Q}$ is compact as an operator on $L^2(\Omega)$. This means that if $\{v_i\}_{i\in\mathbb{N}}$ is any bounded sequence in $L^2(\Omega)$, then there is a subsequence of $\{R_{q,Q}v_i\}_{i\in\mathbb{N}}$ that converges in $L^2(\Omega)$. Equivalently, if $\{u_i\}_{i\in\mathbb{N}}$ is any sequence in $H^2(\Omega)\cap H^1_0(\Omega)$ such that $\{(\Delta+q-Q)u_i\}_{i\in\mathbb{N}}$ is a bounded subset of $L^2(\Omega)$, then there is a subsequence of $\{u_i\}_{i\in\mathbb{N}}$ that converges in $L^2(\Omega)$.

(c) For any $\lambda\in\mathbb{C}\setminus\{0\}$, the range of $R_{q,Q}-\lambda\mathbb{1} : L^2(\Omega)\to L^2(\Omega)$ is the orthogonal complement of the kernel of $R_{\bar q,Q}-\bar\lambda\mathbb{1}$.

Proof: (a) By hypothesis, $\mathrm{Re}\big(q(x)-Q\big)\le 0$ for almost all $x\in\Omega$. Hence, by part(a) of Lemma 3.5.2, there is a $c>0$ such that $\big|\langle u,u\rangle_{q-Q}\big|\ge c\|u\|^2_{1,\Omega}$ for all $u\in H^1(\Omega)$. So the conclusions of both parts of Problem 3.5.2 are applicable to $q-Q$ and, for each $v\in H^{-1}(\Omega)$, there is a unique $u\in H^1(\Omega)$ such that $(\Delta+q-Q)u=v$ and $Ru=0$. By part (v) of Theorem 2.2.2, the condition $Ru=0$ is equivalent to $u\in H^1_0(\Omega)$, so we may restrict v to $L^2(\Omega)$ and set $R_{q,Q}v=u$. By part (b) of Problem 3.5.2,

$$\|R_{q,Q}v\|_{1,\Omega}\le\sqrt{C}\|v\|_{-1,\Omega}\le\sqrt{C}\|v\|_{L^2(\Omega)}$$

By Problem 3.5.3, with $\ell=2$, $F=v$, $f=0$ and q replaced by $q-Q$, we have that $u\in H^2(\Omega)$ and

$$\|R_{q,Q}v\|_{2,\Omega}\le C'\big(\|v\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)}\big)\le C'\big(\|v\|_{L^2(\Omega)}+\|u\|_{1,\Omega}\big)\le C'\big(1+\sqrt{C}\,\big)\|v\|_{L^2(\Omega)}$$

By construction, $(\Delta+q-Q)R_{q,Q}v=v$ for all $v\in L^2(\Omega)$. For the last claim, observe that if $u\in H^2(\Omega)\cap H^1_0(\Omega)$, then $(\Delta+q-Q)u\in L^2(\Omega)$ and $w=R_{q,Q}(\Delta+q-Q)u$ obeys $(\Delta+q-Q)w=(\Delta+q-Q)u$, $Rw=0$. As u obeys the same boundary value problem and as the boundary value problem has a unique solution, $R_{q,Q}(\Delta+q-Q)u=w=u$.

(b) Set $u_i=R_{q,Q}v_i$. By part (a),

$$\|u_i\|_{1,\Omega}=\|R_{q,Q}v_i\|_{1,\Omega}\le\sqrt{C}\,\|v_i\|_{L_2(\Omega)}$$

so that $\{u_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $H^1_0(\Omega)$. By Rellich's theorem (Problem 2.1.15) there is a subsequence of $\{u_i\}_{i\in\mathbb{N}}$ that converges in $L^2(\Omega)$.

To get the second formulation, just set $v_i=(\Delta+q-Q)u_i$. Then $u_i=R_{q,Q}v_i$, by the construction of $R_{q,Q}$ in the proof of part (a).

(c) The adjoint operator, A^*, of any bounded operator $A:L^2(\Omega)\to L^2(\Omega)$ is determined by

$$\langle Ax,y\rangle_{L^2(\Omega)}=\langle x,A^*y\rangle_{L^2(\Omega)}\qquad\text{for all }x,y\in L^2(\Omega)$$

We now show that $\mathcal{R}^*_{q,Q}=\mathcal{R}_{\bar q,Q}$. Then the claim will be an immediate consequence of part (b) of Problem 3.5.4, below.

Two applications of part (c) of Problem 3.4.1 (the divergence theorem) gives

$$\begin{aligned}
\langle R_{q,Q}v, u\rangle_{L^2(\Omega)} &= \langle R_{q,Q}v, (\Delta + \bar{q} - Q)R_{\bar{q},Q}u\rangle_{L^2(\Omega)} \\
&= \int_\Omega R_{q,Q}v(x)\ \overline{[\Delta + \overline{q(x)} - Q]R_{\bar{q},Q}u(x)}\ d^nx \\
&= \int_\Omega [\Delta + q(x) - Q]R_{q,Q}v(x)\ \overline{R_{\bar{q},Q}u(x)}\ d^nx \\
&= \langle (\Delta + q - Q)R_{q,Q}v, R_{\bar{q},Q}u\rangle_{L^2(\Omega)} \\
&= \langle v, R_{\bar{q},Q}u\rangle_{L^2(\Omega)}
\end{aligned}$$

which is what we wanted to show. The two boundary terms vanished because both $R_{q,Q}v$ and $R_{\bar{q},Q}u$ are in $H^2(\Omega) \cap H^1_0(\Omega)$. ■

Problem 3.5.4 Let $\mathcal{H}$ be any Hilbert space. A linear operator $C : \mathcal{H} \to \mathcal{H}$ is said to be compact if for each bounded sequence $\{v_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$, there is a subsequence of $\{Cv_i\}_{i\in\mathbb{N}}$ that is convergent. See Appendix A for an introduction to compact operators. Let C be a compact operator on $\mathcal{H}$.

(a) Set, for each $\lambda \in \mathbb{C}$, $\mathcal{E}_\lambda = \{\ v \in \mathcal{H}\ \big|\ Cv = \lambda v\ \}$. If $\mathcal{E}_\lambda \neq \{0\}$, then λ is called an eigenvalue of C and the nonzero elements of $\mathcal{E}_\lambda$ are called the eigenvectors of C with eigenvalue λ. Prove that if $\varepsilon > 0$, then $\#\{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| \geq \varepsilon, \mathcal{E}_\lambda \neq \{0\}\ \} < \infty$ and furthermore, that if $\lambda \neq 0$, then $\dim \mathcal{E}_\lambda < \infty$.

(b) Prove that if $\lambda \neq 0$ then the range of $C - \lambda\mathbb{1}$ is the orthogonal complement of

$$\mathcal{E}_{*,\bar{\lambda}} = \{\ v \in \mathcal{H}\ \big|\ C^*v = \bar{\lambda}v\ \}$$

Here C^*, the adjoint operator of C, is determined by

$$\langle Cx, y\rangle_{\mathcal{H}} = \langle x, C^*y\rangle_{\mathcal{H}} \qquad \text{for all } x, y \in \mathcal{H}$$

(c) Give an example of a compact operator C whose range is not the orthogonal complement of $\mathcal{E}_{*,0}$.

Lemma 3.5.6 (Eigenvalues of L_q) *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Set*

$$\mathcal{E}_\lambda = \{\ u \in H^1_0(\Omega)\ \big|\ L_qu = \lambda u\ \}$$

For any compact subset K of $\mathbb{C}$, there are only finitely many $\lambda \in K$ such that $\mathcal{E}_\lambda \neq \{0\}$. Furthermore, $\mathcal{E}_\lambda$ is finite dimensional for every $\lambda \in \mathbb{C}$.

Proof: Let K be any compact subset of $\mathbb{C}$ and let $\{\lambda_i\}_{i\in\mathbb{N}}$ be any sequence in K. It suffices for us to exhibit a contradiction to the assumption that there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subset H_0^1(\Omega)$ of independent vectors such that $L_q u_i = \lambda_i u_i$ for all $i \in \mathbb{N}$. By Problem 3.5.3, $\{u_i\}_{i\in\mathbb{N}} \subset H^2(\Omega) \cap H_0^1(\Omega)$.

Choose Q as in Lemma 3.5.5 and set $\mu_i = \lambda_i - Q$. Then $L_{q-Q} u_i = \mu_i u_i$. Since L_{q-Q} is injective $\mu_i \neq 0$. Apply the Gram–Schmidt orthogonalization process to $\{u_i\}_{i\in\mathbb{N}}$ to construct a sequence $\{y_i\}_{i\in\mathbb{N}} \subset H_0^1(\Omega)$ that is orthonormal in $L^2(\Omega)$ with each y_i of the form $y_i = \sum\limits_{j=1}^{i} c_{i,j} u_j$. Setting $v_i = \sum\limits_{j=1}^{i} \frac{\mu_i}{\mu_j} c_{i,j} u_j$ we have

$$L_{q-Q} v_i = \sum_{j=1}^{i} \tfrac{\mu_i}{\mu_j} c_{i,j} L_{q-Q} u_j = \sum_{j=1}^{i} \mu_i c_{i,j} u_j = \mu_i y_i$$

so that $\{(\Delta + q - Q)v_i\}_{i\in\mathbb{N}}$ is a bounded subset of $L^2(\Omega)$. On the other hand, if $\ell < i$,

$$v_i - v_\ell - y_i = \sum_{j=1}^{i} \tfrac{\mu_i}{\mu_j} c_{i,j} u_j - \sum_{j=1}^{\ell} \tfrac{\mu_\ell}{\mu_j} c_{\ell,j} u_j - \sum_{j=1}^{i} c_{i,j} u_j = \sum_{j=1}^{i-1} \big(\tfrac{\mu_i}{\mu_j} - 1\big) c_{i,j} u_j - \sum_{j=1}^{\ell} \tfrac{\mu_\ell}{\mu_j} c_{\ell,j} u_j$$

is in the span of $\big\{u_1, \cdots, u_{i-1}\big\}$ and consequently is orthogonal to y_i in $L^2(\Omega)$. Thus

$$\|v_i - v_\ell\|_{L^2(\Omega)}^2 = \|v_i - v_\ell - y_i\|_{L^2(\Omega)}^2 + \|y_i\|_{L^2(\Omega)}^2 \geq \|y_i\|_{L^2(\Omega)}^2 = 1$$

and $\{v_i\}_{i\in\mathbb{N}}$ may not contain any subsequence that converges in $L^2(\Omega)$, in contradiction to part (b) of Lemma 3.5.5. ■

Problem 3.5.5 Let $\mathcal{H}$ be a Hilbert space and $C : \mathcal{H} \to \mathcal{H}$ a compact operator that obeys $C = C^*$.
(a) Prove that all eigenvalues of C are real.
(b) Let φ_1 and φ_2 be eigenvectors of C with eigenvalues λ_1 and λ_2, respectively. Prove that if $\lambda_1 \neq \lambda_2$, then $\varphi_1 \perp \varphi_2$.
(c) Prove that if $\mathcal{H} \neq \{0\}$, then there is an eigenvalue λ of C that obeys $|\lambda| = \|C\|$.
(d) Prove that there is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors for C.

Problem 3.5.6 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$ be real valued. In the following "eigenvalue" and "eigenvector" always refer to eigenvalues and eigenvectors of $\Delta + q$ with Dirichlet boundary conditions on $\partial\Omega$, as in Definition 3.5.4.
(a) Prove that all eigenvalues are real.
(b) Let φ_1 and φ_2 be eigenvectors of with eigenvalues λ_1 and λ_2, respectively. Prove that if $\lambda_1 \neq \lambda_2$, then $\varphi_1 \perp \varphi_2$.
(c) Prove that there is an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of $\Delta + q$.

Proposition 3.5.7 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Let*

$$\mathcal{K}_q = \left\{\ u \in H_0^1(\Omega)\ \middle|\ L_q u = 0\ \right\}$$
$$\mathcal{K}_{\bar q} = \left\{\ u \in H_0^1(\Omega)\ \middle|\ L_{\bar q} u = 0\ \right\}$$

Then there is a constant c such that the following hold.

(a) $\dim(\mathcal{K}_q) = \dim(\mathcal{K}_{\bar q}) < \infty$.

(b) If $F \in L^2(\Omega)$, then there exists a $u \in H_0^1(\Omega)$ such that $L_q u = F$ if and only if F is orthogonal to $\mathcal{K}_{\bar q}$ in $L^2(\Omega)$, in which case u is unique modulo $\mathcal{K}_q$.

(c) If $\mathcal{K}_q = \{0\}$, then there is a constant c such that, for each $F \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ obeying $L_q u = F$ and this u also obeys

$$\|u\|_{1,\Omega} \le c\|F\|_{L^2(\Omega)}$$

(d) If $\mathcal{K}_q = \{0\}$, then there is a constant c such that, for each $F \in H^{-1}(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ obeying $L_q u = F$ and this u also obeys

$$\|u\|_{1,\Omega} \le c\|F\|_{-1,\Omega}$$

Proof: (a) Since

$$L_q u = 0 \iff 0 = \overline{L_q u} = L_{\bar q}\bar u$$

We have $\overline{\mathcal{K}_q} = \mathcal{K}_{\bar q}$, where, by definition, the complex of a subset S of $H_0^1(\Omega)$ is just $\bar S = \left\{\ \bar u\ \middle|\ u \in S\ \right\}$. Thus, for any index set I, $\{u_i\}_{i\in I}$ is a basis for $\mathcal{K}_q$ if and only if $\{\overline{u_i}\}_{i\in I}$ is a basis for $\mathcal{K}_{\bar q}$. In particular, $\dim(\mathcal{K}_q) = \dim(\mathcal{K}_{\bar q})$.

We now prove by contradiction that $\dim(\mathcal{K}_q) < \infty$. If $\mathcal{K}_q$ is infinite dimensional, it contains an infinite sequence of independent elements. By applying Gram–Schmidt orthogonalization, we may construct another infinite set $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{K}_q \subset H_0^1(\Omega) \cap H^2(\Omega)$ such that

- ○ $\|u_i\|_{L^2(\Omega)} = 1$ for all $i \in \mathbb{N}$
- ○ $\langle u_i, u_j\rangle_{L^2(\Omega)} = 0$ for all $i, j \in \mathbb{N}$ with $i \ne j$

Choose Q as in Lemma 3.5.5. Then $\{(\Delta + q - Q)u_i = L_q u_i - Qu_i = -Qu_i\}_{i\in\mathbb{N}}$ is a bounded subset of $L^2(\Omega)$. By part (b) of Lemma 3.5.5, there is a subsequence of $\{u_i\}_{i\in\mathbb{N}}$ that converges in $L^2(\Omega)$. But that's impossible because $\|u_i - u_j\|_{L^2(\Omega)} = \sqrt{2}$ for all $i \ne j$.

(b) *Necessity:* If $L_q u = F \in L^2(\Omega)$ and $v \in H_0^1(\Omega)$, then

$$\langle F, v\rangle_{L^2(\Omega)} = (L_q u)(\bar v)$$

By part (a) of Lemma 3.5.1 and two applications of the divergence theorem,

$$\begin{aligned}(L_q u)(\bar v) &= \int_\Omega \overline{v(x)}\,\big[\Delta u(x) + q(x)u(x)\big]\ d^n x = \overline{\int_\Omega \big[\Delta v(x) + \overline{q(x)}v(x)\big]\overline{u(x)}\ d^n x} \\ &= \overline{(L_{\bar q} v)(\bar u)}\end{aligned}$$

for all $u, v \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$,

$$(L_q u)(\bar v) = \overline{(L_{\bar q} v)(\bar u)}$$

for all $u, v \in H_0^1(\Omega)$, by continuity. If $v \in \mathcal{K}_{\bar q}$, then $L_{\bar q} v = 0$ and $\langle F, v\rangle_{L^2(\Omega)} = 0$, so F is orthogonal to $\mathcal{K}_{\bar q}$.

Sufficiency: Let F be orthogonal to $\mathcal{K}_{\bar q}$ in $L^2(\Omega)$. Choose $Q > 0$ as in Lemma 3.5.5 and observe that, for $u \in H_0^1(\Omega)$,

$$\begin{aligned} L_q u = F &\iff L_{q-Q} u = F - Qu \iff R_{q,Q}(F - Qu) = u \\ &\iff \big[R_{q,Q} + \tfrac{1}{Q}\big]u = \tfrac{1}{Q} R_{q,Q} F \end{aligned} \tag{3.5.1}$$

By part (c) of Lemma 3.5.5, the range of $R_{q,Q} + \frac{1}{Q}$ is exactly the orthogonal complement, in $L^2(\Omega)$, of the kernel of $R_{\bar q,Q} + \frac{1}{Q}$.

We now verify that $\frac{1}{Q} R_{q,Q} F$ is in the orthogonal complement of the kernel of $R_{\bar q,Q} + \frac{1}{Q}$. If v is in the kernel of $R_{\bar q,Q} + \frac{1}{Q}$, then

$$\begin{aligned} \big[R_{\bar q,Q} + \tfrac{1}{Q}\big]v = 0 &\implies R_{\bar q,Q}(-Qv) = v \implies L_{\bar q - Q} v = -Qv \implies L_{\bar q} v = 0 \\ &\implies v \in \mathcal{K}_{\bar q} \end{aligned} \tag{3.5.2}$$

(since $v = -QR_{\bar q,Q} v$, $v \in H_0^1(\Omega)$ is automatic) and

$$\langle R_{q,Q} F, v\rangle_{L^2(\Omega)} = \langle F, R_{\bar q,Q} v\rangle_{L^2(\Omega)} = -\tfrac{1}{Q}\,\langle F, v\rangle_{L^2(\Omega)} = 0$$

since $F \perp \mathcal{K}_{\bar q}$ by assumption. So $R_{q,Q} F$ is orthogonal to the kernel of $R_{\bar q,Q} + \frac{1}{Q}$. Consequently, there is a $u \in L^2(\Omega)$ obeying $\big[R_{q,Q} + \frac{1}{Q}\big]u = \frac{1}{Q} R_{q,Q} F$ and this same u, which is automatically in $H_0^1(\Omega)$ since $u = R_{q,Q}(F - Qu)$, obeys $L_q u = F$.

Uniqueness: For $u, u' \in H_0^1(\Omega)$,

$$L_q u = L_q u' \iff L_q(u - u') = 0 \iff u - u' \in \mathcal{K}_q$$

(c) If $\mathcal{K}_q = \{0\}$, then $\mathcal{K}_{\bar q} = \{0\}$ by part (a). So, by part (b), for each $F \in L^2(\Omega)$, there is a unique $u \in H_0^1$ obeying $L_q u = F$. The proof of the bound is by contradiction. If the bound is not true, there is a sequence of unit vectors $u_i \in H^1_0(\Omega)$ such that $L_q u_i$ converges to 0 in $L^2(\Omega)$. Choose $Q > 0$ as in Lemma 3.5.5. Set

$$v_i = L_{q-Q} u_i = L_q u_i - Q u_i$$

Then $\{v_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $L^2(\Omega)$. By part(b) of Lemma 3.5.5, there is a subsequence $\{u_{i_\ell}\}_{\ell\in\mathbb{N}}$ that converges in $L^2(\Omega)$, say to $u \in L^2(\Omega)$. Then $\{v_{i_\ell}\}_{\ell\in\mathbb{N}}$ converges to $-Qu$ in $L^2(\Omega)$ and, by part (a) of Lemma 3.5.5, $u_{i_\ell} = R_{q,Q} v_{i_\ell}$ converges in $H^1_0(\Omega)$ to $-QR_{q,Q}u$. Thus $u \in H^1_0(\Omega)$ and $u = -QR_{q,Q}u$ so that $R_{q,Q} + \frac{1}{Q}$ has a nontrivial kernel. But this is a contradiction because (3.5.2), with q replaced by $\bar{q}$, proves that the kernel of $R_{q,Q} + \frac{1}{Q}$ is trivial.

(d) If $L_q u_1 = L_q u_2 = F$, with $u_1, u_2 \in H^1_0(\Omega)$, then $L_q(u_1 - u_2) = 0$ and $u_1 - u_2 \in \mathcal{K}_q$, so solutions must be unique. The proofs of surjectiveness and of the bound are motivated by the “first resolvent identity”, which is a special case of

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

By part (b) of Problem 3.5.2 (with $f = 0$ and q replaced by $q - Q$), the linear map $L_{q-Q} : H^1_0(\Omega) \to H^{-1}(\Omega)$ has a bounded inverse $\mathcal{L}_Q : H^{-1}(\Omega) \to H^1_0(\Omega)$. By part (c) of this Proposition, $L_q : \left\{\ u \in H^1_0(\Omega)\ \middle|\ L_q u \in L^2(\Omega)\ \right\} \to L^2(\Omega)$ has a bounded inverse $\mathcal{L} : L^2(\Omega) \to H^1_0(\Omega)$. If $F \in H^{-1}(\Omega)$, set

$$u = \mathcal{L}_Q F - Q\mathcal{L}\mathcal{L}_Q F$$

Then,

$$\begin{aligned} L_q u &= L_q \mathcal{L}_Q F - Q L_q \mathcal{L}\mathcal{L}_Q F = L_{q-Q}\mathcal{L}_Q F + Q\mathcal{L}_Q f - Q L_q \mathcal{L}\mathcal{L}_Q F \\ &= F + Q\mathcal{L}_Q f - Q\mathcal{L}_Q F = F \end{aligned}$$

so the solution exists and, using $\|\mathcal{L}\|$ and $\|\mathcal{L}_Q\|$ to denote the operator norms of the maps $\mathcal{L}_Q : H^{-1}(\Omega) \to H^1_0(\Omega)$ and $\mathcal{L}_Q : H^{-1}(\Omega) \to H^1_0(\Omega)$, respectively,

$$\begin{aligned} \|u\|_{1,\Omega} &\le \big\|\mathcal{L}_Q F\big\|_{1,\Omega} + Q\big\|\mathcal{L}\mathcal{L}_Q F\big\|_{1,\Omega} \\ &\le \big\|\mathcal{L}_Q F\big\|_{1,\Omega} + Q\big\|\mathcal{L}\big\|\,\big\|\mathcal{L}_Q F\big\|_{L^2(\Omega)} \\ &\le \Big\{1 + Q\big\|\mathcal{L}\big\|\Big\}\big\|\mathcal{L}_Q F\big\|_{1,\Omega} \\ &\le \Big\{1 + Q\big\|\mathcal{L}\big\|\Big\}\big\|\mathcal{L}_Q\big\|\,\|F\|_{-1,\Omega} \end{aligned}$$

so the desired bound is satisfied.

■

Theorem 3.5.8 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Let $L_q : H^1(\Omega) \to H^{-1}(\Omega)$ be the Schrödinger operator with potential q defined in part (a) of Lemma 3.5.1 and $R : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ be the restriction map of Theorem 2.2.2. Assume that 0 is not an eigenvalue for L_q with Dirichlet boundary conditions. That is, assume that $\mathcal{K}_q = \left\{ \ u \in H_0^1(\Omega) \ \middle| \ L_q u = 0 \ \right\} = \{0\}$.*

(a) The map

$$\begin{aligned} H^1(\Omega) &\to H^{-1}(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \\ u &\mapsto \quad \big(L_q u \ , \ Ru\big) \end{aligned}$$

is an isomorphism.

(b) Let $\ell \in \mathbb{N}$. If $\ell \ge 3$, assume that $q \in C^{\ell-2}(\overline{\Omega})$. There is a constant C, depending only on ℓ, Ω and q such that, for all $u \in H^1(\Omega)$, $F \in H^{\ell-2}(\Omega)$ and $f \in H^{\ell-\frac{1}{2}}(\partial\Omega)$ obeying

$$L_q u = F \qquad Ru = f$$

we have $u \in H^\ell(\Omega)$ and

$$\|u\|_{\ell,\Omega} \le C\Big(\|F\|_{\ell-2,\Omega} + \|f\|_{\ell-\frac{1}{2},\partial\Omega}\Big)$$

(c) Let, for each $f \in H^{3/2}(\partial\Omega)$, $u_f \in H^2(\Omega)$ be the solution of

$$\begin{aligned} (L_q u_f)(x) &= 0 \qquad && \text{if } x \in \Omega \\ u_f(x) &= f(x) && \text{if } x \in \partial\Omega \end{aligned}$$

provided by part (b) of this Theorem, with $\ell = 2$. There is a unique bounded linear map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ such that

$$\mathcal{L}_{\Lambda_q f}(g) = \sum_{i=1}^n \int_{\partial\Omega} \tfrac{\partial u_f}{\partial x_i}(x) \hat{n}^i(x) \ g(x) \ d\sigma(x)$$

for all $f \in H^{3/2}(\partial\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Furthermore, if $f, g \in H^{1/2}(\partial\Omega)$ and $v_g \in H^1(\Omega)$ obeys $Rv_g = g$, then

$$\mathcal{L}_{\Lambda_q f}(g) = \int_\Omega \Big[\sum_{i=1}^n \tfrac{\partial u_f}{\partial x_i} \tfrac{\partial v_g}{\partial x_i} - q(x) u_f(x) v_g(x) \Big] \ d^n x$$

Here $\hat{n}^i(x)$ refers to the $i^{\rm th}$ component of $\hat{n}(x)$, the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$, $d\sigma(x)$ refers to the surface measure on $\partial\Omega$ and $\mathcal{L} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)^*$ refers to the isomorphism, from Problem 2.1.24, that is determined by

$$\mathcal{L}_f g = \int_{\partial\Omega} f(x) g(x) \ d\sigma(x) \qquad \text{for all } f, g \in C^\infty(\partial\Omega)$$

Proof: (a) *Boundedness* was part (a) of Problem 3.5.2.
Injectiveness: If $Ru = 0$, then $u \in H^1_0(\Omega)$ by part (v) of Theorem 2.2.2. If $u \in H^1_0(\Omega)$ and $L_q u = 0$, then $u \in \mathcal{K}_q$ and so $u = 0$ by hypothesis.
Surjectiveness and boundedness of the inverse: Let $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$. By parts (iii) and (iv) of Theorem 2.2.2, there is a $w \in H^1(\Omega)$ such that $Rw = f$ and $\|w\|_{1,\Omega} \le C'\|f\|_{1/2,\partial\Omega}$. By part (d) of Proposition 3.5.7, there is a $v \in H^1_0(\Omega)$ such that $L_q v = F - L_q w$ and $\|v\|_{1,\Omega} \le c\|F - L_q w\|_{-1,\Omega}$. Then $u = v + w$ obeys

$$L_q u = L_q v + L_q w = F \qquad Ru = Rv + Rw = 0 + f = f$$

and

$$\begin{aligned}\|u\|_{1,\Omega} &\le \|v\|_{1,\Omega} + \|w\|_{1,\Omega} \le c\|F - L_q w\|_{-1,\Omega} + \|w\|_{1,\Omega}\\ &\le c\|F\|_{-1,\Omega} + cc_1\|w\|_{1,\Omega} + \|w\|_{1,\Omega}\\ &\le c\|F\|_{-1,\Omega} + (1 + cc_1)C'\|f\|_{1/2,\partial\Omega}\end{aligned}$$

(b) The case $\ell = 1$ was proven in part (a). The remaining cases follow from Problem 3.5.3.

(c) The proof of this part of the Theorem is Problem 3.5.7.

■

Problem 3.5.7 Prove part (c) of Theorem 3.5.8.

Problem 3.5.8 Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $q \in L^\infty(\Omega)$. Let $L_q : H^1(\Omega) \to H^{-1}(\Omega)$ be the Schrödinger operator with potential q defined in part (a) of Lemma 3.5.1 and $R : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ be the restriction map of Theorem 2.2.2. Do *not* assume that 0 is not an eigenvalue for L_q with Dirichlet boundary conditions. Let $v \in H^1(\Omega)$ obey $L_q v = 0$.

(a) Prove that if $w,\ w' \in H^1(\Omega)$ obey $Rw = Rw'$, then

$$\int_\Omega \big[\nabla w(x)\cdot\nabla v(x) - q(x)w(x)v(x)\big]\ d^n x = \int_\Omega \big[\nabla w'(x)\cdot\nabla v(x) - q(x)w'(x)v(x)\big]\ d^n x$$

(b) Prove that

$$h \in H^{1/2}(\partial\Omega) \mapsto \int_\Omega \big[\nabla w_h(x)\cdot\nabla v(x) - q(x)w_h(x)v(x)\big]\ d^n x \quad \text{with } w_h \in H^1(\Omega),\ Rw_h = h$$

is a well–defined, bounded linear functional on $H^{1/2}(\Omega)$ with norm at most

$$C\big[1 + \|q\|_{L^\infty(\Omega)}\big]\|v\|_{1,\Omega}$$

(c) Prove that if $v \in H^2(\Omega)$, then the linear functional of part (b) is

$$h \in H^{1/2}(\partial\Omega) \mapsto \int_{\partial\Omega} h(x)\tfrac{\partial v}{\partial\nu}(x)\ d\sigma(x)$$

Theorem 3.5.9 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary, $\gamma \in C^\infty(\overline{\Omega})$ be strictly positive and set $q = -\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$.*

(a) If $f \in H^{1/2}(\partial\Omega)$, then $u \in H^1(\Omega)$ is a solution of

$$\nabla \cdot \gamma\nabla u = 0 \ \text{in } \Omega \qquad u = f \ \text{on } \partial\Omega$$

if and only if $w = \gamma^{1/2}u$ is a solution of

$$\Delta w + qw = 0 \ \text{in } \Omega \qquad w = \gamma^{1/2}f \ \text{on } \partial\Omega$$

(b) Let Λ_q be the Dirichlet to Neumann map for the Schrödinger operator $\Delta + q$ on Ω, defined in part (c) of Theorem 3.5.8 and let Λ_γ be the Dirichlet to Neumann map for isotropic conductivity equation $\nabla \cdot \gamma\nabla u = 0$ on Ω, defined in Theorem 3.4.1. These two Dirichlet to Neumann maps are related by

$$\Lambda_q(g) = \gamma^{-1/2}\Lambda_\gamma\big(\gamma^{-1/2}g\big) + \sum_{j=1}^n \tfrac{1}{2}\gamma^{-1}\tfrac{\partial\gamma}{\partial x_j}\hat{n}^j\ g\Big|_{\partial\Omega}$$

Proof: (a) By the product rule

$$\begin{aligned}
\Delta w &= \nabla\cdot\nabla w = \nabla\cdot\nabla\big(\gamma^{1/2}u\big) = \nabla\cdot\big(\gamma^{1/2}\nabla u + u\nabla\gamma^{1/2}\big) \\
&= \gamma^{1/2}\Delta u + 2\nabla\gamma^{1/2}\cdot\nabla u + u\Delta\gamma^{1/2} \\
&= \gamma^{1/2}\Delta u + \gamma^{-1/2}\nabla\gamma\cdot\nabla u + w\gamma^{-1/2}\Delta\gamma^{1/2} \\
&= \gamma^{1/2}\Delta u + \gamma^{-1/2}\nabla\gamma\cdot\nabla u - wq \\
&= \gamma^{1/2}\Delta u + \gamma^{-1/2}\big(\nabla\cdot(\gamma\nabla u) - \gamma\Delta u\big) - wq \\
&= \gamma^{-1/2}\nabla\cdot(\gamma\nabla u) - wq
\end{aligned}$$

Hence

$$\gamma^{1/2}\big(\Delta w + qw\big) = \nabla\cdot\gamma\nabla u$$

so that

$$\nabla\cdot\gamma\nabla u = 0 \iff \Delta w + qw = 0$$

(b) Set $f = \gamma^{-1/2}g$ and $u_f = \gamma^{-1/2}w_g$. Then, by part (a), u_f is the solution of $\nabla\cdot\gamma\nabla u_f = 0$ in Ω, $u_f = f$ on $\partial\Omega$ and

$$\begin{aligned}
\Lambda_q(g) &= \sum_{j=1}^n \tfrac{\partial w_g}{\partial x_j}\hat{n}^j\Big|_{\partial\Omega} = \sum_{j=1}^n \hat{n}^j\tfrac{\partial}{\partial x_j}(\gamma^{1/2}u_f)\Big|_{\partial\Omega} \\
&= \sum_{j=1}^n \gamma^{1/2}\tfrac{\partial u_f}{\partial x_j}\hat{n}^j\Big|_{\partial\Omega} + \sum_{j=1}^n \tfrac{1}{2}u_f\gamma^{-1/2}\tfrac{\partial\gamma}{\partial x_j}\hat{n}^j\Big|_{\partial\Omega} \\
&= \gamma^{-1/2}\Lambda_\gamma(f) + \sum_{j=1}^n \tfrac{1}{2}f\gamma^{-1/2}\tfrac{\partial\gamma}{\partial x_j}\hat{n}^j\Big|_{\partial\Omega} \\
&= \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2}g) + \sum_{j=1}^n \tfrac{1}{2}g\gamma^{-1}\tfrac{\partial\gamma}{\partial x_j}\hat{n}^j\Big|_{\partial\Omega}
\end{aligned}$$

■

In the next chapter, we prove that if two isotropic conductivities γ_1 and γ_2 in Ω give rise to the same boundary measurements (i.e., $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$), then $\gamma_1 = \gamma_2$. Actually, because constant coefficient differential operators, like $-\Delta$, are easier to deal with than variable coefficient differential operators, like $\nabla \cdot \gamma \nabla$, we first prove the corresponding result for Schrödinger operators. Then we use Theorem 3.5.9 to transer the result over to the orginal conductivity problem. A more detailed outline of this procedure is provided at the beginning of §4.