

Appendix SA: Problem Solutions for Appendix A

Problem A.1 Let $a < b$ and $c < d$ and let $C : [c, d] \times [a, b] \rightarrow \mathbb{C}$ be continuous. Use the Arzelà–Ascoli theorem to prove that the integral operator

$$(Cf)(y) = \int_a^b C(y, x)f(x) dx$$

is compact as an operator from $\mathcal{X} = C[a, b]$ to $\mathcal{Y} = C[c, d]$.

Solution. Let $\{f_i\}_{i \in \mathbb{N}}$ be a bounded sequence in $C[a, b]$. Then

$$\sup_i \|f\|_{C[a,b]} = \sup_{\substack{i \in \mathbb{N} \\ a \leq x \leq b}} |f_i(x)| \leq M < \infty$$

Since $C(y, x)$ is continuous and $[c, d] \times [a, b]$ is compact, $C(y, x)$ is bounded. Set

$$K = \sup_{\substack{a \leq x \leq b \\ c \leq y \leq d}} |C(y, x)|$$

Then

$$|(Cf_i)(y)| \leq \int_a^b |C(y, x)| |f_i(x)| dx \leq \int_a^b K M dx \leq KM(b - a)$$

so the sequence $\{Cf_i\}_{i \in \mathbb{N}}$ is uniformly bounded. Again, since $C(y, x)$ is continuous and $[c, d] \times [a, b]$ is compact, $C(y, x)$ is uniformly continuous. In particular, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|K(y, x) - K(y', x)| < \varepsilon$ for all $a \leq x \leq b$ and all $c \leq y, y' \leq d$ with $|y - y'| < \delta$. Consequently

$$\begin{aligned} |y - y'| < \delta \implies |(Cf_i)(y) - (Cf_i)(y')| &\leq \int_a^b |C(y, x) - C(y', x)| |f_i(x)| dx \\ &\leq \int_a^b \varepsilon M dx \leq M(b - a)\varepsilon \end{aligned}$$

and the sequence $\{Cf_i\}_{i \in \mathbb{N}}$ is equicontinuous. By the Arzelà–Ascoli theorem [F, Theorem 4.43], there is a subsequence of $\{Cf_i\}_{i \in \mathbb{N}}$ that converges uniformly on $[c, d]$. ■

Problem A.2 Prove that any Hilbert–Schmidt operator is bounded.

Solution. If $f \in L^2(Y, d\nu)$ and $g \in L^2(X, d\mu)$, then by Cauchy–Schwarz, applied to the product measure $d\mu(x)d\nu(y)$

$$\begin{aligned} |\langle Kf, g \rangle| &= \left| \int_{X \times Y} k(x, y) f(y) \overline{g(x)} d\mu(x) d\nu(y) \right| \\ &\leq \int_{X \times Y} |f(y)| |g(x)| |k(x, y)| d\mu(x) d\nu(y) \\ &\leq \|f\|_{L^2(Y, d\nu)} \|g\|_{L^2(X, d\mu)} \|k\|_{L^2(X \times Y, d\mu d\nu)} \end{aligned}$$

This implies that K is a bounded operator with norm at most $\|k\|_{L^2(X \times Y, d\mu d\nu)}$. ■

Problem A.3 Let \mathcal{H} be a Hilbert Space. A sequence $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{H}$ is said to converge weakly to $f \in \mathcal{H}$ if

$$\lim_{i \rightarrow \infty} \langle f_i, g \rangle = \langle f, g \rangle$$

for all $g \in \mathcal{H}$.

(a) Give an example of a sequence that converges weakly but not strongly.

(b) Prove that if $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to f , then $\|f\| \leq \liminf_{i \rightarrow \infty} \|f_i\|$. Prove that if $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to f and $\|f\| = \lim_{i \rightarrow \infty} \|f_i\|$, then $\{f_i\}_{i \in \mathbb{N}}$ converges strongly to f .

(c) Prove that \mathcal{H} is weakly sequentially compact. That is, every bounded sequence in \mathcal{H} has a weakly convergent subsequence.

Solution. (a) Let \mathcal{H} be any infinite dimensional Hilbert space and let $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{H}$ be any orthonormal infinite sequence. If $g \in \mathcal{H}$, then

$$\sum_{i=1}^{\infty} |\langle g, f_i \rangle|^2 \leq \|g\|^2 < \infty \implies \lim_{i \rightarrow \infty} \langle g, f_i \rangle = 0$$

So the sequence $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to zero. On the other hand $\|f_i - f_j\| = \sqrt{2}$ for all $i \neq j$, so the sequence cannot converge strongly.

(b) Assume that $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to f . Then, in particular,

$$\|f\|^2 = \lim_{i \rightarrow \infty} \langle f, f_i \rangle \leq \liminf_{i \rightarrow \infty} \|f\| \|f_i\| = \|f\| \liminf_{i \rightarrow \infty} \|f_i\| \implies \|f\| \leq \liminf_{i \rightarrow \infty} \|f_i\|$$

Assume, on the other hand, that $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to f and $\|f\| = \lim_{i \rightarrow \infty} \|f_i\|$. Then

$$\begin{aligned} \lim_{i \rightarrow \infty} \|f - f_i\|^2 &= \lim_{i \rightarrow \infty} \|f\|^2 + \|f_i\|^2 - \langle f, f_i \rangle - \langle f_i, f \rangle \\ &= -\|f\|^2 + \lim_{i \rightarrow \infty} \|f_i\|^2 \quad \text{by weak convergence} \\ &= 0 \end{aligned}$$

This is the definition of “ $\{f_i\}$ converges strongly to f ”.

(c) Let $\{f_i\}_{i \in \mathbb{N}}$ be a bounded sequence in \mathcal{H} and let $\{e_j\}_{j \in J}$ be an orthonormal basis for the closure of the span of $\{f_i\}_{i \in \mathbb{N}}$. Since there are countably many f_i 's, the index set J is either finite or countable, even if \mathcal{H} is not separable. For each $j \in J$, $\{\langle f_i, e_j \rangle\}_{i \in \mathbb{N}}$ is a bounded sequence in \mathbb{C} and so has a convergent subsequence. By taking subsequences of subsequences, we can find a subsequence $\{f_{i_\ell}\}_{\ell \in \mathbb{N}}$ such that

$$g_j = \lim_{\ell \rightarrow \infty} \langle f_{i_\ell}, e_j \rangle$$

exists for all $j \in J$. Furthermore, by Fatou's lemma,

$$\sum_{j \in J} |g_j|^2 = \sum_{j \in J} \lim_{\ell \rightarrow \infty} |\langle f_{i_\ell}, e_j \rangle|^2 \leq \liminf_{\ell \rightarrow \infty} \sum_{j \in J} |\langle f_{i_\ell}, e_j \rangle|^2 \leq \liminf_{\ell \rightarrow \infty} \|f_{i_\ell}\|^2 < \infty$$

so that $g = \sum_{j \in J} g_j e_j$ is a well defined element of \mathcal{H} .

We claim that $\{f_{i_\ell}\}_{\ell \in \mathbb{N}}$ converges weakly to g . To prove this, we must show that for each $h \in \mathcal{H}$ and each $\varepsilon > 0$, there is an $L > 0$ such that $|\langle g, h \rangle - \langle f_{i_\ell}, h \rangle| < \varepsilon$ for all $\ell > L$. Write

$$\langle g, h \rangle - \langle f_{i_\ell}, h \rangle = \sum_{j \in J_0} \langle g - f_{i_\ell}, e_j \rangle \langle e_j, h \rangle + \sum_{j \in J \setminus J_0} \langle g - f_{i_\ell}, e_j \rangle \langle e_j, h \rangle$$

with J_0 a finite subset of J , to be specified shortly. The second sum is bounded in magnitude by

$$\sqrt{\sum_{j \in J \setminus J_0} |\langle g - f_{i_\ell}, e_j \rangle|^2} \sqrt{\sum_{j \in J \setminus J_0} |\langle e_j, h \rangle|^2} \leq \|g - f_{i_\ell}\| \sqrt{\sum_{j \in J \setminus J_0} |\langle e_j, h \rangle|^2}$$

Since

$$\sum_{j \in J} |\langle e_j, h \rangle|^2 \leq \|h\|^2 < \infty$$

we can always choose the finite set J_0 sufficiently large that

$$\sqrt{\sum_{j \in J \setminus J_0} |\langle e_j, h \rangle|^2} \leq \frac{\varepsilon}{2 \sup_{\ell} \|g - f_{i_\ell}\|}$$

We choose such a J_0 . Then the second sum is bounded by $\frac{\varepsilon}{2}$. The first sum is bounded by

$$\sum_{j \in J_0} |\langle g - f_{i_\ell}, e_j \rangle \langle e_j, h \rangle| \leq \|h\| \sum_{j \in J_0} |g_j - \langle f_{i_\ell}, e_j \rangle|$$

By the definition of g_j , there is an $L > 0$ such that

$$\ell > L \implies |g_j - \langle f_{i_\ell}, e_j \rangle| < \frac{\varepsilon}{2|J_0|\|h\|}$$

for each j in the finite set J_0 . Then, if $\ell > L$, the first sum is strictly smaller than $\frac{\varepsilon}{2}$ and $|\langle g, h \rangle - \langle f_{i_\ell}, h \rangle| < \varepsilon$. ■

Problem A.4 Prove that any nuclear operator is compact.

Solution. Let

$$Kx = \sum_{i=1}^{\infty} c_i x'_i(x) y_i$$

be a nuclear operator with $\{y_i\}_{i \in \mathbb{N}}$ a bounded sequence in the Banach space \mathcal{Y} , $\{x'_i\}_{i \in \mathbb{N}}$ a bounded sequence in the dual space \mathcal{X}' of the Banach space \mathcal{X} and $\{c_i\}_{i \in \mathbb{N}}$ a set of complex numbers obeying $\sum_i |c_i| < \infty$.

Let $\{x_j\}_{j \in \mathbb{N}}$ be a bounded sequence in \mathcal{X} . For each fixed $i \in \mathbb{N}$, $\{x'_i(x_j)\}_{j \in \mathbb{N}}$ is a bounded sequence in \mathbb{C} and so has a convergent subsequence. By taking subsequences of subsequences, we can find a subsequence $\{x_{j_\ell}\}_{\ell \in \mathbb{N}}$ such that

$$g_i = \lim_{\ell \rightarrow \infty} x'_i(x_{j_\ell})$$

exists for all $i \in \mathbb{N}$. Write

$$X = \sup_j \|x_j\|_{\mathcal{X}} \quad X' = \sup_i \|x'_i\|_{\mathcal{X}'} \quad Y = \sup_i \|y_i\|_{\mathcal{Y}} \quad C = \sum_i |c_i|$$

Since

$$(SA.1) \quad |x'_i(x_j)| \leq X'X < \infty$$

we have

$$\sum_{i=1}^{\infty} |c_i| |g_i| \|y_i\| \leq X'XYC < \infty$$

and $\sum_{i=1}^{\infty} c_i g_i y_i$ converges strongly to a vector $g \in \mathcal{Y}$.

We claim that $g = \lim_{\ell \rightarrow \infty} Kx_{j_\ell}$. For each $I \in \mathbb{N}$

$$\begin{aligned} \|g - Kx_{j_\ell}\| &= \left\| \sum_{i=1}^{\infty} c_i (g_i - x'_i(x_{j_\ell})) y_i \right\| \\ &\leq \sum_{i=1}^I |c_i| |g_i - x'_i(x_{j_\ell})| \|y_i\|_{\mathcal{Y}} + \sum_{i=I+1}^{\infty} |c_i| [|g_i| + |x'_i(x_{j_\ell})|] \|y_i\|_{\mathcal{Y}} \\ &\leq CY \max_{1 \leq i \leq I} |g_i - x'_i(x_{j_\ell})| + 2X'XY \sum_{i=I+1}^{\infty} |c_i| \end{aligned}$$

by (SA.1). Let $\varepsilon > 0$. Since the series $\sum_{i=1}^{\infty} |c_i|$ converges there is an $I > 0$ such that $\sum_{i=I+1}^{\infty} |c_i| \leq \frac{\varepsilon}{4X'XY}$. Since $g_i = \lim_{\ell \rightarrow \infty} x'_i(x_{j_\ell})$ for all $i \in \mathbb{N}$ and since I is finite, we can find an $L \in \mathbb{N}$ such that $|g_i - x'_i(x_{j_\ell})| < \frac{\varepsilon}{2CY}$ for all $\ell > L$ and all $1 \leq i \leq I$. Then, for all $\ell > L$,

$$\|g - Kx_{j_\ell}\| < CY \frac{\varepsilon}{2CY} + 2X'XY \frac{\varepsilon}{4X'XY} = \varepsilon$$

as desired. ■

Problem A.5 Prove that compact operators are necessarily bounded.

Solution. Let C be a compact operator from the Banach space \mathcal{X} to the Banach space \mathcal{Y} . If C is not bounded, then, for every $n \in \mathbb{N}$ there is a unit vector $x_n \in \mathcal{X}$ such that $\|Cx_n\|_{\mathcal{Y}} \geq n$. But then $\{x_n\}_{n \in \mathbb{N}}$ is a bounded subset of \mathcal{X} and $\lim_{n \rightarrow \infty} \|Cx_n\|_{\mathcal{Y}} = \infty$ so no subsequence of $\{Cx_n\}_{n \in \mathbb{N}}$ can converge. ■

Problem A.6 Let $\mathcal{H} = L^2(X, \mu)$ for some measure space $\langle X, \mu \rangle$. Let $f : X \rightarrow \mathbb{C}$ be a bounded measurable function on X . Let A be the bounded linear operator on \mathcal{H} given by multiplication by $f(x)$.

(a) Prove that $\lambda \in \sigma(A)$ if and only if

$$\forall \varepsilon > 0 \quad \mu\{x \in X \mid |f(x) - \lambda| < \varepsilon\} > 0$$

(b) Prove that λ is an eigenvalue of A if and only if

$$\mu\{x \in X \mid f(x) = \lambda\} > 0$$

(c) Let X be the open interval $(0, 1)$, μ be Lebesgue measure on $(0, 1)$ and $f(x) = x$. Find the spectrum of A , the operator on \mathcal{H} given by multiplication by x . Also find all of the eigenvalues of A .

Solution. (a) *Proof that “ $\forall \epsilon > 0 \quad \mu\{x \in X \mid |f(x) - \lambda| < \epsilon\} > 0$ ” implies “ $\lambda \in \sigma(A)$ ”:* The proof is by contradiction. Suppose that $\lambda \notin \sigma(A)$. Then the operator inverse of $A - \lambda\mathbb{1}$ exists and is a bounded operator, say with norm a . Let S be a set of finite nonzero measure such that $|f(x) - \lambda| < \frac{1}{2a}$ for all $x \in S$ and let ϕ be the characteristic function of S . Then $\|\phi\| = \sqrt{\mu(S)}$ and, since $|(f(x) - \lambda)\phi(x)| \leq \frac{1}{2a}\phi(x)$, $\|(A - \lambda\mathbb{1})\phi\| \leq \frac{1}{2a}\sqrt{\mu(S)}$. Consequently, $\|(A - \lambda\mathbb{1})\phi\| \leq \frac{1}{2a}\|\phi\|$. If $\psi = (A - \lambda\mathbb{1})\phi$, then $\phi = (A - \lambda\mathbb{1})^{-1}\psi$ and $\|(A - \lambda\mathbb{1})^{-1}\psi\| = \|\phi\| \geq 2a\|(A - \lambda\mathbb{1})\phi\| = 2a\|\psi\|$. This contradicts $\|A - \lambda\mathbb{1}\| = a$.

Proof that “ $\lambda \in \sigma(A)$ ” implies “ $\forall \epsilon > 0 \quad \mu\{x \in X \mid |f(x) - \lambda| < \epsilon\} > 0$ ”: The proof is again by contradiction. Suppose that $\bar{\epsilon} > 0$ but $\mu\{x \in X \mid |f(x) - \lambda| < \bar{\epsilon}\} = 0$. Then $|f(x) - \lambda| \geq \bar{\epsilon}$ for almost all x and the operator of multiplication by $\frac{1}{f(x) - \lambda}$ is a bounded operator with operator norm at most $\frac{1}{\bar{\epsilon}}$. Furthermore, it is the operator inverse of $A - \lambda\mathbb{1}$. So λ is not in the spectrum of A .

(b) *Proof that “ $\mu\{x \in X \mid f(x) = \lambda\} > 0$ ” implies “ λ is an eigenvalue of A ”:* Let S be a set of finite nonzero measure such that $f(x) = \lambda$ for all $x \in S$ and let ϕ be the characteristic function of S . Then ϕ is a nonzero vector but $(f(x) - \lambda)\phi(x) = 0$ for all $x \in X$. Thus $A\phi = \lambda\phi$ and ϕ is an eigenfunction of A of eigenvalue λ .

Proof that “ λ is an eigenvalue of A ” implies “ $\mu\{x \in X \mid f(x) = \lambda\} > 0$ ”: Let ϕ is an eigenfunction of A of eigenvalue λ . Since $A\phi = \lambda\phi$, the function $(f(x) - \lambda)\phi(x)$ must be zero almost everywhere. If $\mu\{x \in X \mid f(x) = \lambda\} = 0$, $\phi(x)$ must itself be zero almost everywhere. But then ϕ is the zero vector in the Hilbert space, which is forbidden.

(c) The measure of $\{x \in X \mid |x - \lambda| < \epsilon\}$ is exactly the length of the intersection of the interval $(0, 1)$ with the disk in \mathbb{C} of radius ϵ and centre λ . For any fixed $\lambda \in \mathbb{C}$, this measure is nonzero for all $\epsilon > 0$ if and only if the distance from λ to $(0, 1)$ is zero. That is, if and only if $\lambda \in [0, 1]$. Hence $\sigma(A) = [0, 1]$.

For any complex number λ , the set $\{x \in X \mid x = \lambda\}$ consists either of a single point (if $\lambda \in (0, 1)$) or of the empty set (if $\lambda \notin (0, 1)$). Both have measure zero. So A has no eigenvalues. ■

Problem A.7 Let \mathcal{X} be a Banach space and \mathcal{Y} a proper closed subspace of \mathcal{X} . Let $0 < \rho < 1$. Prove that there is a unit vector $x \in \mathcal{X} \setminus \mathcal{Y}$ whose distance from \mathcal{Y} is at least ρ .

Solution. Let \tilde{x} be any vector in $\mathcal{X} \setminus \mathcal{Y}$ and denote by $\tilde{\rho}$ its distance

$$\tilde{\rho} = \inf \{ \|\tilde{x} - \tilde{y}\| \mid \tilde{y} \in \mathcal{Y} \}$$

from \mathcal{Y} . Since $\tilde{x} \in \mathcal{X} \setminus \mathcal{Y}$ and \mathcal{Y} is closed, $\tilde{\rho} > 0$. Then there is a vector $y \in \mathcal{Y}$ such that

$$\tilde{\rho} \leq \|\tilde{x} - y\| \leq \frac{\tilde{\rho}}{\rho}$$

Then $\tilde{x} - y$ is again a vector in $\mathcal{X} \setminus \mathcal{Y}$ whose distance from \mathcal{Y} is precisely $\tilde{\rho}$ and $x = \frac{\tilde{x} - y}{\|\tilde{x} - y\|}$ is a unit vector in $\mathcal{X} \setminus \mathcal{Y}$ whose distance from \mathcal{Y} is precisely $\frac{\tilde{\rho}}{\|\tilde{x} - y\|} \geq \rho$. ■

Problem A.8 Let \mathcal{X} be an infinite dimensional Banach space. Prove that the identity operator on \mathcal{X} is not compact.

Solution. Since \mathcal{X} is infinite dimensional, it contains an infinite sequence $\{x_n\}_{n \in \mathbb{N}}$ of independent vectors. Denote by \mathcal{X}_n the span of $\{x_1, x_2, \dots, x_n\}$. By Problem A.7, there is, for each $n \geq 2$, a unit vector $y_n \in \mathcal{X}_n$ whose distance from \mathcal{X}_{n-1} is at least $\frac{1}{2}$. If $n > m$, $\|y_n - y_m\| \geq \frac{1}{2}$ since $y_m \in \mathcal{X}_m \subset \mathcal{X}_{n-1}$. But then $\{y_n\}_{n \in \mathbb{N}}$ is a bounded sequence in \mathcal{X} and $\{y_n\}_{n \in \mathbb{N}}$ contains no convergent subsequence. So $\mathbb{1}$ cannot be compact. ■

Problem A.9 Let \mathcal{X} be an infinite dimensional Banach space and $C : \mathcal{X} \rightarrow \mathcal{X}$ a compact operator. Prove that $0 \in \sigma(C)$.

Solution. If $0 \in \rho(C)$, then C has a bounded inverse and, by part (c) of Proposition A.5, $\mathbb{1} = C^{-1}C$ is compact. But, by Problem A.8, the identity operator acting on any infinite dimensional Banach space is never compact, providing a contradiction. ■

Problem A.10 Let \mathcal{H} be a separable Hilbert space and let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for \mathcal{H} . Let $\{\mu_n\}_{n \in \mathbb{N}}$ be any sequence of complex numbers that converges to 0. Prove that the operator defined by

$$C\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{\infty} \mu_n \alpha_n e_{n+1}$$

is compact and has $\sigma(C) = \{0\}$.

Solution. We first prove that C is compact. Set $\nu_n = \sup_{m \geq n} |\mu_m|$. By hypothesis $\lim_{n \rightarrow \infty} \nu_n = 0$. Set, for each $N \in \mathbb{N}$,

$$C_N \left(\sum_{n=1}^{\infty} \alpha_n e_n \right) = \sum_{n=1}^N \mu_n \alpha_n e_{n+1}$$

Then

$$\begin{aligned} \left\| (C - C_N) \left(\sum_{n=1}^{\infty} \alpha_n e_n \right) \right\| &= \left\| \sum_{n=N+1}^{\infty} \mu_n \alpha_n e_{n+1} \right\| = \sqrt{\sum_{n=N+1}^{\infty} |\mu_n \alpha_n|^2} \leq \nu_{N+1} \sqrt{\sum_{n=N+1}^{\infty} |\alpha_n|^2} \\ &\leq \nu_{N+1} \left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\| \end{aligned}$$

so that C_N converges in operator norm to C . As C_N is a nuclear operator (use $y_i = e_{i+1}$, $x'_i(x) = \langle x, e_i \rangle$ and $c_i = \mu_i$ for $1 \leq i \leq N$ and zero otherwise), it is compact, by Problem A.4. By part (d) of Proposition A.5, C is also compact.

We now prove that $\sigma(C) = \{0\}$. By Problem A.9, $0 \in \sigma(C)$. By Proposition A.10, any nonzero $\lambda \in \sigma(C)$ must be an eigenvalue of C . Let $x = \sum_{n=m}^{\infty} \alpha_n e_n$ be a corresponding eigenvector. Choose m so that $\alpha_m \neq 0$. Then

$$\alpha_m = \langle x, e_m \rangle = \left\langle \frac{1}{\lambda} Cx, e_m \right\rangle = \left\langle \frac{1}{\lambda} \sum_{n=m}^{\infty} \mu_n \alpha_n e_{n+1}, e_m \right\rangle = \frac{1}{\lambda} \sum_{n=m}^{\infty} \mu_n \alpha_n \langle e_{n+1}, e_m \rangle = 0$$

which is a contradiction. ■