Operators for Parabolic Block Spin Transformations

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Abstract

This paper is a contribution to a program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It is part of an analysis of the “small field” approximation to the “parabolic flow” which exhibits the formation of a “Mexican hat” potential well. Bounds on the fluctuation integral covariance, as well as on some other linear operators, are an important ingredient in the renormalization group step analysis of [7, 8]. These bounds are proven here.

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1 Introduction

In [7, 8], we exhibit, for a many particle system of weakly interacting Bosons in three space dimensions, the formation of a potential well of the type that typically leads to symmetry breaking in the thermodynamic limit. To do so, we use the block spin renormalization group approach. In previous papers [4, 1, 2, 3] (followed by a simple change of variables) we have written the partition function of such a system on a discrete torus

$$\mathcal{X}_0 = (\mathbb{Z}/L_{tp}\mathbb{Z}) \times (\mathbb{Z}^3/L_{sp}\mathbb{Z}^3)$$

with positive integers $L_{tp}$, $L_{sp}$. Up to corrections which are exponentially small in the coupling constant, and up to a multiplicative normalization factor, this representation is of the form

$$\int \left[ \prod_{x \in \mathcal{X}_0} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \right] e^{A_0(\psi^*, \psi)} \chi_0(\psi)$$

with an action $A_0$ of the form

$$A_0(\psi^*, \psi) = -\langle \psi^*, D_0 \psi \rangle - V_0(\psi^*, \psi) + \mu_0 \langle \psi^*, \psi \rangle + E'_0(\psi^*, \psi)$$

Here

- $D_0 = \mathbb{1} - e^{-h_0} - e^{-h_0} \partial_0$, where $\partial_0$ the forward time derivative (see (2.9) below and $h_0$ is – up to a scaling – the single particle Hamiltonian.
- $V_0(\psi^*, \psi)$ is a quartic monomial that describes the coupling between the particles
- $\mu_0$ is related to the chemical potential of the system
- $E'_0(\psi^*, \psi)$ is perturbatively small
- $\chi_0(\psi)$ is a “small field cut off function”.

See [7, (1.3), (1.4)].

For the block spin renormalization group action, we pick a “block rectangle” of length $L^2$ in the “time direction” and $L$ in “space directions”, where $L$ is a sufficiently large odd positive integer, and a corresponding nonnegative, compactly supported function $q(x)$ on $\mathbb{Z} \times \mathbb{Z}^3$ (the averaging profile). The choice of this kind of rectangle is characteristic of “parabolic scaling”. See [7, Definition 1.3, Remark 1.4, Definition 1.11.d]. For simplicity we assume that $L_{sp}$ and $L_{tp}$ are powers of $L$.

The block spin averaging operator, which we denote $Q$, maps functions on the lattice $\mathcal{X}_0$ to functions on the “coarse” lattice $\mathcal{X}_0^{(1)} = (L^2\mathbb{Z}/L_{tp}\mathbb{Z}) \times (L\mathbb{Z}^3/L_{sp}\mathbb{Z}^3)$.

---

1 All bounds achieved so far are uniform in the volume of this torus.
After each renormalization group step we scale, to again give functions on a unit lattice. After the first RG step this unit lattice is $\mathcal{X}^{(1)}_0 = (\mathbb{Z}/\frac{1}{L}L_{tp}\mathbb{Z}) \times (\mathbb{Z}^3/\frac{1}{L}L_{sp}\mathbb{Z}^3)$. The “scaled” block spin averaging operator maps functions on the lattice $\mathcal{X}_1 = (\mathbb{Z}/\frac{1}{L}L_{tp}\mathbb{Z}) \times (\mathbb{Z}^3/\frac{1}{L}L_{sp}\mathbb{Z}^3)$ to functions on the unit lattice $\mathcal{X}^{(1)}_0$.

In the $n^{th}$ renormalization group step, we end up considering functions on the chain of lattices

$$\mathcal{X}_{n+1}^{(n+1)} \subset \mathcal{X}_n^{(n)} \subset \mathcal{X}_{n-1}^{(n-1)} \subset \cdots \subset \mathcal{X}_1^{(2)} \subset \mathcal{X}_0^{(1)}$$

where, for integers $j \geq -1$ and $n \geq 0$,

$$\mathcal{X}_j^{(n)} = (\mathbb{Z}/\varepsilon_{n+j} L_{tp}\mathbb{Z}) \times (\mathbb{Z}^3/\varepsilon_{n+j} L_{sp}\mathbb{Z}^3) \quad \text{with} \quad \varepsilon_j = \frac{1}{L^{2j}}$$

The subscript in $\mathcal{X}_j^{(n)}$ determines the “coarseness” of the lattice — nearest neighbour points are a distance $\varepsilon_{2j} = \frac{1}{L^{2j}}$ apart in the time direction and a distance $\varepsilon_j = \frac{1}{L^{j}}$ apart in spatial directions. The superscript in $\mathcal{X}_j^{(n)}$ determines the number of points in the lattice — $|\mathcal{X}_j^{(n)}| = |\mathcal{X}_0^{(1)}|/L^{5n}$ for all $j$. We usually write $\mathcal{X}_0^{(0)} = \mathcal{X}_0$. See [7, Definition 1.5.a or Appendix A.1].

The $(n+1)^{st}$ block spin transformation involves the passage from $\mathcal{X}_0^{(n)}$ to its sublattice $\mathcal{X}_{n+1}^{(n+1)}$. The averaging operations determine linear maps

$$Q : \mathcal{H}_j^{(n)} \mapsto \mathcal{H}_{j-1}^{(n+1)} \quad \text{and} \quad Q_n : \mathcal{H}_n = \mathcal{H}_0^{(0)} \mapsto \mathcal{H}_n^{(0)}$$

where $\mathcal{H}_j^{(n)} = L^2(\mathcal{X}_j^{(n)})$ denotes the (finite dimensional) Hilbert space of functions on $\mathcal{X}_j^{(n)}$ with integral $\int_{\mathcal{X}_j^{(n)}} du = \varepsilon_j^2 \sum_{u \in \mathcal{X}_j^{(n)}}$ and the real inner product

$$\langle \alpha_1, \alpha_2 \rangle_j = \int_{\mathcal{X}_j^{(n)}} \alpha_1(u) \alpha_2(u) \ du$$

Again see [7, Definition 1.5.a or Appendix A.3]. In §2 we pick a specific averaging profile and give bounds on the operators $Q, Q_n$, their Fourier transforms, and related operators.

Scaling is performed by the linear isomorphisms

$$\mathbb{L} : \mathcal{X}_j^{(n)} \rightarrow \mathcal{X}_{j-1}^{(n)} \quad (u, \mathbf{u}) \mapsto (L^2 u_0, L \mathbf{u})$$

For a function $\alpha \in \mathcal{H}_j^{(n)}$, define the function $\mathbb{L}_\ast(\alpha) \in \mathcal{H}_{j-1}^{(n)}$ by $\mathbb{L}_\ast(\alpha)(Lu) = \alpha(u)$.

See [7, Appendix A.2]. In particular, after rescaling and multiplication with the “scaling factor” $L^{2n}$, the differential operator $D_0$ in (1.2) becomes the operator

$$D_n = L^{2n} \mathbb{L}_\ast^{-n}(1 - e^{-h_0} - e^{-h_0} \partial_0) \mathbb{L}_\ast^n$$
on $H_n$. This operator is discussed in §3.

As mentioned above, the passage from a functional integral on $X_0^{(n)}$ to a functional integral on $X_{-1}^{(n+1)}$ is an averaging procedure over, roughly speaking, a rectangle of size $L^2$ in the time direction and size $L$ in the spatial directions. This passage is analyzed using stationary phase techniques that involve

- the determination of critical fields on $X_0^{(n)}$ (that are functions of external fields on $X_{-1}^{(n+1)}$) for an appropriate action, and
- a functional integral over “fluctuation fields” around the critical field.

The covariance for the integral over the fluctuation fields has been identified in [7, (1.15)] and is bounded in §4.

The composition of the critical fields of $n$ renormalization group steps is – after rescaling – a field on $X_n$, called the “background field”, that is a function of an external field on $X_0^{(n)}$. It is crucial in our representation of the partition function. See [7, Theorem 1.17]. The “leading order” part of the background field is linear in the external field and has been identified in [7, Proposition 1.14]. It is the composition of an operator, from $H_0^{(n)}$ to $H_n$ determined by the averaging profile $q$, and an operator $S_n$ on $H_n$ which can be viewed as a Green’s function for the differential operator $D_n$ (plus a mass term). This operator, $S_n$, is discussed in §5.

To get bounds on the critical fields in the fluctuation integral at step $n + 1$, we use a well known algebraic relation between these critical fields and the background fields at step $n + 1$ given in [6, Proposition 9] and [7, Proposition 3.4.a]. The operators in the linearization of this relation, and various other linearizations, are studied in §6.

By construction, many of the operators discussed in this paper are linear operators defined on the Hilbert space of functions on a lattice that are invariant under translations with respect to a sublattice. It is natural to use Bloch/Floquet decompositions and Fourier transforms for an analysis of such operators. We use the abstract basic results about such decompositions given in [5].

Most operator estimates we obtain in this paper are with respect to a norm of the following kind.

**Definition 1.1.** For any operator $A : H_j^{(n-j)} \to H_k^{(n-k)}$, with kernel $A(u, u')$, and for any mass $m \geq 0$, we define the norm

$$
\|A\|_m = \max \left\{ \sup_{u \in X_k^{(n-k)}} \int_{X_j^{(n-j)}} \, du' \, e^{m|u-u'|} |A(u, u')|, \sup_{u' \in X_j^{(n-j)}} \int_{X_k^{(n-k)}} \, du \, e^{m|u-u'|} |A(u, u')| \right\}
$$

In the special case that $m = 0$, this is just the usual $\ell^1$–$\ell^\infty$ norm of the kernel.
As we point out in [5, Lemmas 12 and 13] this norm is related to the analyticity properties of the Fourier transform. In this paper we use the following Fourier transform conventions.

The dual lattice of $\mathcal{X}_j^{(n)}$ is

$$\hat{\mathcal{X}}_j^{(n)} = \left( \frac{2\pi}{\varepsilon_{n+jL_{tp}} \varepsilon_j} \mathbb{Z} / \frac{2\pi}{\varepsilon_j} \mathbb{Z} \right) \times \left( \frac{2\pi}{\varepsilon_{n+jL_{sp}} \varepsilon_j} \mathbb{Z}^3 / \frac{2\pi}{\varepsilon_j} \mathbb{Z}^3 \right)$$

For a function $\alpha \in \mathcal{H}_j^{(n)}$

$$\hat{\alpha}(p) = \int_{\mathcal{X}_j^{(n)}} \alpha(u) e^{-ip \cdot u} \, du \quad \alpha(u) = \int_{\hat{\mathcal{X}}_j^{(n)}} \hat{\alpha}(p) e^{iu \cdot p} \, \frac{dp}{(2\pi)^3}$$

where $\int_{\hat{\mathcal{X}}_j^{(n)}} \frac{dp}{(2\pi)^3} = \frac{1}{\varepsilon_{n+jL_{tp}} \varepsilon_{n+jL_{sp}}} \sum_{p \in \hat{\mathcal{X}}_j^{(n)}}$. The maps

$$\mathbb{L} : \hat{\mathcal{X}}_{j-1}^{(n)} \to \hat{\mathcal{X}}_j^{(n)} \quad (q_0, q) \mapsto (L^2 q_0, Lq)$$

are again linear isomorphisms, and, for a function $\alpha \in \mathcal{H}_j^{(n)}$,

$$\mathbb{L}_\ast(\alpha)(q) = L^5 \hat{\alpha}(Lq) \quad (1.3)$$

The quotient map dual to the inclusion $\mathcal{X}_j^{(n)} \subset \mathcal{X}_j^{(n-k)}$ is

$$\hat{\pi}_{n+j}^{(j+k,j)} : \hat{\mathcal{X}}_{j+k}^{(n-k)} \to \hat{\mathcal{X}}_j^{(n)} \quad (1.4)$$

When the indices are clear from the context we suppress them and write $\hat{\pi}$.

Internal Remark
If \( \alpha \in \mathcal{H}_j^{(n)} \), so that \( \mathbb{L}_\epsilon(\alpha) \in \mathcal{H}_{j-1}^{(n)} \), then

\[
\hat{\mathbb{L}}_\epsilon(\alpha)(q) = \int_{\mathcal{X}_{j-1}^{(n)}} \mathbb{L}_\epsilon(\alpha)(v) e^{-iq \cdot v} \, dv
= \int_{\mathcal{X}_{j-1}^{(n)}} \alpha(\mathbb{L}^{-1}v) e^{-iq \cdot v} \, dv
= \varepsilon_{j-1}^5 \sum_{v \in \mathcal{X}_{j-1}^{(n)}} \alpha(\mathbb{L}^{-1}v) e^{-iq \cdot v}
= \varepsilon_{j-1}^5 \sum_{u \in \mathcal{X}_j^{(n)}} \alpha(u) e^{-iq \cdot (\mathbb{L} u)}
= L^5 \int_{\mathcal{X}_j^{(n)}} \alpha(u) e^{-iq \cdot (\mathbb{L} u)} \, du
= L^5 \int_{\mathcal{X}_j^{(n)}} \alpha(u) e^{-i(L q) \cdot u} \, du
= L^5 \hat{\alpha}(\mathbb{L} q)
\]

The estimates of this paper are used in [7, 8]. In particular, the construction of the background fields and the critical fields in [9] uses a contraction mapping argument around the linearizations of §6 and §5 in this paper.

For the readers’ convenience we have included, in Appendix B, a list of most of the operators and lattices that appear in this paper.

**Convention 1.2.** Most estimates in this paper are bounds on norms of operators as in Definition 1.1. The (finite number of) constants that appear in these bounds are consecutively labelled \( \Gamma_1, \Gamma_2, \ldots, \gamma_1, \gamma_2, \ldots, m_1, m_2, \ldots \). All of these constants \( \Gamma_j, \gamma_j, m_j \) are independent of \( L \) and the scale index \( n \). We define \( \Gamma_{\operatorname{op}} \) to be the maximum of the \( \Gamma_j \)'s, and, in [7, 8, 9], refer to the estimates using only this constant \( \Gamma_{\operatorname{op}} \).
2 Block Spin Operators

In this chapter, we analyze the block spin “averaging operators” $Q$ of [7, Definitions 1.1.a and 1.11.d] and $Q_n$ of [7, Definition 1.11.d] as well as the operator $\Omega_n$ of [7, Definition 1.5.b]. Recall that $Q : \mathcal{H}_0^{(n)} \to \mathcal{H}_{-1}^{(n+1)}$ is defined by

$$(Q\psi)(y) = \sum_{x \in \mathbb{Z} \times \mathbb{Z}^3} q(x) \psi(y + [x])$$ (2.1)

where $[x]$ denotes the class of $x \in \mathbb{Z} \times \mathbb{Z}^3$ in the quotient space $\mathcal{X}_0^{(n)}$. The averaging profile $q$ is the $q$-fold convolution of the characteristic function, $1_{\mathbb{Z} \times \mathbb{Z}^3}(x)$, of the rectangle $[-L_2^{-1}, L_2^{-1}] \times [-L_2^{-1}, L_2^{-1}]^3$, normalized to have integral one. That is,

$$q = \frac{1}{L_5^n} \underbrace{1_{\mathbb{Z} \times \mathbb{Z}^3} \ast 1_{\mathbb{Z} \times \mathbb{Z}^3} \ast \cdots \ast 1_{\mathbb{Z} \times \mathbb{Z}^3}}_{q \text{ times}}$$

See [5, Example 8 and Remark 11]. Except where otherwise stated, we shall assume that $q \geq 4$ is a fixed even natural number.$^2$

The operator

$$Q_n = Q^{(1)} \cdots Q^{(n)} = (\mathbb{L}_5^{-1} Q)^n \mathbb{L}_5^n : \mathcal{H}_n = \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$$ (2.2)

where $Q^{(j)} = \mathbb{L}_5^{-j} Q \mathbb{L}_5^j : \mathcal{H}_j^{(n-j)} \to \mathcal{H}_j^{(n-j+1)}$. The operator

$$\Omega_n = a \left( 1 + \sum_{j=1}^{n-1} L_5^j Q_j Q_j^* \right)^{-1}$$

The Fourier transform of the characteristic function $1_{\mathbb{Z}_5^3}$ is $\sigma(L_k)$ with $k \in \hat{\mathcal{X}}_0^{(n)}$ and with

$$\sigma(k) = \sin \left( \frac{1}{2} k_0 \right) \prod_{\nu=1}^3 \sin \left( \frac{1}{2} k_\nu \right)$$ (2.3)

Therefore

$$\hat{q}(k) = u_+(k)^q \quad \text{with} \quad u_+(k) = \frac{\sigma(L_k)}{L_5^3 \sigma(k)}$$ (2.4)

and, by [5, Lemma 9.a]

$$(\mathcal{Q}\psi)(\mathfrak{t}) = \sum_{k \in \hat{\mathcal{X}}_0^{(n)} \atop \hat{\sigma}(k) = \mathfrak{t}} \hat{q}(k) \hat{\psi}(k)$$ (2.5)

for all $\psi \in \mathcal{H}_0^{(n)}$ and $\mathfrak{t} \in \hat{\mathcal{X}}_{-1}^{(n+1)}$.

$^2$See Remark 2.7 for a discussion of the condition $q > 2$. The condition that $q$ be even is imposed purely for convenience.
**Remark 2.1.**

(a) Since \( q \) is even, \( \sigma(k)^q \) is an entire function of \( k \in \mathbb{C} \times \mathbb{C}^3 \) that is periodic with respect to the lattice \( 2\pi(\mathbb{Z} \times \mathbb{Z}^3) \).

**Internal Remark**

We will apply this for ratios of \( \sigma(k)^q \)'s and then we have periodicity for \( q \) odd too.

Also

\[
\sigma(p_j)^q = \sigma(\hat{\pi}_{n}^{(j,0)}(p_j))^q \quad \text{for all } p_j \in \hat{X}_j^{(n-j)}
\]

(b) For all \( \phi \in \mathcal{H}_n \) and \( k \in \hat{X}_0^{(n)} \),

\[
\hat{(Q_n \phi)}(k) = \sum_{p \in \hat{X}_n} u_n(p)^q \hat{\phi}(p) \quad \text{with} \quad u_n(p) = \varepsilon_n \frac{\sigma(p)}{\sigma(\mathbb{L}^{-n}p)}
\]

(c) For all \( \psi \in \mathcal{H}_0^{(n)} \) and \( k \in \hat{X}_0^{(n)} \), \( \hat{\Omega}_n \hat{\psi}(k) = \hat{\Omega}_n(k) \hat{\psi}(k) \) where

\[
\hat{\Omega}_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{p_j \in \hat{X}_j^{(n-j)}} \frac{1}{L^{2j}} u_j(p_j)^{2q} \right]^{-1}
\]

(d) The functions \( u_n(p) \) and \( u_+ (p) \) are entire in \( p \) and are invariant under \( p_\nu \to -p_\nu \) for each \( 0 \leq \nu \leq 3 \) and under \( p_\nu \leftrightarrow p_{\nu'} \) for all \( 1 \leq \nu, \nu' \leq 3 \).

(e) Set, with the notation of (1.4), the “single period” lattices and their duals

\[
\mathcal{B}^+ = (\mathbb{Z}/L^2 \mathbb{Z}) \times (\mathbb{Z}^3/L\mathbb{Z}^3) \quad \mathcal{B}^+_j = (\hat{\pi}_n^{(j,0)})^{-1}
\]

\[
\mathcal{B}_j = (\varepsilon_j^2 \mathbb{Z}/\mathbb{Z}) \times (\varepsilon_j \mathbb{Z}^3/\mathbb{Z}^3) \quad \mathcal{B}_j = (\hat{\pi}_n^{(j,0)})^{-1}
\]

for each integer \( j \geq 0 \). In this notation, the representations of \( Q, Q_n \) and \( \Omega_n \) of (2.5) and parts (b) and (c) are

\[
\hat{(Q \psi)}(\ell) = \sum_{\ell \in \mathcal{B}^+] u_+(\ell + \ell)^q \hat{\psi}(\ell + \ell)
\]
\[ (\hat{Q}_n \phi)(k) = \sum_{\ell \in B_n} u_n(k + \ell)^q \hat{\phi}(k + \ell) \]

\[ \hat{Q}_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \ell_j \frac{1}{L^{q_j}} u_j(k + \ell_j)^{2q} \right]^{-1} \]

Here in \( (\hat{Q} \psi)(\mathbf{t}) = \sum_{\ell \in B_1} u_1(\mathbf{t} + \ell)^q \hat{\psi}(\mathbf{t} + \ell) \), for example, \( \mathbf{t} \in \hat{X}^{(n+1)} \) is represented by the element of \( \frac{2\pi}{\varepsilon_1 L_n} \mathbb{Z} \times \frac{2\pi}{\varepsilon_1 L_n} \mathbb{Z}^3 \) having minimal components and \( \ell \) is represented by the element of \( \frac{2\pi}{\varepsilon_1 L_n} \mathbb{Z} \times \frac{2\pi}{\varepsilon_1 L_n} \mathbb{Z}^3 \) having minimal components. Similarly

\[ (\hat{Q}^* \theta)(\mathbf{t} + \ell) = u_+(\mathbf{t} + \ell)^q \hat{\theta}(\mathbf{t}) \]

\[ (\hat{Q}_n^* \psi)(k + \ell_n) = u_n(k + \ell_n)^q \hat{\psi}(k) \]

**Proof.** (a) Any two points of \( \hat{X}^{(n-j)}_j \) with the same image in \( \hat{X}^{(n)}_0 \) under \( \hat{n}^{(j,0)}_n \) differ by \( 2\pi \) times an integer vector. The formula follows.

(b) By (1.3) and (2.5), we have, for \( \alpha \in \mathcal{H}^{(n-j)}_j \) and \( p_{j-1} \in \hat{X}^{(n-j+1)}_{j-1} \)

\[ (\hat{Q}^j \alpha)(p_{j-1}) = \frac{1}{L^q} (QL^j \alpha)(\mathbb{L}^{-j} p_{j-1}) = \frac{1}{L^q} \sum_{k \in \hat{X}^{(n-j)}_0} \hat{\phi}(k)(\mathbb{L}^j \alpha)(k) \]

\[ = \frac{1}{L^q} \sum_{k \in \hat{X}^{(n-j)}_0} \frac{\sigma(\mathbb{L}^j k)^q}{\sigma(k)^q} \hat{\phi}(k) \]

\[ = \frac{1}{L^q} \sum_{\sigma(k) = \mathbb{L}^{-j} p_{j-1}} \frac{\sigma(k)^q}{\sigma(\mathbb{L}^{-j} p_{j-1})^q} \hat{\phi}(p_{j-1}) \]

so that, by part (a),

\[ (\hat{Q}_n \phi)(p_0) = \frac{1}{L^{5m}} \sum_{\sigma(p_j)^q = \mathbb{L}^{-j-1} p_j} \frac{\sigma(p_1)^q}{\sigma(\mathbb{L}^{-1} p_1)^q} \frac{\sigma(\mathbb{L}^{-1} p_2)^q}{\sigma(\mathbb{L}^{-2} p_2)^q} \cdots \frac{\sigma(\mathbb{L}^{-1} p_n)^q}{\sigma(\mathbb{L}^{-n} p_n)^q} \hat{\phi}(p_n) \]

\[ = \frac{1}{L^{5m}} n \sum_{\sigma(p_j)^q = \mathbb{L}^{-j-1} p_j} \frac{\sigma(p_1)^q}{\sigma(\mathbb{L}^{-1} p_1)^q} \frac{\sigma(\mathbb{L}^{-1} p_2)^q}{\sigma(\mathbb{L}^{-2} p_2)^q} \cdots \frac{\sigma(\mathbb{L}^{-1} p_n)^q}{\sigma(\mathbb{L}^{-n} p_n)^q} \hat{\phi}(p_n) \]

\[ = \frac{1}{L^{5m}} \sum_{\sigma(p_j)^q = \mathbb{L}^{-j} p_j} \frac{\sigma(p_1)^q}{\sigma(\mathbb{L}^{-1} p_1)^q} \frac{\sigma(\mathbb{L}^{-1} p_2)^q}{\sigma(\mathbb{L}^{-2} p_2)^q} \cdots \frac{\sigma(\mathbb{L}^{-1} p_n)^q}{\sigma(\mathbb{L}^{-n} p_n)^q} \hat{\phi}(p_n) \]

\[ = \frac{1}{L^{5m}} \sum_{\sigma(p_j)^q = \mathbb{L}^{-j} p_j} \frac{\sigma(p_1)^q}{\sigma(\mathbb{L}^{-1} p_1)^q} \frac{\sigma(\mathbb{L}^{-1} p_2)^q}{\sigma(\mathbb{L}^{-2} p_2)^q} \cdots \frac{\sigma(\mathbb{L}^{-1} p_n)^q}{\sigma(\mathbb{L}^{-n} p_n)^q} \hat{\phi}(p_n) \]
(c) follows from part (b) and [5, Lemma 9.a].

(d) is obvious since \(\frac{\sin z}{\sin m}\) is even and entire for any nonzero integer \(m\).

In Lemmas 2.2 and 2.3 we derive a number of bounds on the kernels \(u_n\) and \(u_+\) that appear in the representations for \(Q_n\) and \(Q\) of Remark 2.1.e. In Proposition 2.4 we analyze the operator \(\hat{Q}_n\). Then in Remark 2.5 and Lemma 2.6 we study how to move derivatives past \(Q\) and \(Q_n\).

When dealing with the asymmetry between “temporal” and “spatial” scaling we set, for convenience,

\[
L_\nu = \begin{cases} 
L^2 & \text{for } \nu = 0 \\
L & \text{for } \nu = 1, 2, 3
\end{cases}
\]

\[
\varepsilon_{n,\nu} = \begin{cases} 
\varepsilon_n^2 = \frac{L}{L^0} = \frac{1}{L^0} & \text{for } \nu = 0 \\
\varepsilon_n = \frac{1}{L_\nu} = \frac{1}{L_\nu^0} & \text{for } \nu = 1, 2, 3
\end{cases}
\]  

(2.6)

**Lemma 2.2.** Let \(q \in \mathbb{N}\). Assume that \(|\Re k_\nu| \leq \pi, |\Im k_\nu| \leq 2\) for each \(0 \leq \nu \leq 3\).

(a) \(|u_n(k + \ell)| \leq \prod_{\nu=0}^{3} |k_\nu| \prod_{\nu=0}^{3} \frac{24}{|k_\nu| + \pi}\) for all \(\ell \in \hat{B}_n\). We use \(|\ell_\nu|\) to denote the magnitude of the smallest representative of \(\ell_\nu\) in its equivalence class, as an element of \(\hat{B}_n\).

(b) \(|u_n(k + \ell)| \leq \left[ \prod_{\nu=0}^{3} |k_\nu| \right] \prod_{\nu=0}^{3} \frac{24}{|k_\nu| + \pi}\) if \(0 \neq \ell \in \hat{B}_n\).

(c) \(|u_n(k) - 1| \leq 4^q|k|^2\).

(d) If \(\ell \in \hat{B}_n\) and \(\ell_\nu \neq 0\) for some \(0 \leq \tilde{\nu} \leq 3\), then \(u_n(k + \ell) = \sin \left(\frac{1}{2}k_{\tilde{\nu}}\right)v_{n,\tilde{\nu}}(k + \ell)\) with \(|v_{n,\tilde{\nu}}(k + \ell)| \leq \prod_{\nu=0}^{3} \frac{24}{|k_\nu| + \pi}\).

(e) For all \(\ell \in \hat{B}_n\),

\[
|\Im u_n(k + \ell)| \leq 12 |\Im k| \prod_{\nu=0}^{3} \frac{24}{|k_\nu| + \pi}
\]

\[
|\Im u_n(k + \ell)^q| \leq 12 q |\Im k| \left[ \prod_{\nu=0}^{3} \frac{24}{|k_\nu| + \pi}\right]^q
\]

(f) Recall that \(|\Re k_\nu| \leq \pi, |\Im k_\nu| \leq 2\) for each \(0 \leq \nu \leq 3\). We have

\[
\frac{1}{4\pi} \leq |u_n(k)| \leq 4\pi^4
\]

If, in addition, \(k\) is real

\[
\left(\frac{2}{\pi}\right)^4 \leq u_n(k) \leq \left(\frac{\pi}{2}\right)^4
\]

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Proof. Set \( s(x) = \frac{\sin x}{x} \). By the definitions of \( u_n(p) \) in Remark 2.1.b, \( \sigma(k) \) in (2.3) and \( \varepsilon_{n,\nu} \) in (2.6),

\[
u_n(p) = \frac{\sin \frac{1}{2} p_0}{\frac{1}{\varepsilon_n} \sin \frac{1}{2} \varepsilon_n^2 p_0} \prod_{\nu=1}^3 \frac{\sin \frac{1}{2} \varepsilon_n p_\nu}{\frac{1}{\varepsilon_n} \sin \frac{1}{2} \varepsilon_n p_\nu} = \prod_{\nu=0}^3 \frac{s(p_\nu/2)}{s(\varepsilon_{n,\nu} p_\nu/2)} \tag{2.7}
\]

(a) We may assume without loss of generality that \( \ell_\nu \) is bounded, as a real number, by \( \frac{\pi}{\varepsilon_{n,\nu}} - \pi \). (Recall that \( \frac{1}{\varepsilon_{n,\nu}} \) is an odd natural number.) So \( |Re k_\nu + \ell_\nu| \) is always bounded by \( \frac{\pi}{\varepsilon_{n,\nu}} \). Consequently, the hypotheses of Lemma A.1.c, with \( x + iy = k_\nu + \ell_\nu \) and \( \varepsilon = \varepsilon_{n,\nu} \), are satisfied and

\[
\left| \frac{\sin \frac{1}{2} (k_\nu + \ell_\nu)}{\varepsilon_{n,\nu} \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu)} \right| \leq \begin{cases} 
4 & \text{if } \ell_\nu = 0 \\
8 |Re k_\nu + \ell_\nu| & \text{if } |\ell_\nu| \geq 2\pi 
\end{cases} 
\leq \frac{24}{|\ell_\nu| + \pi}
\]

since \( |Re k_\nu| \leq \pi \) and \( \ell_\nu \leq 2\pi \mathbb{Z} \).

When \( q > 1 \), \( |u_n(k + \ell)|^q \) is summable in \( \ell \) and the bound on \( \|Q_n\|_{m=1} \) follows from [5, Lemma 12.c]. When \( q = 1 \), we use that the action of

\[
Q_n : \mathcal{H}_n^{(0)} \to \mathcal{H}_0^{(n)}
\]

in position space is

\[
(Q_n \phi)(x) = \sum_{v \in \varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3} u_n(v)\phi(x + [v]) \tag{2.8}
\]

where \([v] \) denotes the class of \( v \in \varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3 \) in the quotient space

\[
\mathcal{X}_n^{(0)} = (\varepsilon_n^2 \mathbb{Z} / \varepsilon_n L_{t^p} \mathbb{Z}) \times (\varepsilon_n \mathbb{Z}^3 / \varepsilon_n L_{s^p} \mathbb{Z}^3)
\]

and \( x \) runs over

\[
\mathcal{X}_0^{(n)} = (\mathbb{Z} / \varepsilon_n L_{t^p} \mathbb{Z}) \times (\mathbb{Z}^3 / \varepsilon_n L_{s^p} \mathbb{Z}^3)
\]

When \( q = 1 \), the averaging profile \( u_n \) is the characteristic function, \( 1_{00}(v) \) (the dependence on \( n \) is suppressed in the notation), of the rectangle

\[
\left[ \left[ -\frac{1}{2} \varepsilon^2, \frac{1}{2} \varepsilon^2 \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right]^3 \right] \cap (\varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3)
\]

Note that \( u_n = 1_0 \) is already normalized to have integral one. The \( L^1 - L^\infty \) norm of \( u_n \) (with mass zero) is exactly one. The \( L^1 - L^\infty \) norm, with mass \( m = 1 \), of \( u_n \) is bounded by \( \exp \left\{ \left[ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right] \right\} = e \).
Internal Remark

To verify that $u_n = 1$, observe that, by [5, Example 8(continued)], the Fourier transform of the characteristic function $1_\star$ is

$$u_{LT}(\varepsilon_T p_0) \prod_{\nu=1}^{3} u_{LX}(\varepsilon_X p_\nu)$$

with $\varepsilon_T = \varepsilon_n^2$, $\varepsilon_X = \varepsilon_n$, $L_T = \frac{1}{\varepsilon_n}$, $L_X = \frac{1}{\varepsilon_n}$, and

$$u_L(\omega) = \begin{cases} \frac{1}{2} \frac{\sin \frac{L \omega}{2}}{\sin \frac{\omega}{2}} & \text{if } \omega \notin 2\pi \mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$$

Recalling that $\sigma(k) = \sin \left( \frac{1}{2} k_0 \right) \prod_{\nu=1}^{3} \sin \left( \frac{1}{2} k_\nu \right)$, we have

$$u_n(p) = \frac{1}{\varepsilon_n^3} \sigma(\varepsilon_n^2 p_0, \varepsilon_n p_n)$$

as in (2.7).

(b) If $\ell_\nu \neq 0$,

$$\frac{1}{2} |k_\nu + \ell_\nu| \geq \frac{1}{2} |\text{Re} k_\nu + \ell_\nu| \geq \frac{1}{6} (\pi + |\ell_\nu|)$$

so that, by Lemma A.1.a, the denominator

$$\frac{1}{\varepsilon_{n,\nu}} \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu) \geq \frac{1}{\varepsilon_{n,\nu}} \frac{\sqrt{2}}{6} \varepsilon_{n,\nu} (\pi + |\ell_\nu|) = \frac{1}{3\sqrt{2}} (\pi + |\ell_\nu|)$$

On the other hand, the numerator, by Lemma A.1.b,

$$|\sin \frac{1}{2} (k_\nu + \ell_\nu)| = |\sin \left( \frac{1}{2} k_\nu \right)| \leq |k_\nu|$$

As $\ell \neq 0$, there is at least one $\nu$ with $\ell_\nu \neq 0$. For each $\nu$ with $\ell_\nu \neq 0$ bound the factor

$$\left| \frac{\sin \frac{1}{2} (k_\nu + \ell_\nu)}{\frac{1}{\varepsilon_{n,\nu}} \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu) \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu)} \right| \leq \frac{3\sqrt{2} \pi |k_\nu|}{|\ell_\nu| + \pi} \leq \frac{24 |k_\nu|}{|\ell_\nu| + \pi}$$

Bound the remaining factors, with $\nu$ having $\ell_\nu = 0$, by $\frac{24}{|\ell_\nu| + \pi}$ as in part (a). All together

$$\prod_{\nu=0}^{3} \left| \frac{\sin \frac{1}{2} (k_\nu + \ell_\nu)}{\frac{1}{\varepsilon_{n,\nu}} \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu)} \right| \leq \left[ \prod_{0 \leq \nu \leq 3} |k_\nu| \prod_{\ell_\nu \neq 0} \frac{24}{|\ell_\nu| + \pi} \right]^{3}$$

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(c) For both \( z = \frac{1}{2} \varepsilon_{n, \nu} k_{\nu} \) and \( z = \frac{1}{2} k_{\nu}, |z|^2 \leq \frac{\pi^2}{4} + 1 < 4 \) so that, by Lemma A.1.e,
\[
\left| \frac{\sin z}{z} - 1 \right| \leq \frac{1}{2} |z|^2
\]

Using \( \frac{a}{b} - 1 = \frac{(a-1)-(b-1)}{b} \) we have, by Lemma A.1.a,
\[
\left| \frac{\sin \left( \frac{1}{2} k_{\nu} \right)}{\frac{1}{2} \varepsilon_{n, \nu} \sin \left( \frac{1}{2} \varepsilon_{n, \nu} k_{\nu} \right)} \right| - 1 \leq \frac{\frac{1}{2} |\frac{1}{2} k_{\nu}|^2 + \frac{1}{2} |\frac{1}{2} \varepsilon_{n, \nu} k_{\nu}|^2}{|\sin \left( \frac{1}{2} \varepsilon_{n, \nu} k_{\nu} \right)| / |\frac{1}{2} \varepsilon_{n, \nu} k_{\nu}|} \leq \frac{\pi}{\sqrt{2}} \frac{1}{8} (1 + \varepsilon_{n, \nu}^2) |k_{\nu}|^2 \leq |k_{\nu}|^2
\]

Finally, using
\[
\prod_{\nu=0}^{3} A_{\nu} - 1 = (A_0 - 1) \prod_{\nu=1}^{3} A_{\nu} + (A_1 - 1) \prod_{\nu=2}^{3} A_{\nu} + (A_2 - 1) A_3 + (A_3 - 1)
\]
we have, by Lemma A.1.c,
\[
|u_n(k) - 1| \leq 4^3 |k_0|^2 + 4^2 |k_1|^2 + 4 |k_2|^2 + |k_3|^2 \leq 4^3 |k|^2
\]

(d) The proof is the same as that for part (b), except that the factor \( \sin \left( \frac{1}{2} k_{\bar{\nu}} \right) \) in the numerator
\[
\sin \left( \frac{1}{2} (k_{\bar{\nu}} + \ell_{\bar{\nu}}) \right) = (-1)^{\bar{\nu}} \sin \left( \frac{1}{2} k_{\bar{\nu}} \right)
\]
is pulled out of \( u_n \), leaving \( v_{n, \bar{\nu}} \), rather than being bounded by \( |k_{\bar{\nu}}| \).

(e) By Lemma A.1.c
\[
|\text{Im} \frac{\frac{1}{2} (k_{\nu} + \ell_{\nu})}{\frac{1}{2} \varepsilon_{n, \nu} (k_{\nu} + \ell_{\nu})} | \leq 6 |\text{Im} k_{\nu}| \frac{24}{|\ell_{\nu}| + \pi}
\]

In general, for any complex numbers \( z_j = r_j e^{i \theta_j}, 1 \leq j \leq J, \)
\[
|\text{Im} \prod_{j=1}^{J} z_j | = |\sin \left( \sum_{j=1}^{J} \theta_j \right) | \prod_{j=1}^{J} r_j
\]
Repeatedly using
\[
|\sin(\theta + \theta')| = |\sin(\theta) \cos(\theta') + \cos(\theta) \sin(\theta')| \leq |\sin(\theta)| + |\sin(\theta')|
\]
we have
\[
|\text{Im} \prod_{j=1}^{J} z_j | \leq \sum_{j=1}^{J} |\sin(\theta_j)| \prod_{j=1}^{J} r_j = \sum_{j=1}^{J} |\text{Im} z_j| \prod_{j \neq j}^{J} |z_j|
\]

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So

\[ |\text{Im} u_n(k + \ell)| \leq \left( \sum_{\nu=0}^{3} 6|\text{Im} k_\nu| \right) \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu|+\pi} \leq 12 |\text{Im} k| \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu|+\pi} \]

and

\[ |\text{Im} u_n(k + \ell)|^q \leq 12 q |\text{Im} k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu|+\pi} \right]^q \]

(f) Just apply Lemma A.1.a,b separately to all of the numerators and denominators in the right hand side of (2.7). When \( k \) is real, apply Lemma A.1.d instead.

\[ \Box \]

**Lemma 2.3.** Let \( q \in \mathbb{N} \). Assume that \( |\text{Re} \ell_\nu| \leq \frac{\pi}{L_\nu} \) and \( |\text{Im} \ell_\nu| \leq \frac{2 \pi}{L_\nu} \) for each \( 0 \leq \nu \leq 3 \).

(a) \( |u_+ (t + \ell)| \leq \prod_{\nu \in I_+} L_\nu |\ell_\nu| \left[ \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi} \right] \) for all \( \ell \in \hat{B}^+ \). We use \( |\ell_\nu| \) to denote the magnitude of the smallest representative of \( \ell_\nu \) in its equivalence class, as an element of \( \hat{B}^+ \).

(b) \( |u_+ (t + \ell)| \leq \left[ \prod_{\nu \in I_+} L_\nu |\ell_\nu| \right] \left[ \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi} \right] \) for all \( \ell \in \hat{B}^+ \) with \( \ell \neq 0 \). Here \( I_+ \) is any subset of \( \{ \nu \mid 0 \leq \nu \leq 3, \ell_\nu \neq 0 \} \).

(c) \( |u_+ (t) - 1| \leq 4^3 \prod_{\nu=0}^{3} L_\nu^2 |\ell_\nu|^2 \).

(d) If \( \ell \in \hat{B}^+ \) and \( \ell_\nu \neq 0 \) for some \( 0 \leq \nu \leq 3 \), then \( u_+ (t + \ell) = \sin \left( \frac{L_\nu \ell_\nu}{2} \right) v_{+\nu} (t + \ell) \) with \( |v_{+\nu} (t + \ell)| \leq \frac{3}{L_\nu} \frac{24}{|\ell_\nu|+\pi} \).

(e) For all \( \ell \in \hat{B}^+ \),

\[ |\text{Im} u_+ (t + \ell)| \leq 12 |\text{Im} \mathbb{L} t| \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi} \]

\[ |\text{Im} u_+ (t + \ell)|^q \leq 12 q |\text{Im} \mathbb{L} t| \left[ \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi} \right]^q \]

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Proof. (a) We may assume without loss of generality that $\ell_\nu$ is bounded, as a real number, by $\pi - \frac{\pi}{L_\nu}$. (Recall that $L$ is an odd natural number.) So we will always have $|\text{Re} \mathfrak{t}_\nu + \ell_\nu| \leq \pi$. So, by Lemma A.1.c with $\varepsilon = \frac{1}{L_\nu}$ and $x + iy = L_\nu(\mathfrak{t}_\nu + \ell_\nu)$,

$$
|\sin \frac{L_\nu}{2}(\mathfrak{t}_\nu + \ell_\nu)| \leq \begin{cases} \frac{4}{L_\nu|x_\nu + iy_\nu|} & \text{if } \ell_\nu = 0 \\ \frac{8}{\pi} & \text{if } |L_\nu \ell_\nu| \geq 2\pi \end{cases} \leq \frac{24}{L_\nu|\ell_\nu| + \pi}
$$

since $|\text{Re} \mathfrak{t}_\nu| \leq \frac{\pi}{L_\nu}, |\text{Im} \mathfrak{t}_\nu| \leq \frac{2}{L_\nu}$ and $\ell_\nu \in \frac{2\pi}{L_\nu} \mathbb{Z}$.

(b) If $\ell_\nu \neq 0$,

$$
\frac{1}{2}|\mathfrak{t}_\nu + \ell_\nu| \geq \frac{1}{2}|\text{Re} \mathfrak{t}_\nu + \ell_\nu| \geq \frac{1}{6}\left(\frac{\pi}{L_\nu} + |\ell_\nu|\right)
$$

and $\frac{1}{2}|\text{Re} \mathfrak{t}_\nu + \ell_\nu| \leq \frac{\pi}{2}$ so that, by Lemma A.1.a, the denominator

$$
L_\nu |\sin \frac{1}{2}(\mathfrak{t}_\nu + \ell_\nu)| \geq \frac{\sqrt{7}}{6\pi}(\pi + L_\nu|\ell_\nu|)
$$

On the other hand, the numerator, by Lemma A.1.b,

$$
|\sin \frac{L_\nu}{2}(\mathfrak{t}_\nu + \ell_\nu)| = |\sin \left(\frac{L_\nu}{2} \mathfrak{t}_\nu\right)| \leq 2\frac{L_\nu}{2} |\mathfrak{t}_\nu| = L_\nu|\mathfrak{t}_\nu|
$$

As $\ell \neq 0$, there is at least one $\nu$ with $\ell_\nu \neq 0$. Bound each factor with $\nu \in \mathbb{Z}_+$ by

$$
\left|\frac{\sin \frac{L_\nu}{2}(\mathfrak{t}_\nu + \ell_\nu)}{L_\nu \sin \frac{1}{2}(\mathfrak{t}_\nu + \ell_\nu)}\right| \leq \frac{6\pi L_\nu |\mathfrak{t}_\nu|}{\sqrt{2}L_\nu|\ell_\nu| + \pi}
$$

Bound the remaining factors by $\frac{24}{L_\nu|\ell_\nu| + \pi}$.

(c) has the same proof as that of Lemma 2.2.c, with $k_\nu$ replaced by $L_\nu \mathfrak{t}_\nu$ and $\frac{1}{\varepsilon_n,\nu}$ replaced by $L_\nu$.

**Internal Remark**

For both $z = \frac{1}{2}L_\nu \mathfrak{t}_\nu$ and $z = \frac{1}{2} \mathfrak{t}_\nu$, $|z|^2 \leq \frac{\pi^2}{4} + 1 < 4$ so that, by Lemma A.1.e,

$$
|\sin z - 1| \leq \frac{1}{2}|z|^2
$$

Using $a^2 - 1 = \frac{(a-1)-(b-1)}{b}$ we have, by Lemma A.1.a,

$$
\left|\frac{\sin \left(\frac{L_\nu}{2} \mathfrak{t}_\nu\right)}{L_\nu \sin \left(\frac{1}{2} \mathfrak{t}_\nu\right)} - 1\right| \leq \frac{1}{2}\left|\frac{L_\nu}{2} \mathfrak{t}_\nu\right|^2 + \frac{1}{2}\left|\frac{1}{2} \mathfrak{t}_\nu\right|^2 \leq \frac{\pi}{\sqrt{2} \delta_8}(L_\nu^2 + 1)|\mathfrak{t}_\nu|^2
$$
Finally, using
\[ \prod_{\nu=0}^{3} A_{\nu} - 1 = (A_0 - 1) \prod_{\nu=1}^{3} A_{\nu} + (A_1 - 1) \prod_{\nu=2}^{3} A_{\nu} + \cdots + (A_3 - 1) \]
we have, by Lemma A.1.c,
\[ |u_+(\xi) - 1| \leq \frac{\pi}{\sqrt{2}} \left[ 4^3 \left( L_0^2 + 1 \right) |\xi_0|^2 + 4^3 \left( L_1^2 + 1 \right) |\xi_1|^2 + \cdots + 4^3 \left( L_3^2 + 1 \right) |\xi_3|^2 \right] \]
\[ \leq \frac{\pi}{\sqrt{2}} 4 \sum_{\nu=0}^{3} L^2_{\nu} |\xi_\nu|^2 \]
\[ \leq 4 \sum_{\nu=0}^{3} L^2_{\nu} |\xi_\nu|^2 \]

(d) The proof is similar to that for part (b), except that the factor \( \sin \left( \frac{L_{\nu}}{2} \xi_{\nu} + \ell_{\nu} \right) \) in the numerator
\[ \sin \frac{L_{\nu}}{2} \left( \xi_{\nu} + \ell_{\nu} \right) = (-1)^{\frac{L_{\nu} \ell_{\nu}}{2\pi} \sin \left( \frac{L_{\nu}}{2} \xi_{\nu} \right) \}

is pulled out of \( u_+ \), leaving \( v_+_\zeta, \nu \), rather than being bounded by \( L_{\nu} |\xi_{\nu}| \). Also, each \( \frac{\sin \frac{L_{\nu}}{2} (\xi_{\nu} + \ell_{\nu})}{L_{\nu} \sin \frac{1}{2} (\xi_{\nu} + \ell_{\nu})} \) with \( \nu \neq \zeta \) is bounded by \( \frac{24}{L_{\nu} |\xi_{\nu}| + \pi} \).

(e) By Lemma A.1.c
\[ \left| \Im \frac{\sin \frac{L_{\nu}}{2} (\xi_{\nu} + \ell_{\nu})}{L_{\nu} \sin \frac{1}{2} (\xi_{\nu} + \ell_{\nu})} \right| \leq 8 |\Im L_{\nu} \xi_{\nu}| \frac{24}{L_{\nu} |\xi_{\nu}| + \pi} \]

The proof now continues as in Lemma 2.2.e, just by substituting \( k_{\nu} \rightarrow L_{\nu} \xi_{\nu}, \ell_{\nu} \rightarrow L_{\nu} \ell_{\nu} \) and \( \varepsilon_{n,\nu} = \frac{1}{L_{\nu}} \).

**Internal Remark**

Note that
\[ L_{\nu} \xi_{\nu} = \begin{cases} L^2 & \text{for } \nu = 0 \\ L & \text{for } \nu = 1, 2, 3 \end{cases} \]
\[ \xi_{\nu} = (\mathbb{I} \xi)_{\nu} \]
Proposition 2.4. Let \( q \in \mathbb{N} \). There are constants \( \Gamma_2 \), depending only on \( q \), and \( \Gamma_3 \), depending only on \( a \), such that the following hold for all \( L > \Gamma_2 \).

(a) On the domain \( \{ k \in \mathbb{C} \times \mathbb{C}^3 \mid \|\text{Im } k_\nu\| < 2 \text{ for each } 0 \leq \nu \leq 3 \} \) \( \hat{\Omega}_n(k) \) is analytic, and invariant under \( k_\nu \to -k_\nu \) for each \( 0 \leq \nu \leq 3 \) and under \( k_\nu \leftrightarrow k_\nu' \) for all \( 1 \leq \nu, \nu' \leq 3 \), and obeys

\[
\frac{5}{6} a \leq |\hat{\Omega}_n(k)| \leq \frac{5}{7} a \quad \text{Re } \hat{\Omega}_n(k) \geq \frac{a}{7}
\]

If \( k \) is real \( \frac{5}{6} a \leq \hat{\Omega}_n(k) \leq a \).

(b) If \( |\text{Re } k_\nu| \leq \pi \) and \( |\text{Im } k_\nu| \leq 2 \) for each \( 0 \leq \nu \leq 3 \), then

\[
|\hat{\Omega}_n(k) - a_n| \leq \frac{a}{3005} |k|^2 \quad \text{where } a_n = a \frac{1 - L^{-2}}{1 - L^{-2n}}
\]

(c) \( \|\hat{\Omega}_n\|_{m=1} \leq \Gamma_3 \)

Proof. (a) Recall, from Remark 2.1.e, that \( \hat{\Omega}_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2j}} u_j(k + \ell_j)^{2q} \right]^{-1} \).

By Lemma 2.2.a,

\[
\sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2j}} |u_j(k + \ell_j)|^{2q} \leq \sum_{j=1}^{\infty} \frac{1}{L^{2j}} \sum_{\ell \in 2\pi \mathbb{Z} \times 2\pi \mathbb{Z}^3} \prod_{\nu=0}^{3} (\frac{24}{|\ell_\nu| + 1})^{2q} = \frac{c_q}{L^2 - 1}
\]

where \( c_q = \left[ \sum_{j \in \mathbb{Z}} \left( \frac{24}{|2j| + 1} \right)^{2q} \right]^4 \). Just pick \( L \) large enough that \( \frac{c_q}{L^2 - 1} < \frac{1}{5} \) and use that, if \( |z| \leq \frac{1}{5} \)

\[
\text{Re } \frac{1}{1+z} = \frac{\text{Re } (1+z)}{|1+z|^2} \geq \frac{4/5}{(6/5)^2} = \frac{20}{36}
\]

(b) Using \( O(|k|^2) \) to denote any function that is bounded by a constant, depending only on \( q \),

\[
1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2j}} u_j(k + \ell_j)^{2q} = \sum_{j=0}^{n-1} \frac{1}{L^{2j}} + \sum_{j=1}^{n-1} \frac{1}{L^{2j}} [u_j(k)^{2q} - 1]
\]

\[
\leq \frac{1 - L^{-2n}}{1 - L^{-2}} + \sum_{j=1}^{n-1} \frac{1}{L^{2j}} O(|k|^2) \quad \text{by Lemma 2.2.b,c}
\]

\[
\leq \frac{1 - L^{-2n}}{1 - L^{-2}} + \frac{1}{L^2} O(|k|^2)
\]
So
\[
\hat{Q}_n(k) = a\left[\frac{1}{1-L^2/2} + \frac{1}{L^2}O(|k|^2)\right]^{-1} = a_n\left[1 + \frac{1}{L^2}\frac{1-L^2}{1-2L^2}\frac{1}{O(|k|^2)}\right]^{-1}
\]
and it suffices to choose \(\Gamma_2\) large enough that
\[
\frac{9}{10} \leq \frac{1-L^2}{1-2L^2} \leq \frac{11}{10} \quad \left| \frac{1}{L^2}\frac{1-L^2}{1-2L^2}O(|k|^2) \right| \leq \frac{10}{22}\frac{1}{500}\frac{1}{|k|^2} \quad \left| \frac{1}{L^2}\frac{1-L^2}{1-2L^2}O(|k|^2) \right| \leq \frac{1}{2}
\]
for all allowed \(k\)’s and \(L\)’s.

(c) follows immediately from part (a) and [5, Lemma 12.b] with \(X_{\text{fin}} = X_{\text{crs}} = X_0^{(n)}\). □

We now define operators \(Q_{n,\nu}^{(\pm)}\) and \(Q_{+,\nu}^{(\pm)}\) so that the next remark holds.

**Remark 2.5.** Let \(0 \leq \nu \leq 3\). We have
\[
\partial_{\nu}Q_{n,\nu}^{*} = Q_{n,\nu}^{(\pm)}\partial_{\nu} \quad \partial_{\nu}Q_{n} = Q_{n,\nu}^{(\pm)}\partial_{\nu} \quad \partial_{\nu}Q^{*} = Q_{+,\nu}^{(\pm)}\partial_{\nu} \quad \partial_{\nu}Q = Q_{+,\nu}^{(\pm)}\partial_{\nu}
\]
If \(S : \mathcal{H}_n \to \mathcal{H}_n\) and \(T : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}\) are linear operators that are translation invariant with respect to \(X_n\) and \(X_0^{(n)}\), respectively, then
\[
Q_{n,\nu}^{(-)}SQ_{n,\nu}^{(+)} = Q_nS\hat{Q}_n \quad Q_{+,\nu}^{(-)}TQ_{+,\nu}^{(+)} = QT\hat{Q}
\]
To prepare for the definitions, recall that the forward derivatives of \(\alpha \in \mathcal{H}_j^{(n)}\) are defined by
\[
(\partial_{\nu}\alpha)(x) = \frac{1}{\varepsilon_{n,\nu}}[\alpha(x + \varepsilon_{j,\nu}e_{\nu}) - \alpha(x)] \quad (2.9)
\]
where \(e_{\nu}\) is a unit vector in the \(\nu\)th direction. The Fourier transforms
\[
\hat{(\partial_{\nu}\phi)}(p) = 2ie^{i\varepsilon_{n,\nu}p_{\nu}/2} \frac{\sin(p_{\nu}/2)}{\varepsilon_{n,\nu}} \hat{\phi}(p) \quad \text{for all } \phi \in \mathcal{H}_n \quad \text{and } p \in \hat{X}_n
\]
\[
\hat{(\partial_{\nu}\psi)}(k) = 2ie^{ik_{\nu}/2} \frac{\sin(k_{\nu}/2)}{k_{\nu}} \hat{\psi}(k) \quad \text{for all } \psi \in \mathcal{H}_0^{(n)} \quad \text{and } k \in \hat{X}_0^{(n)} \quad (2.10)
\]
\[
\hat{(\partial_{\nu}\theta)}(\xi) = 2ie^{iL_{\xi}e_{\nu}/2} \frac{\sin(L_{\xi}e_{\nu}/2)}{L_{\xi}e_{\nu}} \hat{\theta}(\xi) \quad \text{for all } \theta \in \mathcal{H}_n^{(n+1)} \quad \text{and } \xi \in \hat{X}_n^{(n+1)}
\]
Set
\[
u_n^{(+)}(p) = \prod_{0 \leq \nu \leq 0} \frac{\sin \frac{1}{2}p_{\nu}}{\varepsilon_{n,\nu}} \quad \nu_n^{(-)}(p) = \prod_{0 \leq \nu \leq 0} \frac{\sin \frac{1}{2}p_{\nu}}{\varepsilon_{n,\nu}}
\]
\[
u_n^{(+)}(k) = \prod_{0 \leq \nu \leq 0} \frac{\sin \frac{1}{2}k_{\nu}}{L_{\nu} \sin \frac{1}{2}k_{\nu}} \quad \nu_n^{(-)}(k) = \prod_{0 \leq \nu \leq 0} \frac{\sin \frac{1}{2}k_{\nu}}{L_{\nu} \sin \frac{1}{2}k_{\nu}}
\]

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Define the operators $Q_{n,\nu}^{(+)} : \mathcal{H}_0^{(n)} \to \mathcal{H}_n$ and $Q_{n,\nu}^{(-)} : \mathcal{H}_n \to \mathcal{H}_0^{(n)}$ by

$$
(Q_{n,\nu}^{(+)} \psi)(k + \ell_n) = \zeta_{n,\nu}^{(+)}(k, \ell_n) u_{n,\nu}^{(+)}(k + \ell_n) u_n(k + \ell_n)^{q-1} \dot{\psi}(k)
$$

$$
(Q_{n,\nu}^{(-)} \phi)(k) = \sum_{\ell_n \in \mathbb{B}_n} \zeta_{n,\nu}^{(-)}(k, \ell_n) u_{n,\nu}^{(-)}(k + \ell_n) u_n(k + \ell_n)^{q-1} \dot{\phi}(k + \ell_n)
$$

(2.11)

and the operators $Q_{+,\nu}^{(+)} : \mathcal{H}_{-1}^{(n+1)} \to \mathcal{H}_0^{(n)}$ and $Q_{+,\nu}^{(-)} : \mathcal{H}_0^{(n)} \to \mathcal{H}_{-1}^{(n+1)}$ by

$$
(Q_{+,\nu}^{(+)} \theta)(\ell + \ell) = \zeta_{+,\nu}^{(+)}(\ell, \ell) u_{+,\nu}^{(+)}(\ell + \ell) u_+(\ell + \ell)^{q-1} \dot{\theta}(\ell)
$$

$$
(Q_{+,\nu}^{(-)} \psi)(\ell) = \sum_{\ell \in \mathbb{B}_+} \zeta_{+,\nu}^{(-)}(\ell, \ell) u_{+,\nu}^{(-)}(\ell + \ell) u_+(\ell + \ell)^{q-1} \dot{\psi}(\ell + \ell)
$$

(2.12)

**Proof of Remark 2.5.** For the “$Q_n^{(+)}$” and “$Q_n^{(-)}$” cases, it suffices to observe that

$$
(2ie^{\epsilon_{n,\nu}(k+\ell_n)/2} \sin(\epsilon_{n,\nu}(k+\ell_n)/2)) u_n(k + \ell_n) = \zeta_{n,\nu}^{(+)}(k, \ell_n) u_{n,\nu}^{(+)}(k + \ell_n)(2ie^{ik\nu/2} \sin(k\nu/2))
$$

and

$$
(2ie^{ik\nu/2} \sin(k\nu/2)) u_n(k + \ell_n) = \zeta_{n,\nu}^{(-)}(k, \ell_n) u_{n,\nu}^{(-)}(k + \ell_n)(2ie^{i\epsilon_{n,\nu}(k+\ell_n)/2} \sin(\epsilon_{n,\nu}(k+\ell_n)/2))
$$

and

$$
\zeta_{n,\nu}^{(-)}(k, \ell_n) u_{n,\nu}^{(-)}(k + \ell_n) \zeta_{n,\nu}^{(+)}(k, \ell_n) u_{n,\nu}^{(+)}(k + \ell_n) = u_n(k + \ell_n)^2
$$

for all $k$, $\ell$, $\nu$. We remark that “$Q_n^{(-)} S Q_n^{(+)}$” should not be surprising since $\partial_{\nu} Q_n^{(-)} S Q_n^{(+)} = Q_n^{(-)} S Q_n^{(+)} \partial_{\nu}$ and $Q_n^{(-)} S Q_n^{(+)}$ is translation invariant on the unit scale and so commutes with $\partial_{\nu}$. The proof for the “$Q^{(-)}$” and “$Q^{(+)}$” cases are virtually identical. \(\square\)

**Lemma 2.6.** Let $0 \leq \nu \leq 3$, $\ell \in \mathbb{B}_+$ and $\ell_n \in \mathbb{B}_n$.

(a) $\zeta_{n,\nu}^{(+)}(k, \ell_n) u_{n,\nu}^{(+)}(k + \ell_n)$ and $\zeta_{n,\nu}^{(-)}(k, \ell_n) u_{n,\nu}^{(-)}(k + \ell_n)$ are entire in $k$ and $\zeta_{+,\nu}^{(+)}(\ell, \ell) u_{+,\nu}^{(+)}(\ell + \ell)$ and $\zeta_{+,\nu}^{(-)}(\ell, \ell) u_{+,\nu}^{(-)}(\ell + \ell)$ are entire in $\ell$.
(b) Assume that $|\text{Re} k_{\nu'}| \leq \pi$, $|\text{Im} k_{\nu'}| \leq 2$, $|\text{Re} \ell_{\nu'}| \leq \frac{\pi}{L_{\nu'}}$ and $|\text{Im} \ell_{\nu'}| \leq \frac{2}{L_{\nu'}}$ for each $0 \leq \nu' \leq 3$. Then
\[ \left| \zeta_{n,\nu}(+)^{(\pm)}(k, \ell_{n}) u_{n,\nu}^{(+)}(k + \ell_{n}) \right| \leq e \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{|\ell_{n,\nu'}| + \pi} \]
\[ \left| S_{+,\nu'}(\ell, \ell) u_{+,\nu'}^{(+)}(\ell + \ell) \right| \leq e \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{|\ell_{\nu'}| + \pi} \]
\[ \left| \zeta_{n,\nu}^{(-)}(k, \ell_{n}) u_{n,\nu}^{(-)}(k + \ell_{n}) \right| \leq \frac{24e}{|\ell_{n,\nu'}| + \pi} \prod_{\nu' = 0}^{3} \frac{24}{|\ell_{n,\nu'}| + \pi} \]
\[ \left| S_{+,\nu'}^{(-)}(\ell, \ell) u_{+,\nu'}^{(-)}(\ell + \ell) \right| \leq \frac{24e}{|\ell_{\nu'}| + \pi} \prod_{\nu' = 0}^{3} \frac{24}{|\ell_{\nu'}| + \pi} \]
\[(c)\ There\ is\ a\ constant\ \Gamma_{4},\ depending\ only\ on\ q,\ such\ that\ \|Q_{n,\nu}^{(\pm)}\|_{m=1} \leq \Gamma_{4}.\]

**Proof.** (a) The proof is virtually identical to that of Remark 2.1.d.
(b) The proof is virtually identical to that of Lemmas 2.2.a and 2.3.a.
(c) By (2.11), the Fourier transform of $Q_{n,\nu}^{(\pm)}$ is $\zeta_{n,\nu}^{(\pm)}(k, \ell_{n}) u_{n,\nu}^{(\pm)}(k + \ell_{n}) u_{n}(k + \ell_{n}) q^{-1}$, which by part (b) and Lemma 2.2.a, is bounded in magnitude by
\[ e \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{|\ell_{n,\nu'}| + \pi} \prod_{\nu = 0}^{3} \left( \frac{24}{|\ell_{\nu'}| + \pi} \right)^{q^{-1}} \]
As $q > 2$, the claim now follows by [5, Lemma 12.c]. \hfill \Box

**Remark 2.7.** The principle obstruction to allowing $q = 1$ arises when a differential operator $\partial_{\nu}$ is intertwined with the block spin averaging operator $Q_{n}$, as happens in Remark 2.5. See, for example, the proof of Lemma 2.6.c. We use the condition $q > 1$ starting at Lemma 4.3 in §4 and in §5, 6. (See Lemma 5.5.) We use the condition $q > 2$ in Proposition 6.1 and Lemma 2.6.c.

**Internal Remark**
The condition $q > 1$ is used in the proof of Lemma 4.3. This had nothing to do with derivatives. The failure of the proof when $q = 1$ could be an artificiality of working in momentum space. I’m not sure.

One main place in [7, 8] where we would have problems dealing with $q = 1$ is the following. $\mathcal{E}$ is a function of the derivative fields $\partial_{\nu} \psi$. These derivative fields have been assigned the weight factor $\kappa'(n)$. During the course of a renormalization group
step $\psi$ is replaced by the critical field $\psi_n(\theta)$. The dominant contribution to $\psi_n(\theta)$ is a linear operator acting on $\theta$. If we cannot move the derivative past the linear operator, then the weight factor $\kappa'(n)$ (for $\partial_\nu \psi$) is effectively replaced by a constant times the weight factor $\bar{\kappa}(n)$ (for $\theta$). This substantially increases the norm of $\mathcal{E}$. 


3 Differential Operators

In [7, Definition 1.5.a] we associated to an operator \( h_0 \) on \( L^2(\mathbb{Z}^3/L_{sp}\mathbb{Z}^3) \) the operators

\[
D_n = L^{2n} \mathbb{L}_n^{-n} (1 - e^{-h_0} - e^{-h_0} \partial_0) \mathbb{L}_n^n
\]  

(3.1)

Here \( \partial_0 \) is the forward time derivative of (2.9). In this chapter we assume that \( h_0 \) is the periodization (see [5, §3]) of a translation invariant operator \( h_0 \) on \( L^2(\mathbb{Z}^3) \) whose Fourier transform \( \hat{h}_0(p) \)

- is entire in \( p \) and invariant under \( p_\nu \rightarrow -p_\nu \) for each \( 1 \leq \nu \leq 3 \)
- is nonnegative when \( p \) is real and is strictly positive when \( p \in \mathbb{R}^3 \setminus 2\pi\mathbb{Z}^3 \)
- obeys \( \hat{h}_0(0) = \frac{\partial^2 \hat{h}_0}{\partial p_\mu \partial p_\nu}(0) = 0 \) for \( 1 \leq \nu \leq 3 \) and has strictly positive Jacobian matrix

\[
H = \left[ \frac{\partial^2 \hat{h}_0}{\partial p_\mu \partial p_\nu}(0) \right]_{1 \leq \mu, \nu \leq 3}.
\]

Remark 3.1.

(a) The operator \( D_n \) is the periodization of a translation invariant operator \( D_n \), acting on \( L^2(\epsilon_n \mathbb{Z} \times \epsilon_n^2 \mathbb{Z}^3) \), whose Fourier transform \( \hat{D}_n(p) \)

\[
\hat{D}_n(p) = \frac{1}{2} \epsilon_n^2 p_0 e^{-\hat{h}_0(\epsilon_n p)} \left[ \frac{\sin \frac{1}{2} \epsilon_n^2 p_0}{\frac{1}{2} \epsilon_n^2 p_0} \right]^2 + \frac{p_0^2(1 - e^{-\hat{h}_0(\epsilon_n p)})}{\epsilon_n^2 p_0^2} - ip_0 e^{-\hat{h}_0(\epsilon_n p)} \frac{\sin \epsilon_n^2 p_0}{\epsilon_n^2 p_0}
\]

with \( p = (p_0, p) \in \mathbb{C} \times \mathbb{C}^3 \).

(b) \( \hat{D}_n(p) \) is entire in \( p \) and invariant under \( p_\nu \rightarrow -p_\nu \) for each \( 1 \leq \nu \leq 3 \).

(c) \( \hat{D}_n(p) \) has nonnegative real part when \( p \) is real.

Proof. (a) follows from (1.3) and the observation, by (2.10), that the Fourier transform of \( \partial_0 \), on \( \mathbb{Z} \), is

\[
2ie^{ik_0/2} \sin(k_0/2) = -2\sin^2(k_0/2) + i\sin(k_0)
\]

(b) and (c) are obvious. \( \square \)

Lemma 3.2. There are constants \( \gamma_1, \Gamma_5 \) and a function \( \bar{m}(c) > 0 \) that depend only on \( \hat{h}_0 \) and in particular are independent of \( n \) and \( L \), such that the following hold.
(a) For all \( p \in \mathbb{R} \times \mathbb{R}^3 \),
\[
|\hat{D}_n(p)| \geq \gamma_1(|p_0| + \sum_{\nu=1}^3 |p_\nu|^2)
\]

We use \(|p_0|, |p_\nu|\) and \(|p|\) to refer to the magnitudes of the smallest representatives of \( p_0 \in \mathbb{C}, p_\nu \in \mathbb{C} \) and \( p \in \mathbb{C}^3 \) in \( \mathbb{C}/\mathbb{Z}, \mathbb{C}/\mathbb{Z} \) and \( \mathbb{C}^3/\mathbb{Z}^3 \), respectively.

(b) For all \( p \in \mathbb{C} \times \mathbb{C}^3 \) with \( \varepsilon_n^2 |p_0|, \varepsilon_n p \) having modulus less than one,
\[
\hat{D}_n(p) = -ip_0 + \frac{1}{2} \varepsilon_n^2 p_0^2 + \frac{1}{2} \sum_{\nu \neq \nu'} H_{\nu,\nu'} p_\nu p_{\nu'} + O \left( \varepsilon_n |p|^3 + \varepsilon_n^4 |p_0|^3 \right)
\]
The higher order part \( O(\cdot) \) is uniform in \( n \) and \( L \).

(c) We have, for all \( p \in \mathbb{C} \times \mathbb{C}^3 \) with \( \varepsilon_n^2 |\text{Im } p_0| \leq 1 \) and \( \varepsilon_n |\text{Im } p| \leq 1 \),
\[
|\hat{D}_n(p)| \leq \Gamma_5 (|p_0| + \sum_{\nu=1}^3 |p_\nu|^2)
\]
and
\[
\left| \frac{\partial \hat{D}_n(p)}{\partial p_\ell_i} \right| \leq \Gamma_5 \left\{ \frac{1}{[1+|p_0|+|p|^2]^{1/2}} \quad \text{if } \nu = 0, \ell_i = 1, 2 \right. \\
\left. \frac{1}{[1+|p_0|+|p|^2]^{1/2-1}} \quad \text{if } 1 \leq \nu \leq 3, 1 \leq \ell_i \leq 4 \right.
\]

(d) For all \( c > 0 \) and \( p \in \mathbb{C} \times \mathbb{C}^3 \), with \( |p_0| + |p| \geq c \) and \( |\text{Im } p| \leq \bar{\text{m}}(c) \),
\[
|\hat{D}_n(p)| \geq \gamma_1 \left( |p_0| + \sum_{\nu=1}^3 |p_\nu|^2 \right)
\]

(e) For all \( c > 0 \) and all \( p \) in the set
\[
\{ \ p \in \mathbb{C} \times \mathbb{C}^3 \ | \ |\text{Im } p| \leq \bar{\text{m}}(c), \ |p| \geq c \} \\
\cup \{ \ p \in \mathbb{C} \times \mathbb{C}^3 \ | \ |\text{Im } p| \leq \bar{\text{m}}(c), \ |\varepsilon_n p_0| \geq c \}
\]
we have
\[
\text{Re } \hat{D}_n(p) \geq \gamma_1 \left( \varepsilon_n^2 |p_0|^2 + \sum_{\nu=1}^3 |p_\nu|^2 \right)
\]
Proof. (a) By the hypotheses on \( \hat{h} \),
\[
1 - e^{-h_0(\varepsilon_n \mathbf{p})} = \frac{\hat{h}_0(\varepsilon_n \mathbf{p}) + O(h_0(\varepsilon_n \mathbf{p})^2)}{\varepsilon_n^2 \mathbf{p}^2} = \frac{1}{2} \varepsilon_n^2 \mathbf{p} \cdot H \mathbf{p} + O(|\varepsilon_n \mathbf{p}|^3)
\]
for \( \varepsilon_n \mathbf{p} \) is a real neighbourhood of \( 0 \). This is strictly positive and bounded away from 0 on some real neighbourhood of 0, uniformly in \( \varepsilon_n \). Since \( \hat{h} \) is continuous and strictly positive on \( \mathbb{R}^3 \setminus 2\pi \mathbb{Z}^3 \), there is a constant \( \gamma' > 0 \), independent of \( \varepsilon_n \), such that
\[
1 - e^{-h_0(\varepsilon_n \mathbf{p})} \geq \gamma' \quad \text{and} \quad e^{-h_0(\varepsilon_n \mathbf{p})} \geq \gamma'
\]
for all \( \mathbf{p} \in \mathbb{R}^3 \). The claim now follows from Lemma A.1.a.

(b) Expanding \( \frac{\sin z}{z} = 1 + O(|z|^2) \), for \( |z| \leq 1 \), and
\[
e^{-\hat{h}(\varepsilon_n \mathbf{p})} = 1 - \frac{1}{2} \varepsilon_n^2 \mathbf{p}^2 + \frac{1}{3} \sum_{\nu, \nu'} \frac{3}{\nu} \mathbf{p}_\nu \mathbf{p}_{\nu'} + O\left( (\varepsilon_n |\mathbf{p}|)^3 \right)
\]
gives, using Remark 3.1.a,
\[
\hat{D}_n(p) = \left[ -i p_0 + \frac{1}{2} \varepsilon_n^2 p_0^2 \right] \left[ 1 + O\left( (\varepsilon_n |\mathbf{p}|)^2 + (\varepsilon_n^2 p_0^2) \right) \right] + \frac{1}{2} \sum_{\nu, \nu'} \frac{3}{\nu} \mathbf{p}_\nu \mathbf{p}_{\nu'} + O\left( \varepsilon_n |\mathbf{p}|^3 \right)
\]
\[
= -i p_0 + \frac{1}{2} \varepsilon_n^2 p_0^2 + \frac{1}{2} \sum_{\nu, \nu'} \frac{3}{\nu} \mathbf{p}_\nu \mathbf{p}_{\nu'} + O\left( \varepsilon_n |\mathbf{p}|^3 + \varepsilon_n^2 |p_0| |\mathbf{p}|^2 + \varepsilon_n^4 |p_0|^3 \right)
\]
The claim now follows from
\[
\varepsilon_n^2 |p_0| |\mathbf{p}|^2 \leq \left( \varepsilon_n^4 |p_0|^3 \right)^{1/3} \left( \varepsilon_n |\mathbf{p}|^3 \right)^{2/3}
\]
(c) Since \( \hat{D}_n(p) \) is periodic with respect to \( \frac{2\pi}{\varepsilon_n} \mathbb{Z} \times \frac{2\pi}{\varepsilon_n} \mathbb{Z}^3 \), we may assume that \( \varepsilon_n \text{Re} p_0 \) and each \( \varepsilon_n \text{Re} \mathbf{p}_\nu \), \( 1 \leq \nu \leq 3 \) is bounded in magnitude by \( \pi \). By Lemma A.1.b, \( \frac{\sin z}{z} \leq 2 \) for all \( z \in \mathbb{C} \) with \( |\text{Im} z| \leq 1 \). Since \( \varepsilon_n \mathbf{p} \) runs over a compact set (independently of \( n \) and \( L \)), \( e^{-\hat{h}(\varepsilon_n \mathbf{p})} \) is bounded. So
\[
\left| \frac{1}{2} \varepsilon_n^2 p_0^2 e^{-h_0(\varepsilon_n \mathbf{p})} \left[ \sin \frac{\varepsilon_n^2 p_0}{2} \right]^2 \right| \leq \text{const}|p_0|
\]
and
\[
\left| \frac{1 - e^{-h_0(\varepsilon_n \mathbf{p})}}{\varepsilon_n^2} \right| \leq \text{const}\frac{|\varepsilon_n \mathbf{p}|^2}{\varepsilon_n^4} \leq \text{const}|\mathbf{p}|^2
\]
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This gives the bound on $|\hat{D}_n(p)|$.

The bounds
\[
\frac{\partial}{\partial p_0} \varepsilon_n^2 \left[ \frac{\sin(\frac{1}{\varepsilon_n} p_0)}{\frac{1}{\varepsilon_n}} \right]^2 = 2 \sin(\varepsilon_n^2 p_0) = O(1) \quad \frac{\partial}{\partial p_0} \left[ \frac{\sin(\varepsilon_n^2 p_0)}{\varepsilon_n^2} \right] = \cos(\varepsilon_n^2 p_0) = O(1)
\]
\[
\frac{\partial^2}{\partial p_0^2} \varepsilon_n^2 \left[ \frac{\sin(\frac{1}{\varepsilon_n} p_0)}{\frac{1}{\varepsilon_n}} \right]^2 = 2 \varepsilon_n^2 \cos(\varepsilon_n^2 p_0) = O(\varepsilon_n^2) \quad \frac{\partial^2}{\partial p_0^2} \left[ \frac{\sin(\varepsilon_n^2 p_0)}{\varepsilon_n^2} \right] = -\varepsilon_n^2 \sin(\varepsilon_n^2 p_0) = O(\varepsilon_n^2)
\]
\[
\frac{\partial}{\partial p_\nu} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(|p|) \quad \frac{\partial^2}{\partial p_\nu^2} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(1 + \varepsilon_n^2 |p|^2) = O(1)
\]
\[
\frac{\partial^3}{\partial p_\nu^3} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(\varepsilon_n^2 |p| + \varepsilon_n^4 |p|^3)
\]
\[
\frac{\partial^4}{\partial p_\nu^4} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(\varepsilon_n^2 + \varepsilon_n^4 |p|^2 + \varepsilon_n^6 |p|^4)
\]

together with
\[
\varepsilon_n^2 \leq \frac{\text{const}}{1 + |p_0| + |p|^2} \quad \varepsilon_n^2 |p_0| \leq \text{const}
\]
yield the bounds on the derivatives.

(d) Write $p = p + iQ$ with $p = (p_0, p)$, $Q = (Q_0, Q) \in \mathbb{R} \times \mathbb{R}^3$. We may choose $\bar{m}(c)$ sufficiently small that, if $|p + iQ| \geq c$ and $|Q| \leq \bar{m}(c)$, then
\[
|p_0 + iQ_0| + \sum_{\nu=1}^3 |P_\nu + iQ_\nu|^2 \\
\leq \left( |p_0| + \sum_{\nu=1}^3 |P_\nu|^2 \right) + \left( |Q_0| + \sum_{\nu=1}^3 |Q_\nu|^2 \right) \\
\leq 2 \left( |p_0| + \sum_{\nu=1}^3 |P_\nu|^2 \right)
\]

If $\gamma_1 > 0$ is chosen small enough and $\Gamma_5$ is chosen large enough, then, for all such $p, Q$, we have, by parts (a) and (c),
\[
|\hat{D}_n(p)| \geq 4\gamma_1 (|p_0| + \sum_{\nu=1}^3 |P_\nu|^2) \\
\geq 2\gamma_1 (|p_0 + iQ_0| + \sum_{\nu=1}^3 |P_\nu + iQ_\nu|^2) \\
|\hat{D}_n(p + iQ) - \hat{D}_n(p)| \leq 2\Gamma_5 \left( 1 + |p_0 + iQ_0| + \sum_{\nu=1}^3 |P_\nu + iQ_\nu|^2 \right) |Q|
\]

Recalling that $|p_0 + iQ_0| + \sum_{\nu=1}^3 |P_\nu + iQ_\nu|^2 \geq \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon'}{4} \right\}$, it now suffices to choose $\bar{m}(c)$ small enough that
\[
2\Gamma_5 \bar{m}(c) \leq \frac{1}{2} \gamma_1 \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon'}{4} \right\} \quad \text{and} \quad 2\Gamma_5 \bar{m}(c) \leq \frac{1}{2} \gamma_1
\]

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(e) Again write $p = p + i q$ with $p = (p_0, p)$, $q = (q_0, q) \in \mathbb{R} \times \mathbb{R}^3$. Since $\hat{D}_n(p + i q)$ is periodic with respect to $p \in \frac{2\pi}{\varepsilon_n} \mathbb{Z} \times \frac{2\pi}{\varepsilon_n} \mathbb{Z}^3$, we may assume that $|\varepsilon_n p_0| \leq \pi$ and $|\varepsilon_n p_\nu| \leq \pi$, for each $1 \leq \nu \leq 3$. If the constant $\gamma_1$ was chosen small enough, then, as in part (a),

$$\text{Re} \hat{D}_n(p) = \frac{1}{2} \varepsilon_n^2 p_0^2 \hat{e} - h_0(\varepsilon_n p) \left[ \sin \frac{1}{2} \varepsilon_n^2 p_0 \right]^2 + p^2 \frac{1 - e^{-\hat{h}_0(\varepsilon_n p)}}{\varepsilon_n^2 p^2} \geq 2 \gamma_1 (\varepsilon_n^2 p_0^2 + |p|^2)$$

Hence it suffices to prove that it is possible to choose $\tilde{m} = \tilde{m}(c)$ so that

$$\left| \text{Re} \hat{D}_n(p + i q) - \text{Re} \hat{D}_n(p) \right| \leq \gamma_1 |p|^2 \quad \text{when } |p| \geq c \text{ and } |q| \leq \tilde{m} \quad (3.2)$$

and that

$$\left| \text{Re} \hat{D}_n(p + i q) - \text{Re} \hat{D}_n(p) \right| \leq \gamma_1 c^2 \quad \text{when } |p| \leq c, |\varepsilon_n p_0| \geq c \text{ and } |q| \leq \tilde{m} \quad (3.3)$$

This is a consequence of the following bounds on the derivatives of the real parts of the three terms making up $\hat{D}_n(p + i q)$ in Remark 3.1.a. For the first term,

$$\left| \frac{d}{dt} \left[ \varepsilon_n^2 (p_0 + it q_0)^2 e^{-\hat{h}_0(\varepsilon_n p)} \right] \right| \leq \text{const} \left[ \varepsilon_n^2 |p_0 + i t q_0| |q_0| + \varepsilon_n^2 |p_0 + it q_0|^2 \varepsilon_n^2 |q_0| \right] \leq \text{const} \tilde{m}$$

$$\left| \frac{d^2}{dt^2} \varepsilon_n^2 (p_0 + it q_0)^2 \left[ e^{-\hat{h}_0(\varepsilon_n p + it e_n q)} - e^{-\hat{h}_0(\varepsilon_n p)} \right] \left[ \sin \frac{1}{2} \varepsilon_n^2 (p_0 + it q_0) \right]^2 \right|_{t=0} = 0$$

$$\left| \frac{d^2}{dt^2} \varepsilon_n^2 (p_0 + it q_0)^2 \left[ e^{-\hat{h}_0(\varepsilon_n p + it e_n q)} - e^{-\hat{h}_0(\varepsilon_n p)} \right] \left[ \sin \frac{1}{2} \varepsilon_n^2 (p_0 + it q_0) \right]^2 \right| \leq \text{const} \left[ \varepsilon_n^2 |q_0|^2 + \varepsilon_n^2 |p_0 + i t q_0| |q_0| (|\varepsilon_n| |q_0| + \varepsilon_n^2 |q_0|) + \varepsilon_n^2 |p_0 + i t q_0|^2 (|\varepsilon_n| |q_0| + \varepsilon_n^2 |q_0|) \right] \leq \text{const} \tilde{m}^2$$

For the second term,

$$\left| \frac{d}{dt} \text{Re} \frac{1 - e^{-\hat{h}_0(\varepsilon_n p + it e_n q)}}{\varepsilon_n^2} \right|_{t=0} = 0$$

$$\left| \frac{d^2}{dt^2} \frac{1 - e^{-\hat{h}_0(\varepsilon_n p + it e_n q)}}{\varepsilon_n^2} \right| \leq \text{const} \frac{1}{\varepsilon_n^2} (|\varepsilon_n| |q_0|)^2 \leq \text{const} \tilde{m}^2$$

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For the third term,

\[
\left| \frac{d}{dt} \text{Re} i(p_0 + itQ_0) e^{-\hat{h}_0(\varepsilon_n\mathbf{P})} \frac{\sin \varepsilon_n^2(p_0 + itQ_0)}{\varepsilon_n^2(p_0 + itQ_0)} \right|
\leq \text{const} |Q_0| + \text{const} |p_0 + iQ_0| \varepsilon_n^2|Q_0|
\leq \text{const} \tilde{m}
\]

\[
\left| \frac{d}{dt} \text{Re} i(p_0 + itQ_0) \left[ e^{-\hat{h}_0(\varepsilon_n\mathbf{P} + it\varepsilon_n\mathbf{Q})} - e^{-\hat{h}_0(\varepsilon_n\mathbf{P})} \frac{\sin \varepsilon_n^2(p_0 + itQ_0)}{\varepsilon_n^2(p_0 + itQ_0)} \right]_{t=0} \right|
\leq \text{const} \varepsilon_n^2 |p_0| |\mathbf{P}| \tilde{m}
\leq \text{const} |\mathbf{P}| \tilde{m}
\]

\[
\left| \frac{d^2}{dt^2} i(p_0 + itQ_0) \left[ e^{-\hat{h}_0(\varepsilon_n\mathbf{P} + it\varepsilon_n\mathbf{Q})} - e^{-\hat{h}_0(\varepsilon_n\mathbf{P})} \frac{\sin \varepsilon_n^2(p_0 + itQ_0)}{\varepsilon_n^2(p_0 + itQ_0)} \right] \right|
\leq \text{const} \left[ |Q_0| (\varepsilon_n |\mathbf{Q}| + \varepsilon_n^2 |Q_0|) + |p_0 + iQ_0| (\varepsilon_n |\mathbf{Q}| + \varepsilon_n^2 |Q_0|)^2 \right]
\leq \text{const} \tilde{m}^2
\]

Now choose \( \tilde{m} = \tilde{m}(c) \) small enough that (3.2) and (3.3) are satisfied. \( \square \)
4 The Covariance

The covariance for the fluctuation integral in [8] is

\[ C^{(n)} = (\frac{a}{L^2} Q^* Q + \Delta^{(n)})^{-1} \]

where

\[ \Delta^{(n)} = \begin{cases} (1 + \Omega_n Q_n D^{-1} Q_{n}^*)^{-1} \Omega_n & \text{if } n \geq 1 \\ D_0 & \text{if } n = 0 \end{cases} : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)} \]

See [7, (1.15) and (1.14)]. In Lemma 4.2 we study the properties of \( \Delta^{(n)} \) and in Corollary 4.5 we study the properties of \( C^{(n)} \) and its square root.

Remark 4.1. Let \( n \geq 1 \).

(a) The operator \( \Delta^{(n)} \) is the periodization of a translation invariant operator \( \Delta^{(n)} \), acting on \( L^2(\mathbb{Z} \times \mathbb{Z}^3) \), whose Fourier transform is

\[ \hat{\Delta}^{(n)}(k) = \hat{\Omega}_n(k)
\left( 1 + \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell) 2^q \hat{D}_n^{-1}(k + \ell) \right)^{-1} \]

with \( k \in \mathbb{C} \times \mathbb{C}^3 \), where \( u_n(p) \) and \( \hat{B}_n \) were defined in parts (b) and (e) of Remark 2.1, respectively.

(b) \( \hat{\Delta}^{(n)}(k) \) is invariant under \( k_\nu \to -k_\nu \) for each \( 1 \leq \nu \leq 3 \).

(c) \( \hat{\Delta}^{(n)}(k) \) has nonnegative real part when \( k \) is real.

Lemma 4.2. There are constants \( m_1 > 0, \gamma_2, \Gamma_6 \) and \( \Gamma_7 \).

Internal Remark

They depend only on \( q, h_0 \) and \( a \), and in particular are independent of \( n \) and \( L > \Gamma_2 \) (where \( \Gamma_2 \) is the constant of Proposition 2.4).
such that, for $L > \Gamma_2$, the following hold.

(a) $\hat{\Delta}^{(n)}(k)$ is analytic on $|\text{Im } k| < 3m_1$.

(b) For all $k \in \mathbb{C} \times \mathbb{C}^3$ with $|\text{Im } k| \leq 3m_1$.

$$\hat{\Delta}^{(n)}(k) = -ik_0 + \left(\frac{1}{a_n} + \frac{\varepsilon^2}{2}\right)k_0^2 + \frac{1}{2} \sum_{\nu, \nu' = 1}^3 H_{\nu, \nu'}k_\nu k_{\nu'} + O(|k|^3)$$

$\hat{\Delta}^{(n)}(k) \hat{D}^{-1}_n(k) = 1 + \frac{ik_0}{a_n} + O(|k|^2)$

The higher order part $O(\cdot)$ is uniform in $n$ and $L$.

(c) $|\hat{\Delta}^{(n)}(k)| \leq 2a$ and $|\frac{\partial}{\partial k_\nu} \hat{\Delta}^{(n)}(k)|, |\frac{\partial^2}{\partial k_\nu \partial k_{\nu'}} \hat{\Delta}^{(n)}(k)| \leq \Gamma_7$ for all $0 \leq \nu, \nu' \leq 3$ and $k \in \mathbb{C} \times \mathbb{C}^3$ with $|\text{Im } k| < 3m_1$.

(d) There is a function $\rho(c) > 0$, which is defined for all $c > 0$ and which depends only on $m_1, q, \hat{h}_0$ and $a$ and, in particular, is independent of $n$ and $L$, such that

$$\text{Re } \hat{\Delta}^{(n)}(k) \geq \rho(c)$$

for all $k \in \mathbb{C} \times \mathbb{C}^3$ with $|k| \geq c$ and $|\text{Im } k| \leq 3m_1$.

(e) For all $k \in \mathbb{C} \times \mathbb{C}^3$ with $|\text{Im } k| \leq 3m_1$ and $|\text{Re } k_\nu| \leq \pi$ for all $0 \leq \nu \leq 3$,

$$|\hat{\Delta}^{(n)}(k)| \geq \gamma_2 |\hat{D}_n(k)|$$

(f) $\hat{D}^{-1}_n(p) \hat{\Delta}^{(n)}(p)$ is analytic on $|\text{Im } p| \leq 3m_1$. Furthermore, for all $p \in \mathbb{C} \times \mathbb{C}^3$ with $|\text{Im } p| \leq 3m_1$,

$$|\hat{D}^{-1}_n(p) \hat{\Delta}^{(n)}(p)| \leq \frac{\Gamma_6}{1 + |p_0| + \sum_{\nu = 1}^3 |p_{\nu}|^2}$$

Here, as usual, $|p_0|$ and $|p_{\nu}|$ refer to the magnitudes of the smallest representatives of $p_0 \in \mathbb{C}$ and $p_{\nu} \in \mathbb{C}$ in $\mathbb{C}/2\pi\mathbb{Z}$ and $\mathbb{C}/2\pi\mathbb{Z}$ respectively.

(g) $|\frac{\partial}{\partial k_\nu} \hat{\Delta}^{(n)}(k)| \leq \Gamma_7 |k_\nu|$ for all $1 \leq \nu \leq 3$ and $k \in \mathbb{C} \times \mathbb{C}^3$ with $|\text{Im } k| < 3m_1$.

Proof. We first prove part (b). Using that

- $u_n(k)^{2q} = 1 + O(|k|^2)$ by Lemma 2.2.b
\[
\Delta^{(n)}(k) = \frac{\hat{\Omega}_n(k) \hat{D}_n(k)}{\Omega_n(k) u_n(k)^{2q} + \hat{D}_n(k) + O(|k|^2)} \\
= \hat{D}_n(k) \frac{a_n + O(|k|^2)}{a_n + \hat{D}_n(k) + O(|k|^2)} \\
= \hat{D}_n(k) \left( 1 - \frac{1}{a_n} \hat{D}_n(k) + O(|k|^2) \right) \\
= -ik_0 + \frac{1}{2} \hat{D}_n(k)^2 + \frac{3}{2} \sum_{\nu,\nu'} H_{\nu,\nu'} k_{\nu} k_{\nu'} - \frac{1}{a_n} \hat{\Delta}_n(k)^2 + O(|k|^3) \\
= -ik_0 + \left( \frac{1}{a_n} + \frac{3}{2} \right) k_0 + \frac{3}{2} \sum_{\nu,\nu'} H_{\nu,\nu'} k_{\nu} k_{\nu'} + O(|k|^3)
\]

This also shows that, in a neighbourhood of the origin, \(\hat{\Delta}^{(n)}(k)\) is analytic and bounded in magnitude by \(2a\). For the second expansion,

\[
\hat{\Delta}^{(n)}(k) \hat{D}_n^{-1}(k) = \frac{a_n + O(|k|^2)}{a_n + \hat{D}_n(k) + O(|k|^2)} = 1 - \frac{1}{a_n} \hat{D}_n(k) + O(|k|^2) \\
= 1 + \frac{i k_0}{a_n} + O(|k|^2)
\]

(a), (c) We first prove the analyticity and the bound \(|\hat{\Delta}^{(n)}(k)| \leq 2a\) of part (c). We have already done so for a neighbourhood of the origin. So it suffices to consider \(|k| > c_0\), for some suitably small \(c_0\). Since \(\Delta^{(n)}\) is periodic with respect to \(2\pi \mathbb{Z} \times 2\pi \mathbb{Z}^3\), it suffices to consider \(k\) in the set

\[
M(m_1) = \left\{ k \in \mathbb{C} \times \mathbb{C}^3 \mid \text{Re } k_\nu \leq \pi \text{ for all } 0 \leq \nu \leq 3, \text{Im } k \leq 3m_1, \ |k| > c_0 \right\}
\]

Recall, from Remarks 2.1.d and 3.1.b, that \(u_n(p)\) and \(\hat{D}_n(p)\) are entire. If \(m_1\) is small enough, then, by Lemma 3.2.d, the functions \(k \mapsto \hat{D}_n(k + \ell), \ell \in \mathcal{B}_n, \ n \geq 1\), have no zeroes in \(M(m_1)\). Hence each term in the infinite sum

\[
1 + \hat{\Omega}_n(k) \sum_{\ell \in \mathcal{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell)
\]
is analytic and it suffices to prove that, for $k \in M(m_1)$, the sum converges uniformly
and $\left| 1 + \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right| \geq \frac{6}{10}$.

By Proposition 2.4.a, Remark 2.1.d and Lemma 3.2.d, there is an $l > 0$ such that

$$|\hat{Q}_n(k)| \sum_{\ell \in \hat{B}_n, |\ell| \geq l} |u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell)| \leq \frac{5}{4} a \sum_{\ell \in \hat{B}_n, |\ell| \geq l} \frac{3}{\nu^2 \pi} \prod_{\nu=0}^{24} \frac{3}{(\nu^2 + \pi^2)^{2q}} \leq \frac{\text{const}}{l} \leq \frac{2}{10} \quad (4.1)$$

Hence we have uniform convergence and

$$\left| 1 + \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^2 \hat{D}_n^{-1}(k + \ell) \right| \geq \left| 1 + \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n, |\ell| < l} u_n(k + \ell)^2 \hat{D}_n^{-1}(k + \ell) \right| - \frac{2}{10}$$

For real $k$ and every $\ell \in \hat{B}_n$, the real part of $\hat{Q}_n(k) u_n(k + \ell)^2 \hat{D}_n^{-1}(k + \ell)$ is nonnegative. Consequently, if $m_1$ is small enough and $|\text{Im} k| \leq 3m_1$

$$\text{Re} \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n, |\ell| < l} u_n(k + \ell)^2 \hat{D}_n^{-1}(k + \ell) \geq -\frac{2}{10} \quad (4.2)$$

since, for all $k \in M(m_1)$ and $\ell \in \hat{B}_n$ with $|\ell| < l$,

- $\hat{D}_n(k + \ell)$ is bounded away from zero (by Lemma 3.2.d) and has bounded first derivative (by Lemma 3.2.c) and
- $u_n(k + \ell)$ and $\hat{Q}_n(k)$ are bounded with bounded first derivatives (by analyticity, Remark 2.1.d and Proposition 2.4.a).

So, by (4.2),

$$\left| 1 + \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n, |\ell| < l} u_n(k + \ell)^2 \hat{D}_n^{-1}(k + \ell) \right| \geq \frac{8}{10}$$

All of this is uniform in $n$ and $L$.

Shrinking $m_1$ by a factor of 2, the bounds $|\frac{\partial}{\partial k} \hat{A}^{(n)}(k)|, |\frac{\partial^2}{\partial k \partial k^*} \hat{A}^{(n)}(k)| \leq \Gamma_7$, with $

\Gamma_7$ being the maximum of $4a$ divided by the original $3m_1$ (for first order derivatives) and $16a$ divided by the square of the original $3m_1$ (for second order derivatives), follow by the Cauchy integral formula.

(d) Denote the real and imaginary parts of the numerator and denominator by

$$R_n = \text{Re} \hat{Q}_n(k) R_d = \text{Re} \left[ 1 + \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right]$$

$$I_n = \text{Im} \hat{Q}_n(k) I_d = \text{Im} \left[ 1 + \hat{Q}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right]$$

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By Proposition 2.4.a and (4.1), (4.2),
\[ R_n \geq \frac{a}{2}, \quad R_d \geq \frac{6}{10} \]
By Proposition 2.4.a, Remark 2.1.d and Lemma 3.2.d,
\[ I_d \leq \left| \hat{\Omega}_n(k) \sum_{\ell \in \mathcal{B}_n} u_n(k + \ell) 2^{q} \hat{D}_n^{-1}(k + \ell) \right| \leq a \tilde{I}_d(c) \]
where \( \tilde{I}_d(c) = \frac{6}{5} a \sum_{\ell \in \mathcal{B}_n} \frac{1}{\gamma_1 \min\{e/2, e^2/4\}} \prod_{\nu = 0}^{3} \frac{\nu^2 2^{q}}{(|\nu| + \pi)^2} \). When \( k \) is real, \( \hat{\Omega}_n(k) \) is real.
By analyticity and Proposition 2.4.a, the first order derivatives of \( \hat{\Omega}_n(k) \) are bounded. So if \( m_1 \) is small enough, \( |I_n| \leq \frac{2}{10 \tilde{I}_d(c)} \). So
\[ \operatorname{Re} \hat{\Delta}^{(n)}(k) = \frac{R_n R_d + I_n I_d}{R_d + I_d^2} \geq \frac{a(3/10) - a(2/10)}{(1 + a \tilde{I}_d(c))^2} = \frac{a}{10(1 + a \tilde{I}_d(c))^2} \]
(c) By Remark 2.1.d, Proposition 2.4.a and Lemma 3.2.d,
\[ \left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in \mathcal{B}_n} u_n(k + \ell) 2^{q} \hat{D}_n^{-1}(k + \ell) \right| \]
\[ \leq \left| \hat{\Omega}_n(k) u_n(k) 2^{q} \hat{D}_n^{-1}(k) \right| + 1 + \sum_{\ell \notin \mathcal{B}_n} \left| \hat{\Omega}_n(k) u_n(k + \ell) 2^{q} \hat{D}_n^{-1}(k + \ell) \right| \]
\[ \leq \frac{6}{5} a \left( \frac{24}{\pi} \right)^{8q} \left| \hat{D}_n^{-1}(k) \right| + 1 + \frac{6}{5} a \sum_{\ell \notin \mathcal{B}_n} \frac{1}{\gamma_1 \pi} \prod_{\nu = 0}^{3} \left( \frac{\nu^2 2^{q}}{|\nu| + \pi} \right)^2 \]
and so, by Lemma 3.2.c,
\[ \left| \hat{D}_n(k) \right| \left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in \mathcal{B}_n} u_n(k + \ell) 2^{q} \hat{D}_n^{-1}(k + \ell) \right| \]
\[ \leq \frac{6}{5} a \left( \frac{24}{\pi} \right)^{8q} + \Gamma_5 \left( \pi + 1 + 3(\pi + 1)^2 \right) \left[ 1 + \frac{6}{5} a \sum_{\ell \notin \mathcal{B}_n} \frac{1}{\gamma_1 \pi} \prod_{\nu = 0}^{3} \left( \frac{\nu^2 2^{q}}{|\nu| + \pi} \right)^2 \right] \]
(f) By part (c) of this lemma, Remark 3.1.b and Lemma 3.2.d, it suffices to consider \( p = k \) with \( |\operatorname{Im} k| \leq 3m_1 \) and \( |k| < c_0 < \pi \) where \( c_0 \) is sufficiently small. Now
\[ \hat{D}_n^{-1}(k) \hat{\Delta}^{(n)}(k) \]
\[ = \frac{\hat{\Omega}_n(k)}{\hat{\Omega}_n(k) u_n(k) 2^{q} \hat{D}_n(k) \left[ 1 + \hat{\Omega}_n(k) \sum_{\ell \notin \mathcal{B}_n} u_n(k + \ell) 2^{q} \hat{D}_n^{-1}(k + \ell) \right]} \]
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By Proposition 2.4.a, Remark 2.1.d, Lemma 2.2.b and parts (c) and (d) of Lemma 3.2, we may choose $c_0 > 0$ so that
\[ |\hat{D}_n(k)\left[1 + \hat{Q}_n(k) \sum_{\ell \neq n} u_n(k + \ell)^2 q \hat{D}_n^{-1}(k + \ell)\right]| < \frac{a}{5} \]
and
\[ |\hat{Q}_n(k)| u_n(k)^2 - 1| < \frac{a}{5} \]
for all $|k| < c_0$ with $|\text{Im } k| \leq 3 m_1$. That does it.

(g) Fix any $1 \leq \nu \leq 3$. Observe that
\[ \frac{\partial^2 \Delta^{(n)}}{\partial k_{\nu}^2}(k(t)) dt \quad \text{with} \quad k(t)_{\nu'} = \begin{cases} k_{\nu'} & \text{if } \nu' \neq \nu \\ tk_{\nu} & \text{if } \nu = \nu' \end{cases} \]
since $\frac{\partial \Delta^{(n)}}{\partial k_{\nu}}(k(0)) = 0$, by Remark 4.1.b. Now apply the bound on the second derivative from part (c).

We next consider the “resolvents”
\[ R^{(n)}(\zeta) = (\zeta I - \frac{\partial}{\partial k_{\nu}} Q^* Q - \Delta^{(n)})^{-1} \]
in preparation for studying $C_n$ and $\sqrt{C_n}$. We shall use [5, Lemma 14], with
\begin{align*}
\varepsilon_T &= 1 \\
\varepsilon_X &= 1 \\
L_T &= L^2 \\
L_X &= L \\
\mathcal{L}_T &= L_{tp} \\
\mathcal{L}_X &= L_{sp}
\end{align*}
and
\begin{align*}
\mathcal{X}_\text{fin} &= \mathcal{X}^{(n)}_0 \\
\mathcal{Z}_\text{fin} &= \mathbb{Z} \times \mathbb{Z}^3 \\
\mathcal{X}_{\text{crs}} &= \mathcal{X}^{(n+1)}_{-1} \\
\mathcal{Z}_{\text{crs}} &= L^2 \mathbb{Z} \times L\mathbb{Z}^3 \\
\mathcal{B} &= \mathcal{B}^+
\end{align*}

Internal Remark

<table>
<thead>
<tr>
<th>$\mathcal{X}_\text{fin}$</th>
<th>$(\mathbb{Z}/\varepsilon_n^2 L_{tp} \mathbb{Z}) \times (\mathbb{Z}^3 / \varepsilon_n L_{sp} \mathbb{Z}^3)$</th>
<th>$\hat{\mathcal{X}}_\text{fin}$</th>
<th>$(\frac{2\pi}{\varepsilon_n L_{tp}} \mathbb{Z}/2\pi \mathbb{Z}) \times (\frac{2\pi}{\varepsilon_n L_{sp}} \mathbb{Z}^3/2\pi \mathbb{Z}^3)$</th>
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<tbody>
<tr>
<td>$\mathcal{Z}_\text{fin}$</td>
<td>$\mathbb{Z} \times \mathbb{Z}^3$</td>
<td>$\hat{\mathcal{Z}}_\text{fin}$</td>
<td>$(\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$</td>
</tr>
<tr>
<td>$\mathcal{X}_{\text{crs}}$</td>
<td>$(L^2 \mathbb{Z}/\varepsilon_n^2 L_{tp} \mathbb{Z}) \times (L \mathbb{Z}^3 / \varepsilon_n L_{sp} \mathbb{Z}^3)$</td>
<td>$\hat{\mathcal{X}}_{\text{crs}}$</td>
<td>$(\frac{2\pi}{\varepsilon_n L_{tp}} \mathbb{Z}/2\pi \mathbb{Z}) \times (\frac{2\pi}{\varepsilon_n L_{sp}} \mathbb{Z}^3/2\pi \mathbb{Z}^3)$</td>
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<tr>
<td>$\mathcal{B}$</td>
<td>$(\mathbb{Z}/L^2 \mathbb{Z}) \times (\mathbb{Z}^3/L \mathbb{Z}^3)$</td>
<td>$\hat{\mathcal{B}}$</td>
<td>$(\frac{2\pi}{L} \mathbb{Z}/2\pi \mathbb{Z}) \times (\frac{2\pi}{L} \mathbb{Z}^3/2\pi \mathbb{Z}^3)$</td>
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</table>
We wish to apply [5, Lemma 14], with \( A \to D_n = \frac{a}{L^2} Q^* Q + \Delta^{(n)} \) (scaled). By [5, Lemmas 9.b and 5.b], for each \( \ell, \ell' \in \bar{\mathcal{B}} = \bar{\mathcal{B}}^+ \),

\[
a_q(\ell, \ell') \to d_{n,t}(\ell, \ell') = \frac{a}{L^2} u_+ (\ell + \ell')^a u_+ (\ell + \ell')^a + \delta_{\ell,\ell'} \hat{\Delta}^{(n)} (\ell + \ell) \tag{4.3}
\]

Here we are denoting

- momenta dual to the \( L \)-lattice by \( \ell \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3) \)
- momenta dual to the unit lattice \( \mathbb{Z} \times \mathbb{Z}^3 \) by \( k \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3) \)
- \( d \) decompose \( k = \ell + \ell' \)
- \( d \) with \( \ell, \ell' \) in a fundamental cell for \( (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3) \)

We also use \( d_{n,t} \) to denote the \( \bar{\mathcal{B}}^+ \times \bar{\mathcal{B}}^+ \) matrix \([d_{n,t}(\ell, \ell')]_{\ell,\ell' \in \bar{\mathcal{B}}^+} \). Observe, by Remark 2.1.d and Lemma 4.2.a, that \( d_{n,t}(\ell, \ell') \) is analytic in the strip \(|\text{Im } \ell| < 3m_1|\).

Let \([v_\ell]_{\ell \in \bar{\mathcal{B}}^+}\) and \([w_\ell]_{\ell \in \bar{\mathcal{B}}^+}\) be any vectors in \( L^2(\bar{\mathcal{B}}^+) \). Then, if \( \ell = L^{-1}(k) \),

\[
\langle \hat{v}, d_{n,t} w \rangle = a \left| \sum_{\ell \in \bar{\mathcal{B}}^+} u_+(L^{-1}(k)+\ell)^a \hat{v}_\ell \right| \left| \sum_{\ell \in \bar{\mathcal{B}}^+} u_+(L^{-1}(k)+\ell)^a \hat{w}_\ell \right| + \sum_{\ell \in \bar{\mathcal{B}}^+} \hat{\Delta}^{(n)} (L^{-1}(k)+\ell) \hat{v}_\ell \hat{w}_\ell \tag*{(4.3)}
\]

where the “scaled” matrix

\[
d^{(s)}_{n,k}(\ell, \ell') = \frac{a}{L^2} u_+ (L^{-1}(k+\ell))^a u_+ (L^{-1}(k+\ell'))^a + \delta_{\ell,\ell'} \hat{\Delta}^{(n)} (L^{-1}(k+\ell))
\]

**Lemma 4.3.** There are constants \( m_2, \lambda_0, \Gamma_8 > 0 \).

---

**Internal Remark**

They depend only on \( q \geq 2, \hat{h}_0 \) and \( a \), and in particular are independent of \( n \) and \( L > \Gamma_2 \).

such that, for all \( L > \Gamma_2 \) and \( k \in \mathbb{C} \times \mathbb{C}^3 \) with \(|\text{Im } k| < 3m_2|\), the following hold.

(a) Write \( \ell = L^{-1}(k) \). For both the operator and (matrix) \( \ell^1 - \ell^\infty \) norms

\[
\|d_{n,t}\| \leq \Gamma_8 \quad \|d^{(s)}_{n,k}\| \leq \Gamma_8
\]

(b) Let \( \lambda \in \mathbb{C} \) be within a distance \( \frac{\lambda}{L^2} \) of the negative real axis. Then the resolvent

\[
\left\| (\lambda I - d^{(s)}_{n,k})^{-1} \right\| \leq \Gamma_8 L^2
\]

This is true for both the operator and (matrix) \( \ell^1 - \ell^\infty \) norms.
Proof. Since \(d_{n+1}(\ell, \ell') = d_{n}(\ell + p, \ell' + p)\), for all \(p, \ell, \ell' \in \hat{B}^+\), we may always assume that \(|\text{Re} \, \ell_0| \leq \frac{\pi}{2}\) and \(|\text{Re} \, \ell_\nu| \leq \frac{\pi}{2}\) for \(1 \leq \nu \leq 3\).

(a) By Lemmas 2.3.a and 4.2.c, the matrices
\[
\left[ \frac{\alpha}{2\pi} u_+(\ell + \nu)^q u_+(\ell + \nu')^q \right]_{\ell, \ell' \in \hat{B}^+} \quad \text{and} \quad \left[ \frac{\alpha}{2\pi} \left(\mathbb{L}^{-1}(k + \ell)\right)^q u_+(\mathbb{L}^{-1}(k + \ell'))^q \right]_{\ell, \ell' \in \hat{B}^+}
\]
have finite \(L^1-L^\infty\) norms and the matrix elements of
\[
\delta_{\ell, \ell'} \hat{\Delta}^{(n)}(\ell + \nu) \quad \text{and} \quad [\delta_{\ell, \ell'} \hat{\Delta}^{(n)}(\mathbb{L}^{-1}(k + \ell))]_{\ell, \ell' \in \hat{B}^+}
\]
am all bounded.

(b) Case \(|k| \leq c_0\), with \(c_0\) being chosen later in this case: We first consider those diagonal matrix elements of \(d_{n,k}^{(e)}(\ell, \ell')\) having \(k, \ell\) such that \(|\mathbb{L}^{-1}(k + \ell)| < \tilde{c}_0\), where \(\tilde{c}_0 > 0\) is a small number to be chosen shortly. This, and all other constants chosen in the course of this argument are to be independent of \(L\). Then, by Lemma 4.2.b, we have the following.

- If at least one of \(\ell_\nu, 1 \leq \nu \leq 3\) is nonzero, then \(\text{Re} \hat{\Delta}^{(m)}(\mathbb{L}^{-1}(k + \ell)) \geq \frac{\pi}{2} \), provided \(m_2\) and \(\tilde{c}_0\) are chosen small enough. Here \(c_1\) and the constraints on \(m_2\) and \(\tilde{c}_0\) depend only on the largest and smallest eigenvalues of \([H_{\nu, \nu'}]\), \(a_n\) and the \(O(|k|^3)\).

To see this, denote by \(k\) and \(\ell\) the spatial parts of \(k\) and \(\ell\) and observe that
\[
\begin{align*}
\text{Re} \, \mathbb{L}^{-1}(k + \ell) &\geq \frac{\pi}{2} \max \left\{ \pi, \frac{1}{2} |\ell| \right\}, \\
\text{Im} \, \mathbb{L}^{-1}(k + \ell) &\leq \frac{1}{2} |\text{Im} k| \leq \frac{3m_2}{2} \quad \text{and} \\
\text{Im} \, \mathbb{L}^{-1}(k + \ell)_0 &\leq \frac{|\text{Im} k_0|}{2} \leq \frac{3m_2}{2}.
\end{align*}
\]

In controlling the contribution from \(O\left(|\mathbb{L}^{-1}(k + \ell)|^3\right)\) when \(|c_0|\) is larger than \(\ell_L^2\), we have to use that, in this case, the real part of \(\left(\ell_1 + \frac{\alpha^2}{2}\right)^3 \mathbb{L}^{-1}(k + \ell)_2\) is at least a strictly positive constant times \(\ell_L^2\).

---

**Internal Remark**

Note that
\[
\begin{align*}
\text{Re} \sum_{\nu, \nu' = 1}^{3} H_{\nu, \nu'}(\mathbb{L}^{-1}(k + \ell))_{\nu} (\mathbb{L}^{-1}(k + \ell))_{\nu'} &\geq C_H |\mathbb{L}^{-1}(k + \ell)|^2 \\
|\mathbb{L}^{-1}(k + \ell)| &\leq \tilde{c}_0 \\
|\mathbb{L}^{-1}(k + \ell)| &\geq \frac{1}{4} L
\end{align*}
\]

So if at least two of \(\nu_1, \nu_2, \nu_3\) are not zero
\[
\left| (\mathbb{L}^{-1}(k + \ell))_{\nu_1} (\mathbb{L}^{-1}(k + \ell))_{\nu_2} (\mathbb{L}^{-1}(k + \ell))_{\nu_3} \right| \leq \text{const} \tilde{c}_0 \text{Re} \sum_{\nu, \nu' = 1}^{3} H_{\nu, \nu'}(\mathbb{L}^{-1}(k + \ell))_{\nu} (\mathbb{L}^{-1}(k + \ell))_{\nu'}.
\]

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The same is true even when fewer than two of \( \nu_1, \nu_2, \nu_3 \) are spatial if \( \frac{|\ell_0|}{L^2} \leq \frac{q}{L} \). If two or three of the \( \nu_i \)'s are zero and \( \frac{|\ell_0|}{L^2} \geq \frac{q}{L} \), we use

- \( \text{Re} \left( \frac{1}{a_n} + \frac{\epsilon^2}{2} \right) \left( L^{-1}(k + \ell) \right)_0^2 \geq \frac{1}{100 a_n} \frac{\epsilon^2}{L^2} \)

so that

\[
\left| (L^{-1}(k + \ell))_{\nu_1} (L^{-1}(k + \ell))_{\nu_2} (L^{-1}(k + \ell))_{\nu_3} \right| \leq \text{const} \tilde{c}_0 \left( \frac{1}{a_n} + \frac{\epsilon^2}{2} \right) (L^{-1}(k + \ell))^2
\]

- \( \frac{T}{a} |u_+ (L^{-1}(k + \ell))^2| \leq \frac{T}{a} |k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi} \right]^{2q} \)

- If \( \ell = 0 \) then \( \text{Re} \{ \frac{T}{a} u_+ (L^{-1}(k))^2 q + \hat{\Delta}^{(n)} (L^{-1}(k)) \} \geq \frac{T}{a} q \), provided \( m_2 \) and \( \tilde{c}_0 \) are chosen small enough. To see this observe that

\[
\frac{\text{Re} \left( L^{-1}(k + \ell) \right)}{L^{1/2}} \leq \frac{\sqrt{3}}{L},
\]

\[
\frac{\text{Im} \left( L^{-1}(k + \ell) \right)}{L} \leq \frac{3m_2}{L},
\]

- \( \text{sgn} \ell_0 \text{Re} L^{-1}(k + \ell)_0 \geq \frac{1}{L^2} \max \{ \pi, \frac{1}{2} |\ell_0| \} \)

- \( \left| \text{Re} L^{-1}(k + \ell)_0 \right| \leq \frac{2T}{L} |\ell_0| \) and

- \( \left| \text{Im} L^{-1}(k + \ell)_0 \right| \leq \frac{3m_2}{L^2} \).

- If \( \ell \neq 0 \), then, by parts (a) and (b) of Lemma 2.3,

\[
\frac{T}{a} |u_+ (L^{-1}(k + \ell))^2 | \leq \frac{T}{a} |k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi} \right]^{2q}
\]

- If \( \ell = 0 \) then \( \text{Re} \{ \frac{T}{a} u_+ (L^{-1}(k))^2 q + \hat{\Delta}^{(n)} (L^{-1}(k)) \} \geq \frac{T}{a} q \), provided \( m_2 \) and \( \tilde{c}_0 \) are chosen small enough. To see this observe that

\[
|u_+ (L^{-1}(k)) - 1| \leq 4^3 |k|^2, \quad \text{by Lemma 2.3.c and } |u_+ (L^{-1}(k))| \leq \left( \frac{2L}{a} \right)^4 \text{ by Lemma 2.3.a.}
\]

- \( \text{Re} \hat{\Delta}^{(n)} (L^{-1}(k)) \geq -c_2 \left( \frac{m_2}{L} + \tilde{c}_0 \left( \frac{|k|}{L} \right)^2 \right) \) where \( c_2 \) depends only on \( a_n \), the largest eigenvalue of \( [H_{\nu,\nu}] \), and the \( O(|k|^3) \).

Note that we have now fixed \( \tilde{c}_0 \). Now we consider the remaining matrix elements.

- For the remaining diagonal matrix elements we have \( |L^{-1}(k + \ell)| \geq \tilde{c}_0 \) and then, by Lemma 4.2.d, \( \text{Re} \hat{\Delta}^{(n)} (L^{-1}(k + \ell)) \geq \rho(\tilde{c}_0) \)

- Finally, the off-diagonal matrix elements of \( d_{n,k}^{(s)} \) obey, by parts (a) and (b) of Lemma 2.3,

\[
\frac{T}{a} |u_+ (L^{-1}(k + \ell))^q u_+ (L^{-1}(k + \ell'))^q | \leq \frac{T}{a} |k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi} \right]^q \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu' + \pi|} \right]^q
\]

Hence the off-diagonal part of \( d_{n,k}^{(s)} \) has Hilbert-Schmidt, matrix and, as \( q > 1 \), \( L^1-L^\infty \) norms all bounded by a universal constant times \( \frac{T}{a} |k| \).
Thus $\lambda I - d_{n,k}^{(e)}$ has diagonal matrix elements of magnitude at least

$$\min \left\{ \frac{c_1}{L^2}, \frac{\pi}{2L^2}, \frac{a}{2L^2}, \rho(\tilde{c}_0) \right\} - \frac{a}{L^2} \left| k \right|^2 \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu|+\pi} \right]^{2q} - \frac{\lambda^2}{L^2}$$

and off diagonal part with $L^1-L^\infty$ norm bounded by a universal constant times $\frac{a}{L^2} |k|$. It now suffices to choose $c_0$ and $\lambda_0$ small enough that every diagonal matrix element has magnitude at least $\frac{1}{2L^2} \min \left\{ c_1, \frac{\pi}{2}, \frac{a}{2}, \rho(\tilde{c}_0) \right\}$ and the off diagonal part has $L^1-L^\infty$ norm bounded by $\frac{1}{4L^2} \min \left\{ c_1, \frac{\pi}{2}, \frac{a}{2}, \rho(\tilde{c}_0) \right\}$ and then do a Neumann expansion.

(b) Case $|k| \geq c_0$, with the $c_0$ just chosen: We may assume that $|\text{Re} k_\nu| \leq \pi$ for each $0 \leq \nu \leq 3$.

- If $|L^{-1}(k+\ell)| < \tilde{c}_0$ and $\ell_\nu \neq 0$ for at least one $1 \leq \nu \leq 3$ or if $|L^{-1}(k+\ell)| \geq \tilde{c}_0$, then $\text{Re} \Delta^{(n)}(L^{-1}(k+\ell)) \geq \min \{ \rho(\tilde{c}_0), \frac{\pi}{2} \}$. The proof of this given in the case $|k| \leq c_0$ applies now too.

- If $|L^{-1}(k+\ell)| < \tilde{c}_0$, $\ell_\nu = 0$ for all $\nu \geq 1$ then

$$\text{Re} \Delta^{(n)}(L^{-1}(\text{Re} k + \ell)) \geq 0$$

$$\left| \Delta^{(n)}(L^{-1}(k+\ell)) - \Delta^{(n)}(L^{-1}(\text{Re} k + \ell)) \right| \leq 4\pi \Gamma \frac{1}{L^2} |\text{Im} k|$$

The first bound follows immediately from Remark 4.1.c. The second bound follows from Lemma 4.2.c (for the $\frac{1}{L} \text{Im} k_0$ contribution) and Lemma 4.2.g (for the $\frac{1}{L} \text{Im} k_\nu$ contribution, with $1 \leq \nu \leq 3$, — note that on the line segment from $L^{-1}(\text{Re} k + \ell)$ to $L^{-1}(k + \ell)$, $|\frac{\partial \Delta^{(n)}}{\partial k_\nu}|$ is bounded by $\Gamma \frac{1}{L} |L^{-1}(k+\ell)_\nu| = \frac{\Gamma}{L} |k_\nu| \leq \Gamma \frac{\pi + 3m^2}{L}$). Furthermore

$$-\text{sgn} \ell_0 \text{ Im} \Delta^{(n)}(L^{-1}(\text{Re} k + \ell)) \geq \frac{\pi}{2L^2} \quad \text{if } \ell_0 \neq 0$$

$$\left| \Delta^{(n)}(L^{-1}(\text{Re} k)) \right| \geq \gamma_1 \gamma_2 \min \left\{ \frac{c_0}{\sqrt{2}}, \frac{\tilde{c}_0}{2} \right\} \frac{1}{L^2} \quad \text{if } \ell_0 = 0$$

In the case $\ell_0 \neq 0$, the proof of the bound given in the case $|k| \leq c_0$ applies now too. (Just apply it to $\text{Re} k$.) The bound for the case $\ell_0 = 0$ follows from Lemma 4.2.e and Lemma 3.2.a.

- For all $\ell$, by parts (a) and (e) of Lemma 2.3,

$$|u_+(L^{-1}(k+\ell))^q| \leq \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu|+\pi} \right]^{q}$$

$$|\text{Im} u_+(L^{-1}(k+\ell))^q| \leq 16q |\text{Im} k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\nu|+\pi} \right]^{q}$$
We split $\lambda \mathbb{1} - d_{n,k}^{(s)}$ into three pieces

$$\left( \lambda \mathbb{1} - d_{n,k}^{(s)} \right)(\ell, \ell') = D(\ell, \ell') - P(\ell, \ell') + I(\ell, \ell')$$

where

$$D(\ell, \ell') = \delta_{\ell,\ell'} d_{\ell}$$

with $d_{\ell} = \lambda - \left\{ \begin{aligned}
\hat{\Delta}^{(n)}(I_{-1}(k+\ell)) & \quad \ell_{\nu} \neq 0 \text{ for some } \nu \geq 1 \\
\hat{\Delta}^{(n)}(I_{-1}(k+\ell)) & \quad |I_{-1}(k+\ell)| \geq \tilde{c}_0 \\
\hat{\Delta}^{(n)}(I_{-1}(\text{Re } k+\ell)) & \quad \text{otherwise}
\end{aligned} \right.$$

$$\lambda_\lambda = \min\{\text{Re } \lambda, 0\} + i\text{Im } \lambda$$

$$P(\ell, \ell') = \frac{a}{L^2} v(\ell) v(\ell')$$

with $v(\ell) = \text{Re } u_+ (I_{-1}(k+\ell))^q$

We have chosen $\lambda_\lambda$ so that $\text{Re } \lambda_\lambda \leq 0$ and $|\lambda - \lambda_\lambda| \leq \frac{\lambda a}{L^2}$. As $P$ is a rank one operator

$$(D - P)^{-1}(\ell, \ell') = \frac{1}{d_{\ell}} \delta_{\ell,\ell'} + \frac{1}{1 - \kappa} \frac{a}{L^2} \frac{v(\ell) v(\ell')}{d_{\ell}}$$

with

$$\kappa = \sum_{\ell''} \frac{a}{L^2} \frac{1}{d_{\ell''}} v(\ell'')^2$$

Internal Remark

Formally

$$(D - P)^{-1} = D^{-1} (I - PD^{-1})^{-1} = \sum_{n=0}^{\infty} D^{-1} (PD^{-1})^n = D^{-1} + \frac{1}{1 - \kappa} D^{-1} PD^{-1}$$

with

$$\kappa = \sum_{\ell''} \frac{a}{L^2} \frac{1}{d_{\ell''}} v(\ell'')^2$$

since

$$(PD^{-1}P)(\ell, \ell') = \sum_{\ell''} \frac{a}{L^2} v(\ell) v(\ell'') \frac{1}{d_{\ell''}} \frac{a}{L^2} v(\ell'') v(\ell') = \kappa P(\ell, \ell')$$

The formula

$$(D - P)^{-1} = D^{-1} + \frac{1}{1 - \kappa} D^{-1} PD^{-1}$$
is valid as long as all the \( d_\ell \)'s are nonzero and \( \kappa \neq 1 \), since
\[
(D - P)(D^{-1} + \frac{1}{1 - \kappa} D^{-1} P D^{-1}) = \mathbb{I} - PD^{-1} + \frac{1}{1 - \kappa} PD^{-1} - \frac{1}{1 - \kappa} P D^{-1} PD^{-1} \\
= \mathbb{I} - PD^{-1} + \frac{1}{1 - \kappa} PD^{-1} - \frac{\kappa}{1 - \kappa} PD^{-1} \\
= \mathbb{I}
\]
\[
(D^{-1} + \frac{1}{1 - \kappa} D^{-1} PD^{-1})(D - P) = \mathbb{I} - D^{-1}P + \frac{1}{1 - \kappa} D^{-1}P - \frac{1}{1 - \kappa} D^{-1}PD^{-1}P \\
= \mathbb{I} - D^{-1}P + \frac{1}{1 - \kappa} D^{-1}P - \frac{\kappa}{1 - \kappa} D^{-1}P \\
= \mathbb{I}
\]

We have shown above that
\[
|d_\ell| \geq \frac{\lambda_1}{L^2} \\
\text{Re } d_\ell \leq \begin{cases} 
0 & \text{if } |L^{-1}(k + \ell)| < \tilde{c}_0, \ k_\nu = 0 \text{ for all } \nu \geq 1 \\
-\frac{\lambda_1}{L^2} & \text{otherwise}
\end{cases} \\
|v(\ell)| \leq \left[ \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi} \right]^{q} \\
|I(\ell, \ell')| \leq \left( \frac{\lambda_1}{L^2} + 4\pi \Gamma_7 \frac{1}{L^2} |\text{Im } k| \right) \delta_{\ell, \ell'} + 32q |\text{Im } k| \frac{a}{L^2} \left[ \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi} \right]^{q} \left[ \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi} \right]^{q}
\]

provided we choose \( 0 < \lambda_1 \leq \min \left\{ \epsilon_1, \rho(\tilde{c}_0), \frac{\pi}{2}, \gamma_1 \frac{\gamma_2}{\sqrt{2}}, \gamma_1 \frac{\gamma_2}{2}, 2^2 \right\} \).

Since \( \text{Re } z \leq 0 \Rightarrow \text{Re } \frac{1}{z} \leq 0 \), and \( v(\ell) \in \mathbb{R} \) for all \( \ell \), we have that \( \text{Re } \kappa \leq 0 \) so that \( \frac{1}{1 - \kappa} \leq 1 \) and
\[
\| (D - P)^{-1} \|_{\ell_{1} \rightarrow \ell_{\infty}} \leq \frac{L^2}{a} \quad \| I \|_{\ell_{1} \rightarrow \ell_{\infty}} \leq \frac{\Gamma_7}{a^2}(\lambda_0 + |\text{Im } k|)
\]

with the constant \( \lambda_2 > 0 \) depending only on \( a \) and \( \lambda_1 \) and the constant \( \Gamma_7 \) depending only on \( \Gamma_7 \), \( a \) and \( q \). It now suffices to choose \( \lambda_0 \) and \( m_2 \) smaller than \( \frac{\lambda_2}{12a^2} \) and use a Neumann expansion to give
\[
\| (\lambda \mathbb{I} - d^{(s)}_{m,k})^{-1} \|_{\ell_{1} \rightarrow \ell_{\infty}} \leq 2 \frac{L^2}{a} \frac{\lambda_2}{\lambda_1}
\]

\( \square \)

**Proposition 4.4.** Let \( m_2, \lambda_0, \Gamma_8 \) be as in Lemma 4.3 and use \( \mathbb{R}_- \) to denote the negative real axis in \( \mathbb{C} \). Set
\[ \mathcal{O}_C = \{ z \in \mathbb{C} \mid \text{dist}(z, \mathbb{R}_-) > \frac{\lambda_0}{2L^2}, |z| < \Gamma_8 + 1 \} \]
\[ \mathcal{O} = \{ z \in \mathbb{C} \mid \text{dist}(z, \mathbb{R}_-) > \frac{\lambda_0}{L^2}, |z| < \Gamma_8 + 2 \} \]

and let

- \( C = \partial \mathcal{O}_C \), oriented counterclockwise
- \( f : \mathcal{O} \to \mathbb{C} \) be analytic.

Then \( f\left(\frac{n}{L^2}Q^*Q + \Delta^{(n)}(s)\right) \), defined by [5, (12) and Lemma 15.a], exists and there is a constant\(^3 \Gamma_9 \)

**Internal Remark**

It depends only on \( \Gamma_8 \) and \( m_2 \), and, in particular is independent of \( n \) and \( L \).

such that

\[ \|f\left(\frac{n}{L^2}Q^*Q + \Delta^{(n)}(s)\right)\|_{m_2} \leq \Gamma_9 L^7 \sup_{\zeta \in C} |f(\zeta)| \]

**Proof.** Apply [5, Lemma 14] with \( \hat{a}_k(\ell, \ell') = \hat{d}_{n,k}^{(s)}(\ell, \ell') \) and

\[ \mathcal{X}_\text{fin} = \mathcal{X}_1^{(n)} \quad \mathbb{Z}_\text{fin} = \varepsilon_2^2 \mathbb{Z} \times \varepsilon_1 \mathbb{Z}^3 \quad \mathcal{X}_\text{crs} = \mathcal{X}_0^{(n+1)} \quad \mathbb{Z}_\text{crs} = \mathbb{Z} \times \mathbb{Z}^3 \quad \mathcal{B} = \mathcal{B}_1 \]

and \( m = 3m_2 \), \( m' = 2m_2 \) and \( m'' = m_2 \). Then \( \text{vol}_c = 1 \), \( \#\mathcal{B}_1 = L^5 \). Observe in particular that, by Lemma 4.3, the spectrum of \( \hat{d}_{n,k}^{(s)} \) is contained in

\[ \{ z \in \mathbb{C} \mid \text{dist}(z, \mathbb{R}_-) > \frac{\lambda_0}{L^2}, |z| \leq \Gamma_8 \} \]

which is the shaded region in the figure above. So the lemma gives

\[ \|f\left(\frac{n}{L^2}Q^*Q + \Delta^{(n)}(s)\right)\|_{m''} \]
\[ \leq \frac{C_{m'-m''} |\mathcal{O}|}{2\pi \text{vol}_c} \sup_{\zeta \in C} |f(\zeta)| \sup_{|\text{Im } k|=m'-m'' |\ell|, \ell' \in \mathcal{B}_1} \sum_{\ell, \ell'} |(\zeta I - \hat{d}_{n,k}^{(s)})^{-1}(\ell, \ell')| \]
\[ \leq \frac{C_{m'-m''}}{2\pi} (\Gamma_8 + 1)(3\pi + 2) \sup_{\zeta \in C} |f(\zeta)| L^5 \sup_{|\text{Im } k|=m'-m'' |\ell|, \ell' \in \mathcal{B}_1} \sum_{\ell, \ell'} |(\zeta I - \hat{d}_{n,k}^{(s)})^{-1}(\ell, \ell')| \]
\[ \leq \frac{C_{m'-m''}}{2\pi} (\Gamma_8 + 1)(3\pi + 2) \sup_{\zeta \in C} |f(\zeta)| L^5 \Gamma_8 L^2 \]

by Lemma 4.3.b.

\[ \square \]

\(^3\text{Recall Convention 1.2.}\)
Applying Proposition 4.4 with $f(z) = \frac{1}{z}$, $f(z) = \frac{1}{\sqrt{z}}$ and $f(z) = \sqrt{z}$, where $\sqrt{z}$ is the principal value of the square root gives

**Corollary 4.5.** The operators $C^{(n)}$, $\sqrt{C^{(n)}}$ and $(\sqrt{C^{(n)}})^{-1}$ all exist. There is a constant $\Gamma_{10}$ such that

$$\|L^{-1}C^{(n)}L\|_{m_2}, \|\sqrt{L^{-1}C^{(n)}L}\|_{m_2}, \|(\sqrt{L^{-1}C^{(n)}L})^{-1}\|_{m_2} \leq \Gamma_{10}L^9$$

**Internal Remark**

It depends only on the $m_2, \lambda_0, \Gamma_8$ of Lemma 4.3, and, in particular is independent of $n$ and $L$.

We could do the same thing with $f$ being one over the principal value of the square root and the equation being $f(A)f(A) = A^{-1}$. 

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Let $N \in \mathbb{N}$

- $\mathcal{M}$ be the set of all $N \times N$ matrices all of whose spectrum is a distance strictly greater than $\varepsilon > 0$ from the imaginary axis. Note that we are not assuming diagonalizability.
- for each $A \in \mathcal{M}$, $f(A) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta^2 - A} d\zeta$ where $C$ is the contour of Proposition 4.4, and $f$ is the principal value of the square root.

Then

- If $A \in \mathcal{M}$ is diagonal, $f(A)f(A) = A$ by the Cauchy integral formula.
- If $A \in \mathcal{M}$ is diagonalizable, $f(A)f(A) = A$. This is the case since if $A = UDU^{-1}$ with $D$ diagonal, then $f(A) = Uf(D)U^{-1}$.
- If $A \in \mathcal{M}$ is not diagonalizable, we still have $f(A)f(A) = A$ by continuity since diagonalizable matrices are dense and $f(A)$ is continuous in $A$. 

We could do the same thing with $f$ being one over the principal value of the square root and the equation being $f(A)f(A) = A^{-1}$. 

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5 The Green’s Functions

In this chapter, we discuss the inverses of the operators

\[ D_n + Q^*_n \Omega_n Q_n \]

These inverses, and variations thereof, are constituents of the leading part of the power series expansion of the background fields of [7, 8, 9]. See [7, Proposition 1.14] and [9, Proposition 2.1]. In Proposition 5.1, below, we show that for sufficiently small \( \mu \), the operators \( D_n + Q^*_n \Omega_n Q_n - \mu \) are invertible, and we estimate the decay of the kernels \( S_n(\mu)(x,y) \) of their inverses

\[ S_n(\mu) = [D_n + Q^*_n \Omega_n Q_n - \mu]^{-1} \]

By Remark 2.5

\[ \partial_\nu(S_n(\mu)^*)^{-1} = (S_{n,\nu}^{(+)}(\mu))^{-1}\partial_\nu \quad \partial_\nu S_n(\mu)^{-1} = (S_{n,\nu}^{(-)}(\mu))^{-1}\partial_\nu \quad (5.1) \]

where

\[ S_{n,\nu}^{(+)}(\mu) = [D_n^* + Q_{n,\nu}^{(+)} \Omega_n Q_{n,\nu}^{(-)} - \mu]^{-1} \quad S_{n,\nu}^{(-)}(\mu) = [D_n + Q_{n,\nu}^{(-)} \Omega_n Q_{n,\nu}^{(-)} - \mu]^{-1} \quad (5.2) \]

and \( Q_{n,\nu}^{(+)} \), \( Q_{n,\nu}^{(-)} \) were defined in (2.11). We shall write

\[ S_n = S_n(0) \quad S_{n,\nu}^{(\pm)} = S_{n,\nu}^{(\pm)}(0) \quad (5.3) \]

The main result extends the statement of [7, Theorem 1.13]. It is

**Proposition 5.1.** There are constants \( \mu_{\text{up}}, m_3 > 0 \) and \( \Gamma_{11} \), depending only on \( q \), \( h_0 \) and \( a \), and in particular independent of \( n \) and \( L > \Gamma_2 \), such that, for \( |\mu| \leq \mu_{\text{up}} \), the operators \( D_n + Q^*_n \Omega_n Q_n - \mu \) and \( D_n^* + Q_{n,\nu}^{(+) \Omega_n Q_{n,\nu}^{(-)} - \mu} \), \( D_n + Q_{n,\nu}^{(-) \Omega_n Q_{n,\nu}^{(-)} - \mu} \) are invertible, and their inverses \( S_n(\mu) \) and \( S_{n,\nu}^{(+)}(\mu) \), \( S_{n,\nu}^{(-)}(\mu) \), respectively, fulfill

\[ \| S_n(\mu) \|_{m_3}, \| S_{n,\nu}^{(\pm)}(\mu) \|_{m_3} \leq \Gamma_{11} \]

\[ \| S_n(\mu) - S_n \|_{m_3}, \| S_{n,\nu}^{(\pm)}(\mu) - S_{n,\nu}^{(\pm)} \|_{m_3} \leq |\mu| \Gamma_{11} \]

This Proposition is proven following the proof of Lemma 5.5.

**Example 5.2.** As a model computation, we evaluate the inverse transform

\[ s(x) = \int_{\mathbb{R} \times \mathbb{R}^3} \frac{e^{ip \cdot x}}{-ip_0 + m^2 + p_1^2 + p_2^2 + p_3^2} \frac{dp_0 \, dp_1 \, dp_2 \, dp_3}{(2\pi)^4} \]
of $s(p) = \frac{1}{-ip_0 + m^2 + p^2}$. It is designed to mimic the behaviour of $S_n$ in the limit $n \to \infty$. Write $x = (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$. We first compute the $p_0$ integral. Observe that the integrand has exactly one pole, which is at $p_0 = -i(m^2 + \mathbf{p}^2)$, and that the $e^{ip_0 t}$ in the integrand forces us to close the contour in the upper half plane when $t > 0$ and in the lower half plane when $t < 0$. Thus

$$\int_{-\infty}^{\infty} \frac{e^{ip_0 t + ip \mathbf{x} \cdot \mathbf{p}}}{-ip_0 + m^2 + \mathbf{p}^2} \frac{dp_0}{(2\pi)^3} = \begin{cases} 0 & \text{if } t > 0 \\ e^{(m^2 \mathbf{p}^2) t + ip \mathbf{x} \cdot \mathbf{p}} \frac{1}{(2\pi)^3} & \text{if } t < 0 \end{cases}$$

**Internal Remark**

To justify dropping the “semicircular part of the contour at infinity”, parametrize that part of the contour by $p_0 = re^{i(\text{sgn})\theta}$, $0 \leq \theta \leq \pi$. Then $\frac{1}{| -ip_0 + m^2 + \mathbf{p}^2 |} \leq \frac{\text{const}}{r}$, $dp_0 \leq rd\theta$ and $|e^{ip_0 t + ip \mathbf{x} \cdot \mathbf{p}}| \leq e^{-|t|r \sin \theta}$. The integral converges to zero as $r \to \infty$ by the Lebesgue dominated convergence theorem.

Hence $s(x) = 0$ for $t > 0$ and, for $t < 0$,

$$s(x) = \int_{\mathbb{R}^3} e^{-(m^2 + \mathbf{p}^2)|\mathbf{x}|} e^{ip \mathbf{x} \cdot \mathbf{p}} \frac{d^3 \mathbf{p}}{(2\pi)^3} = e^{-m^2|t|} \prod_{j=1}^{3} \int_{-\infty}^{\infty} e^{-|t|^2 \mathbf{p}_j^2} e^{ip_j x_j} \frac{dp_j}{2\pi}$$

$$= \frac{1}{2^{3/2}|t|^{3/2}} e^{-m^2|t|} e^{-\frac{j^2}{4|x|^2}}$$

Here are some observations about $s(x)$.

- Since $m^2|t| + \frac{x_j^2}{4|t|} \geq m|x|$ (the minimum is at $|t| = \frac{|x|}{2m}$), $s(x)$ decays exponentially for large $|x|$ in all directions.

- For $\mathbf{x} \neq 0$, $\lim_{t \to 0} e^{-\frac{x^2}{4|t|}} = 0$, so $s(x)$ is continuous everywhere except at $x = 0$.

- $s(x)$ has an integrable singularity at $x = 0$. There are a number of ways to see this. For example, the inequality $e^{-\frac{x^2}{4|t|}} \leq \text{const}(\frac{|t|}{|x|^2})^\kappa/2$ implies that $|s(x)|$ is bounded near $x = 0$ by a constant times $\frac{1}{|t|(3-\kappa)/2} \frac{1}{|x|^2}$. This is integrable if $1 < \kappa < 3$.

- If we send $x \to 0$ along a curve with $\mathbf{x}^2 = -4\gamma|t| \ln|t|$, $s(x) \approx \text{const} \frac{1}{|t|^{3/2\gamma}}$.

- We see, using $\mathbf{x}^T e^{-\frac{x^2}{4|t|}} \leq \text{const}|t|^2$, that $(t^2 + \sum_{j=1}^{3} x_j^2)|s(x)|$ is bounded and exponentially decaying. Note that $\frac{1}{t^2 + \sum_{j=1}^{3} x_j^2}$ has an integrable singularity at the origin, since $\int_{-\infty}^{\infty} \frac{1}{t^2 + \sum_{j=1}^{3} x_j^2} dt = \text{const} \frac{1}{\sqrt{2\gamma x_j}}$.

As preparation for and in addition to the position space estimates of Proposition 5.1, we also derive bounds on the Fourier transforms of these and related operators.
To convert bounds in momentum space into bounds in position space, we shall use [5, Lemma 12], with

\[ \mathcal{X}_{\text{fin}} = \mathcal{X}_n, \quad \mathcal{Z}_{\text{fin}} = \varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3 \quad \mathcal{X}_{\text{crs}} = \mathcal{X}_0^{(n)} \quad \mathcal{Z}_{\text{crs}} = \mathbb{Z} \times \mathbb{Z}^3 \quad \mathcal{B} = \mathcal{B}_n \]  

(5.4)

We shall routinely use \(|p_0|, |p_\nu|\) and \(|p|\) to refer to the magnitudes of the smallest representatives of \(p_0 \in \mathbb{C}, p_\nu \in \mathbb{C}\) and \(p \in \mathbb{C}^3\) in \(\mathbb{C}/2\pi \varepsilon_n \mathbb{Z}, \mathbb{C}/\varepsilon_n^2 \mathbb{Z}\) and \(\mathbb{C}^3/2\pi \varepsilon_n \mathbb{Z}^3\), respectively.

The operators \(S_n(\mu)\) act on functions on the lattice \(\mathcal{X}_n\), but they are only translation invariant with respect to the sublattice \(\mathcal{X}_0^{(n)}\). An exponentially decaying operator which is fully translation invariant, and has the same local singularity as \(S_n\), is the operator

\[ S'_n = [D_n + a_n \exp\{-\Delta_n\}]^{-1} \quad \text{where} \quad \Delta_n = \partial_0^* \partial_0 + (\partial_1^* \partial_1 + \partial_2^* \partial_2 + \partial_3^* \partial_3) \]  

(5.5)

and \(a_n = a \frac{1}{1 - L^{p_0}}\) as in Proposition 2.4.b, and the forward derivatives \(\partial_\nu\) are defined in (2.9). Obviously \(S'_n\) has Fourier transform

\[ \hat{S}_n'(p) = [\hat{D}_n(p) + a_n \exp\{-\Delta_n(p)\}]^{-1} \quad \text{where} \quad \Delta_n(p) = \left[ \frac{\sin \frac{4\pi^2 p_0}{2\pi \varepsilon_n}}{2\pi \varepsilon_n^2} \right]^2 + \sum_{\nu=1}^3 \left[ \frac{\sin \frac{4\pi^2 p_\nu}{2\pi \varepsilon_n}}{2\pi \varepsilon_n} \right]^2 \]  

Before we discuss the properties of \(S'_n\) and of the difference \(\delta S = S_n - S'_n\) we note

**Remark 5.3.** The Fourier transform \(\Delta_n(p)\) of the four dimensional Laplacian \(\Delta_n\) is entire. For \(p \in \mathbb{R} \times \mathbb{R}^3\),

\[ \Delta_n(p) \geq \frac{2}{\pi^2} \left[ |p_0|^2 + |p|^2 \right] \]

For \(p \in \mathbb{C} \times \mathbb{C}^3\) with \(\varepsilon_n^2 \Im p_0 \leq 1\) and \(\varepsilon_n \Im p \leq 1\)

\[ |\Delta_n(p)| \leq 4 \left[ |p_0|^2 + |p|^2 \right] \quad |\frac{\partial^*}{\partial p_\nu} \Delta_n(p)| \leq 4 |p_\nu| \quad |\frac{\partial^*}{\partial p_\nu} \Delta_n(p)| \leq 4 \varepsilon_n^{\ell - 2} \]  

if \(\ell \geq 2\)

with the \(\varepsilon_{n,\nu}\) of (2.6). For \(p \in \mathbb{C} \times \mathbb{C}^3\) with \(|\Im p| \leq 1\),

\[ \Re \Delta_n(p) \geq -5\pi^2 + \frac{1}{\pi^2} \left[ |p_0|^2 + |p|^2 \right] \]

**Proof.** For the first two claims, just apply parts (a) and (b) of Lemma A.1. For the derivatives, use

\[ \frac{d}{d\theta} \left[ \frac{\sin(n\theta)}{n} \right] = \sin(2n\theta) \quad \Rightarrow \quad \frac{d^\ell}{d\theta} \left[ \frac{\sin(n\theta)}{n} \right] = \pm (2n)^{\ell-1} \frac{1}{n} \begin{cases} \sin(2n\theta) & \text{for } \ell \text{ odd} \\ \cos(2n\theta) & \text{for } \ell \text{ even} \end{cases} \]
For the final claim, write $p = p + i q$ with $p, q \in \mathbb{R} \times \mathbb{R}^3$. Then
\[
\text{Re } \Delta_n(p + i q) \geq \Delta_n(p) - \left| \text{Re } \Delta_n(p + i q) - \Delta_n(p) \right|
\geq \frac{2}{\pi^2} \left( |p_0|^2 + |p|^2 \right) - 4|q||p + i q|
\geq \frac{2}{\pi^2} |p + i q|^2 - 2(2 + \frac{2}{\pi^2}) |q||p + i q| - \frac{2}{\pi^2}|q|^2
\geq \frac{1}{\pi^2} |p + i q|^2 - \left\{ \pi^2 (2 + \frac{2}{\pi^2})^2 + \frac{2}{\pi^2} \right\}|q|^2
\]

By the resolvent identity
\[
\delta S = S_n - S'_n = -S'_n \left[ Q_n^* \Omega_n Q_n - a_n \exp \{-\Delta_n\} \right] S_n
\]
$S_n$ and $\delta S$ are translation invariant with respect to the sublattice $\mathcal{X}^{(n)}_0$ of $\mathcal{X}_n$. By “Floquet theory” (see [5, Lemma 1]), their Fourier transforms $\tilde{S}_n(p, p')$, $\tilde{\delta S}(p, p')$, $p, p' \in \tilde{\mathcal{X}}$, vanish unless $\pi^{(n,0)}_n(p) = \pi^{(n,0)}_n(p')$, i.e. unless there are $k \in \tilde{\mathcal{X}}^{(n)}_0$ and $\ell, \ell' \in \tilde{\mathcal{B}}_n$ such that $p = k + \ell$, $p' = k + \ell'$. The blocks $\tilde{S}_{n,k}^{-1}(\ell, \ell') = \tilde{S}_{n,k}^{-1}(k + \ell, k + \ell')$ and $\tilde{\delta S}_{n,k}(\ell, \ell') = \tilde{\delta S}(k + \ell, k + \ell')$ are given by
\[
\tilde{S}_{n,k}^{-1}(\ell, \ell') = \tilde{D}_n(k + \ell) \delta_{\ell, \ell'} + u_n(k + \ell)^q \tilde{\Omega}_n(k) u_n(k + \ell')^q \\
\tilde{\delta S}_{n,k}(\ell, \ell') = -\sum_{\ell'' \in \tilde{\mathcal{B}}_n} \tilde{S}_{n,k}'(k + \ell) \left[ u_n(k + \ell)^q \tilde{\Omega}_n(k) u_n(k + \ell'')^q - a_n e^{-\Delta_n(k+\ell)} \delta_{\ell, \ell''} \right] \tilde{S}_{n,k}(\ell'', \ell')
\]
(5.6)

where $u_n$ and $\tilde{\Omega}_n$ are given in parts (b) and (e) of Remark 2.1.

**Lemma 5.4.** There are constants\(^4\) $m_4 > 0$ and $\Gamma_{12}$.

**Internal Remark**

They depend only on $q$, $\hat{h}$ and $a$, and in particular are independent of $n$ and $L > \Gamma_2$.

such that the following hold for all $L > \Gamma_2$.

(a) $\tilde{S}_n'(p)$ is analytic in $|\text{Im } p| < 3m_4$ and obeys
\[
|\tilde{S}_n'(p)| \leq \frac{\Gamma_{12}}{1 + |p_0| + |p|^2} \quad \text{and} \quad \left| \frac{\partial^2}{\partial p_0^2} \tilde{S}_n'(p) \right|, \left| \frac{\partial^4}{\partial p_0^4} \tilde{S}_n'(p) \right| \leq \frac{\Gamma_{12}}{(1 + |p_0| + |p|^2)^3}
\]
for $1 \leq \nu \leq 3$ there.

\(^4\)Recall Convention 1.2.
(b) For all \( \ell, \ell' \in \hat{B}_n \), \( \hat{S}_{n,k}(\ell, \ell') \) is analytic in \(|\text{Im} \, k| < 3m_4 \) and obeys

\[
|\hat{S}_{n,k}(\ell, \ell')| \leq \frac{\Gamma_{12}}{1 + |\ell_0| + \Sigma_{\nu=1}^3 |\ell'_\nu|^2} \left\{ \delta_{\ell,\ell'} + \frac{1}{1 + |\ell_0| + \Sigma_{\nu=1}^3 |\ell'_\nu|^2} \prod_{\nu=0}^3 \frac{1}{|\ell'_\nu| + 1} \prod_{\nu=0}^3 \frac{3}{|\ell'_\nu| + 1} \right\}
\]

there.

(c) For all \( \ell, \ell' \in \hat{B}_n \), \( \hat{\delta S}_k(\ell, \ell') \) is analytic in \(|\text{Im} \, k| < 3m_4 \) and obeys

\[
|\hat{\delta S}_k(\ell, \ell')| \leq \Gamma_{12} \exp \left\{ -\frac{\Gamma_{12}}{40} \sum_{\nu=0}^3 |\ell'_\nu|^2 \right\} \delta_{\ell,\ell'}
\]

\[
+ \frac{\Gamma_{12}}{1 + |\ell_0| + \Sigma_{\nu=1}^3 |\ell'_\nu|^2} \left\{ \prod_{\nu=0}^3 \frac{3}{|\ell'_\nu| + 1} \prod_{\nu=0}^3 \frac{3}{|\ell'_\nu| + 1} \right\}
\]

there.

(d) For all \( u, u' \in \mathcal{X}_n \),

\[
|S_n(u, u') - S'_n(u, u')| \leq \Gamma_{12} e^{-2m_4 |u - u'|}
\]

\[
|S'_n(u, u')| \leq \Gamma_{12} \min \left\{ \frac{e^{-2m_4 |u - u'|}}{|u_0 - u'_0|^2 + |u - u'|^2}, L^{5n} \right\}
\]

**Proof.** (a) Obviously \( \hat{S}'_n(p)^{-1} \) is entire. For real \( p \)

\[
|\hat{S}'_n(p)^{-1}| \geq a_n \exp \{-\Delta_n(p)\} + \text{const}\{|p_0| + |p|^2\} \geq \text{const}\{1 + |p_0| + |p|^2\} \quad (5.7)
\]

by Remark 5.3, Lemma 3.2.a, and the fact that \( \Delta_n(p), \Re \hat{D}_n(p) \geq 0 \) for real \( p \). The bound on \( |\frac{\partial}{\partial p_\nu} \hat{D}_n(p)| \) of Lemma 3.2.c and Remark 5.3 shows that (5.7) is valid for all \(|\text{Im} \, p| < 3m_4 \), if \( m_4 \) is chosen sufficiently small.

We now bound the derivatives. For any \( 0 \leq \nu \leq 3 \) and \( \ell \in \mathbb{N} \), \( \frac{\partial}{\partial p_\nu} \hat{S}'_n(p) \) is a finite linear combination of terms of the form

\[
\hat{S}'_n(p)^{1+j} \prod_{i=1}^j \frac{\partial^i_{p_\nu}}{\partial \nu} \left[ \hat{D}_n(p) + a_n \exp \{-\Delta_n(p)\} \right]
\]

with each \( \ell_i \geq 1 \) and \( \sum_{i=1}^j \ell_i = \ell \). By Remark 5.3, all derivatives of \( \exp \{-\Delta_n(p)\} \) of order \( \ell_i \in \mathbb{N} \) are bounded by \( \text{const}_{\ell_i} \). Hence, by Lemma 3.2.c,

\[
\left| \frac{\partial^i_{p_\nu}}{\partial \nu} \left[ \hat{D}_n(p) + a_n \exp \{-\Delta_n(p)\} \right] \right| \leq \text{const} \left\{ \frac{1}{|1 + |p_0| + |p|^2|^{3/2}} \right\} \quad \text{if } \nu = 0, \ell_i = 1, 2
\]

\[
\left| \frac{\partial^i_{p_\nu}}{\partial \nu} \left[ \hat{D}_n(p) + a_n \exp \{-\Delta_n(p)\} \right] \right| \leq \text{const} \left\{ \frac{1}{|1 + |p_0| + |p|^2|^{3/2}} \right\} \quad \text{if } \nu \geq 1, 1 \leq \ell_i \leq 4
\]
As \( |\hat{S}_n^\prime(p)| \leq \frac{\text{const}}{1+|p_0|+|p|^2}, \)

\[
|\hat{S}_n^\prime(p)|^{1+j} \prod_{i=1}^{j} \left( \frac{1}{1+|p_0|+|p|^2} \right)^{\nu_i} \leq \frac{\text{const}}{1+|p_0|+|p|^2} \prod_{i=1}^{j} \left( \frac{1}{1+|p_0|+|p|^2} \right)^{\nu_i}
\]

if \( \nu = 0, \ell_i = 1, 2 \)

if \( \nu \geq 1, 1 \leq \ell_i \leq 4 \)

and the claim follows.

(b) For any \( c''_\ell > 0 \) we have, for \(|k| \geq c''_\ell\) and \(|\text{Im } k_0| < 3m_4\), analyticity and the bound

\[
|\hat{S}_{n,k}(\ell, \ell')| \leq \frac{\Gamma_{19}'}{1+|\ell_0|+\sum_{\nu=1}^{3} |\ell_{\nu}|^2} \delta_{\ell,\ell'} + \frac{\Gamma_{16}'}{1+|\ell_0|+\sum_{\nu=1}^{3} |\ell_{\nu}|^2} \prod_{\nu=0}^{3} \frac{1}{|\ell_{\nu}|+1} \prod_{\nu=0}^{3} \frac{1}{\ell_{\nu}^2}
\]

(with \( \Gamma_{10}' \) depending on \( c''_\ell \)) which follows from the representation

\( S_n = D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1} \)

(see [6, Remark 10.b]) and Lemmas 3.2.d, 2.2.a and 4.2.c.

Internal Remark

\[
\begin{align*}
[D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [(D_n^{-1})^{-1} + Q_n^* \Omega_n Q_n] & \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n] \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n] \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n] \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n]
\end{align*}
\]

and

\[
\begin{align*}
[D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [(D_n^{-1})^{-1} + Q_n^* \Omega_n Q_n] & \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n] \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n] \\
& = [D_n^{-1} - D_n^{-1} Q_n^* \Delta(n) Q_n D_n^{-1}] [D_n^{-1} + \Omega_n Q_n]
\end{align*}
\]

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For \(|k| < c'_0\), with \(c'_0\) to be shortly chosen sufficiently small, and \(|\text{Im}\ k| < 3m_4\) we use the representation
\[
\hat{S}_{n,k}^{-1}(\ell, \ell') = \hat{D}_n(k + \ell) \delta_{\ell, \ell'} + u_n(k + \ell) \hat{\Omega}_n(k) u_n(k + \ell')^q = D_{\ell, \ell'} + B_{\ell, \ell'}
\]
with
\[
D_{\ell, \ell'} = \hat{D}_n(k + \ell) \delta_{\ell, \ell'} + \begin{cases} a_n & \text{if } \ell, \ell' = 0 \\ 0 & \text{otherwise} \end{cases}
\]
\[
B_{\ell, \ell'} = \begin{cases} \hat{\Omega}_n(k) u_n(k)^{2q} - a_n & \text{if } \ell = \ell' = 0 \\ u_n(k + \ell) \hat{\Omega}_n(k) u_n(k + \ell')^q & \text{otherwise} \end{cases}
\]
By parts (c) and (d) of Lemma 3.2, assuming that \(|k| < c'_0\) with \(c'_0\) small enough, \(D\) is invertible and the inverse is a diagonal matrix with every diagonal matrix element obeying
\[
|D^{-1}_{\ell, \ell}| \leq \frac{\Gamma''_{10}}{1 + |\ell_0| + \sum_{\nu=1}^u |\ell_\nu|^2}
\]
for some \(\Gamma''_{10}\) which is independent of \(c'_0\). By parts (b) and (c) of Lemma 2.2 and parts (a) and (b) of Proposition 2.4,
\[
|B_{\ell, \ell'}| \leq \text{const}_{a, q} |k| \prod_{\nu=0}^3 \left( \frac{24}{|\ell_\nu| + \pi} \right)^q \prod_{\nu=0}^3 \left( \frac{24}{|\ell_\nu| + \pi} \right)^q
\]
So if \(|k| < c'_0\) with \(c'_0\) small enough, \(D + B\) is invertible with the inverse given by the Neumann expansion \(D^{-1} + \sum_{p=1}^\infty (-1)^p D^{-1} (BD^{-1})^p\). Since \(D\) and \(B\) are both analytic on \(|\text{Im}\ k| < 2\) and
\[
\sum_{\ell \in 2\pi \mathbb{Z}^4} \text{const}_{a, q} |k| \left( \prod_{\nu=0}^3 \frac{1}{|\ell_\nu| + \pi} \right)^q \prod_{\nu=0}^3 \frac{\Gamma''_{10}}{1 + |\ell_0| + \sum_{\nu=1}^u |\ell_\nu|^2} \left( \prod_{\nu=0}^3 \frac{1}{|\ell_\nu| + \pi} \right)^q < \frac{1}{2}
\]
if \(c'_0\) is small enough, we again get the desired analyticity and bound on \(|\hat{S}_{n,k}(\ell, \ell')|\).

(c) Just apply Remark 5.3 and parts (a) and (b) of this lemma, Lemma 2.2.a and Proposition 2.4.a and the fact that
\[
\sum_{\ell \in 2\pi \mathbb{Z}^4} \left( \prod_{\nu=0}^3 \frac{1}{|\ell_\nu| + \pi} \right)^q \prod_{\nu=0}^3 \frac{1}{1 + |\ell_0| + \sum_{\nu=1}^u |\ell_\nu|^2} \left( \prod_{\nu=0}^3 \frac{1}{|\ell_\nu| + \pi} \right)^q
\]
is bounded uniformly in \(n\) and \(L\) to (5.6).
(d) The bound on $|S_n(u, u') - S_n'(u, u')|$ follows from part (c) by [5, Lemma 12.b] with the replacements (5.4). The bound on $|S_n'(u, u')|$ follows from part (a), noting in particular that $\frac{\Gamma_{12}}{(1+|p_0|+|p|^2)^n} \in L^1(\mathbb{R}^n)$, and

$$|u_\nu - u_\nu'|^2 S_n'(u, u') = \int_{\mathbb{R}^n} \frac{\partial^4 S_n}{\partial p_\nu^4}(p) e^{-ip(u-u')} \frac{d^4p}{(2\pi)^4}$$

and

$$|S_n'(u, u')| \leq \left| \int_{\mathbb{R}^n} \hat{S}_n'(p) e^{-ip(u-u')} \frac{d^4p}{(2\pi)^4} \right| \leq \Gamma_{12} \int_{\mathbb{R}^n} \frac{d^4p}{(2\pi)^4} = \Gamma_{12} L^{5n}$$

Internal Remark

By using the scaling $p_0 = \frac{q_0}{\varepsilon_n}, q = \frac{p}{\varepsilon_n}$, we can get

$$|S_n'(u, u')| \leq \frac{1}{\varepsilon_n^2} \int_{|q| \leq \pi} \frac{\Gamma_{12}}{1 + (|q| + |q'|)^2} \frac{dq_0 dq}{(2\pi)^2} = \frac{1}{\varepsilon_n^2} \int_{|q| \leq \pi} \frac{\Gamma_{12}}{\varepsilon_n^2 + |q| + |q'|^2} \frac{dq_0 dq}{(2\pi)^2} \leq \text{const} L^{3n}$$

since $\frac{1}{q^2}$ is locally integrable in $\mathbb{R}^4$.

We now prove the analog of Lemma 5.4 for the operators $S_{n,\nu}^{(+)}$ and $S_{n,\nu}^{(-)}$ of (5.3). As in (5.5)–(5.6), we decompose

$$S_{n,\nu}^{(+)} = S_n^{(+)} + \delta S_{\nu}^{(+)} \quad S_{n,\nu}^{(-)} = S_n^{(-)} + \delta S_{\nu}^{(-)}$$

with

$$\delta S_{\nu}^{(+)} = S_{n,\nu}^{(+)} - S_n^{(+)} = -S_n^{(+)} [Q_{n,\nu}^{(+)} Q_{n,\nu}^{(-)} - a_n \exp\{-\Delta_n\}] S_{n,\nu}^{(+)}$$

$$\delta S_{\nu}^{(-)} = S_{n,\nu}^{(-)} - S_n^{(-)} = -S_n^{(-)} [Q_{n,\nu}^{(+)} Q_{n,\nu}^{(-)} - a_n \exp\{-\Delta_n\}] S_{n,\nu}^{(-)}$$

They have Fourier representations

$$\widehat{\delta S}_{\nu,k}^{(+)}(\ell, \ell') = -\sum_{\ell'' \in B_n} \widehat{S}_{\nu}^{(+)}(k, l) U_{n,\nu}^{(+)}(k, l') - a_n e^{-\Delta_n(k+l)} \delta_{\ell, \ell''} S_{n,\nu,k}^{(+)}(\ell'', \ell')$$

$$\widehat{\delta S}_{\nu,k}^{(-)}(\ell, \ell') = -\sum_{\ell'' \in B_n} \widehat{S}_{\nu}^{(-)}(k, l) U_{n,\nu}^{(-)}(k, l') - a_n e^{-\Delta_n(k+l)} \delta_{\ell, \ell''} S_{n,\nu,k}^{(-)}(\ell'', \ell')$$

(5.9)
where, by (2.11),
\[ U^{(+)\,}_n(k, \ell) = \zeta^{(+)\,}_n(k, \ell) u^{(+\,)}_n(k + \ell) u_n(k + \ell)^{a-1} \]
\[ U^{(-\,)}_n(k, \ell') = \zeta^{(-\,)}_n(k, \ell') u^{(-\,)}_n(k + \ell') u_n(k + \ell')^{a-1} \]

**Lemma 5.5.** There are constants \( m_5 > 0 \) and \( \Gamma_{13} \)

**Internal Remark**

They depend only on \( q, h_0 \) and \( a \), and in particular are independent of \( n \) and \( L > \Gamma_2 \).

(a) For all \( \ell, \ell' \in \mathcal{B}_n \), \( \hat{S}^{(\pm)}_{n,\nu,k}(\ell, \ell') \) is analytic in \( |\text{Im} k| < 3m_5 \) and obeys
\[
|\hat{S}^{(\pm)}_{n,\nu,k}(\ell, \ell')| \leq \Gamma_{13} \exp \left\{ -\frac{1}{40} \sum_{\nu=0}^3 |\ell'_{\nu}|^2 \right\} \delta_{\ell,\ell'} + \frac{\Gamma_{13}}{1+|\ell_0|+\sum_{\nu=1}^3 |\ell'_\nu|} \prod_{\nu=0}^3 \frac{1}{|\ell'_\nu+1|^3} \prod_{\nu=0}^3 \frac{1}{|\ell'_\nu|^3} \]
there.

(b) For all \( \ell, \ell' \in \mathcal{B}_n \), \( \hat{\delta}S^{(\pm)}_{\nu,k}(\ell, \ell') \) is analytic in \( |\text{Im} k| < 3m_5 \) and obeys
\[
|\hat{\delta}S^{(\pm)}_{\nu,k}(\ell, \ell')| \leq \Gamma_{13} \exp \left\{ -\frac{1}{40} \sum_{\nu=0}^3 |\ell'_{\nu}|^2 \right\} \delta_{\ell,\ell'} + \frac{\Gamma_{13}}{1+|\ell_0|+\sum_{\nu=1}^3 |\ell'_\nu|} \prod_{\nu=0}^3 \frac{1}{|\ell'_\nu+1|^3} \prod_{\nu=0}^3 \frac{1}{|\ell'_\nu|^3} \]
there.

(c) For all \( u, u' \in \mathcal{X}_n \), \( |S^{(\pm)}_{n,\nu}(u, u') - S'_n(u, u')| \leq \Gamma_{13} e^{-2m_5|u-u'|} \).

**Proof.** (a) For any \( c_0'' > 0 \) we have, for \( |k| \geq c_0'' \) and \( |\text{Im} k_0| < 3m_5 \), analyticity and the desired bound follows from the representations (apply [6, Remark 10.b] with \( R = Q^{(-\,)}_{n,\nu} \), \( R_s = Q^{(+\,)}_{n,\nu} \) and use Remark 2.5 to give \( RD^{-1}R_s = Q_-D^{-1}Q_- \))
\[
S^{(+\,)}_{n,\nu} = D^{*-1}_n - D^{*-1}_n Q^{(+\,)}_{n,\nu} \Delta^{(n\,)} Q^{(-\,)}_{n,\nu} D^{*-1}_n \quad S^{(-\,)}_{n,\nu} = D^{-1}_n - D^{-1}_n Q^{(+\,)}_{n,\nu} \Delta^{(n\,)} Q^{(-\,)}_{n,\nu} D^{-1}_n \]
and Lemmas 3.2.d, 2.2.a, 2.6.b and 4.2.c.

**Internal Remark**

Note that
\[
\Delta^{(n\,)} = \Omega_n (\mathbb{I} + Q_n D^{-1}_n Q^{(-\,)}_{n,\nu} \Omega_n)^{-1} = \Omega_n (\mathbb{I} + Q^{(-\,)}_{n,\nu} D^{-1}_n Q^{(+\,)}_{n,\nu} \Omega_n)^{-1} \]
as in Remark 2.5. So

\[
[D_n^{-1} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1}] [D_n^{-1} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1}]
\]

\[
= \mathbb{I} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)} + D_n^{-1}Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1}Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}
\]

\[
= \mathbb{I} - D_n^{-1}Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)} + D_n^{-1}Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}
\]

\[
= \mathbb{I}
\]

\[
[(D_n^{-1})^{-1} + Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}] [D_n^{-1} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1}]
\]

\[
= \mathbb{I} - Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1} + Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}D_n^{-1} - Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1}
\]

\[
= \mathbb{I} - Q_{n,\nu}^{(+)}(\mathbb{I} + \Omega_nQ_{n,\nu}^{(-)}D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1})
\]

\[
= \mathbb{I} - Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}D_n^{-1} + Q_{n,\nu}^{(+)}\Omega_nQ_{n,\nu}^{(-)}D_n^{-1}
\]

\[
= \mathbb{I}
\]

For $|k| < c_0'$, with $c_0'$ to be shortly chosen sufficiently small, and $|\text{Im} k| < 3m_5$ we use the representation

\[
(\hat{S}_{n,\nu,k}^{(-)})^{-1}(\ell, \ell') = \hat{D}_n(k + \ell) \delta_{\ell,\ell'} + U_{n,\nu}^{(+)}(k, \ell) \hat{\Omega}_n(k) U_{n,\nu}^{(-)}(k, \ell') = D_{\ell,\ell'} + B_{\ell,\ell'} \quad (5.10)
\]

with

\[
D_{\ell,\ell'} = \hat{D}_n(k + \ell) \delta_{\ell,\ell'} + \begin{cases} a_n & \text{if } \ell, \ell' = 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
B_{\ell,\ell'} = \begin{cases} \hat{\Omega}_n(k) a_n(k)^{2q} - a_n & \text{if } \ell = \ell' = 0 \\ U_{n,\nu}^{(+)}(k, \ell) \hat{\Omega}_n(k) U_{n,\nu}^{(-)}(k, \ell') & \text{otherwise} \end{cases}
\]

and the obvious analog for $(\hat{S}_{n,\nu,k}^{(+)})^{-1}$ — just replace $\hat{D}_n(k + \ell)$ with its complex conjugate. As in the proof of Lemma 5.4.b, $D$ is invertible and the inverse is a diagonal matrix with every diagonal matrix element obeying

\[
|D_{\ell,\ell'}^{-1}| \leq \frac{\Gamma_{11}}{1 + |\ell_0| + \sum_{|\ell_0| = 1} |\ell_0|^2}
\]

for some $\Gamma_{11}$, which is independent of $c_0'$. By parts (b) and (c) of Lemma 2.2, parts (a) and (b) of Proposition 2.4, and part (b) of Lemma 2.6,

\[
|B_{\ell,\ell'}| \leq \text{const}_{a,q} |k| \prod_{\nu=0}^3 \left(\frac{24}{|\ell_0| + \pi}\right)^{q-1} \prod_{\nu=0}^3 \left(\frac{24}{|\ell_0| + \pi}\right)^q
\]

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So if $|k| < c'_0$ with $c'_0$ small enough, $D + B$ is invertible with the inverse given by the Neumann expansion $D^{-1} + \sum_{p=1}^{\infty} (-1)^p D^{-1} (BD^{-1})^p$. As $q > 1$, the desired analyticity and the desired bound on $|S_{n;\nu;k}(\ell,\ell')|$ follow as in the proof of Lemma 5.4.b

(b) is proven just as Lemma 5.4.c.

(c) follows from part (b) by [5, Lemma 12.b].

Proof of Proposition 5.1. Set $m_3 = \min\{m_4, m_5\}$. As $|u_0|^2 + |u|^4$ is locally integrable in $\mathbb{R}^4$, the pointwise bounds on $|S_n(u, u') - S'_n(u, u')|$ and $|S'_n(u, u')|$, given in Lemma 5.4.d, and on $|S_{n;\nu}(u, u') - S'_n(u, u')|$, given in Lemma 5.5.c, imply

$$
\| S_n \|_{m_3}, \| S_{n;\nu} \|_{m_3} \leq \tilde{\Gamma}_{11}
$$

with $\tilde{\Gamma}_{11}$ a constant, depending only on $m_3$, times $\max\{\Gamma_{12}, \Gamma_{13}\}$. Setting $\mu_{up} = \frac{1}{2\tilde{\Gamma}_{11}}$ and $\Gamma_{11}$ to be the maximum of $2\tilde{\Gamma}_{11}$ (for $\| S_n(\mu) \|_{m_3}$ and $\| S_{n;\nu}(\mu) \|_{m_3}$) and $2\tilde{\Gamma}_{11}^2$ (for $\| S_n(\mu) - S_n \|_{m_3}$ and $\| S_{n;\nu}(\mu) - S_{n;\nu} \|_{m_3}$) a Neumann expansion gives the specified bounds.

We now formulate and prove two more technical lemmas that will be used elsewhere.

Lemma 5.6. There are constants $m_6 > 0$ and $\Gamma_{14}$ such that, for all $L > \Gamma_2$,

$$
\left| (S_n Q^*_n)(y, x) \right| \leq \Gamma_{14} e^{-2m_6|x-y|} \quad \| S_n Q^*_n \|_{m_6} \leq \Gamma_{14}
$$

Proof. From the definitions of $S_n$, in (5.3), and $\Delta^{(n)}$, at the beginning of §4, one sees directly that

$$
S_n^{(s)} Q^*_n = D_n^{(s)-1} Q^*_n \Delta^{(n)(s)} Q^{-1}_n : L^2(\mathcal{X}_{\text{crs}}) \to L^2(\mathcal{X}_{\text{fin}}) \quad (5.11)
$$

The Fourier transform of the kernel, $b(y, x)$, of the operator $S_n Q^*_n$ is
To verify that \( S_n Q_n = D_n^{-1} Q_n \Delta^{(n)} \Omega^{-1} \) we compute
\[
S_n^{-1} D_n^{-1} Q_n \Delta^{(n)} = (D_n + Q_n \Omega_n \Omega_n^{-1}) D_n^{-1} Q_n (I + \Omega_n Q_n D_n^{-1} Q_n) -1 \Omega_n
\]
\[
= Q_n^*(I + \Omega_n Q_n D_n^{-1} Q_n) -1 \Omega_n
\]
\[
= Q_n^*(I + \Omega_n Q_n D_n^{-1} Q_n) -1 \Omega_n
\]

To verify that \( S_n^* Q_n = D_n^{-1} Q_n \Delta^{(n)*} \Omega^{-1} \) we compute
\[
S_n^{-1} D_n^{-1} Q_n \Delta^{(n)*} = (D_n + Q_n^* \Omega_n \Omega_n^{-1}) D_n^{-1} Q_n^* (I + \Omega_n Q_n D_n^{-1} Q_n) -1 \Omega_n
\]
\[
= Q_n^*(I + \Omega_n Q_n D_n^{-1} Q_n) -1 \Omega_n
\]
\[
= Q_n^* \Omega_n
\]

\[
\hat{b}_k(\ell) = \hat{D}_n^{-1}(k + \ell) u_n(k + \ell) \Delta^{(n)} \frac{1}{\Omega_n(k)}
\]

By Lemma 4.2.a,c,f, Remark 3.1.b and Lemma 3.2.d, Proposition 2.4.a, Remark 2.1.d and Lemma 2.2.a, \( \hat{b}_k(\ell) \) is analytic in \(|\text{Im } k| < 3m_6 \) and
\[
|\hat{b}_k(\ell)| \leq \frac{5}{m_6} \Gamma_6 \sum_{\nu=1}^{3} |k_\nu + \ell_\nu|^2 \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_\nu| + \pi} \right)^q
\]
The bound is uniform in \( n \) and \( L \) and is summable in \( \ell \), so the claims follow from [5, Lemma 12.c].

\textbf{Lemma 5.7.} \textit{There are constants } \( m_7 > 0 \) \textit{and } \( \Gamma_15 \)

\textbf{Internal Remark}

They depend only on \( q, h_0 \) and \( a \) and in particular are independent of \( n \) and \( L > \Gamma_2 \), such that, for all \( L > \Gamma_2 \), the operators
\[
(S_n(\mu)Q_n^* \Omega_n)^+ (S_n(\mu)Q_n^* \Omega_n), \quad (S_n(\mu)^*Q_n^* \Omega_n)^+ (S_n(\mu)^*Q_n^* \Omega_n)
\]
\[
(S_n^{(\pm)}(\mu)Q_n^{(\pm)} \Omega_n)^+ (S_n^{(\pm)}(\mu)Q_n^{(\pm)} \Omega_n)
\]
all have bounded inverses. The \( \| \cdot \|_{m_7} \) norms of the inverses are all bounded by \( \Gamma_15 \).
Proof. We first consider the case that \( \mu = 0 \). By (5.11), the operator
\[
S_n Q_n^* \Omega_n = D_n^{-1} Q_n^* \Delta^{(n)} : L^2(\mathcal{X}_{\text{rss}}) \to L^2(\mathcal{X}_{\text{fin}})
\]
has Fourier transform
\[
\tilde{b}_k(\ell) = \hat{D}_n^{-1}(k + \ell)u_n(k + \ell)\hat{\Delta}^{(n)}(k)
\]

Internal Remark
To verify that \( S_n Q_n^* \Omega_n = D_n^{-1} Q_n^* \Delta^{(n)} \) we compute
\[
S_n^{-1} D_n^{-1} Q_n^* \Delta^{(n)} = (D_n + Q_n^* \Omega_n Q_n) D_n^{-1} Q_n^* \Omega_n (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
\[
= Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n + (Q_n^* \Omega_n Q_n - \mu) D_n^{-1} Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
\[
= Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
To verify that \( S_n^* Q_n^* \Omega_n = D_n^{-1} Q_n^* \Delta^{(n)*} \) we compute
\[
S_n^{-1} D_n^{-1} Q_n^* \Delta^{(n)*} = (D_n + Q_n^* \Omega_n Q_n) D_n^{-1} Q_n^* \Omega_n (\mathbb{I} + Q_n D_n^{-1} Q_n^* \Omega_n)^{-1}
\]
\[
= Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n + Q_n^* \Omega_n Q_n D_n^{-1} Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
\[
= Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
To verify that \( S_{n,\nu}^{-1} Q_{n,\nu}^* \Omega_n = D_n^{-1} Q_{n,\nu}^* \Delta^{(n)} \), we compute
\[
(S_{n,\nu}^{-1})^{-1} D_n^{-1} Q_{n,\nu}^* \Delta^{(n)} = (D_n + Q_{n,\nu}^* \Omega_n Q_{n,\nu}) D_n^{-1} Q_{n,\nu}^* \Omega_n (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
\[
= Q_{n,\nu}^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n + Q_{n,\nu}^* \Omega_n Q_n D_n^{-1} Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
\[
= Q_{n,\nu}^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
by Remark 2.5.
To verify that \( S_{n,\nu} Q_{n,\nu}^* \Omega_n = D_n^{-1} Q_{n,\nu}^* \Delta^{(n)*} \), we compute
\[
(S_{n,\nu}^{-1})^{-1} D_n^{-1} Q_{n,\nu}^* \Delta^{(n)*} = (D_n + Q_{n,\nu}^* \Omega_n Q_{n,\nu}) D_n^{-1} Q_{n,\nu}^* \Omega_n (\mathbb{I} + Q_n D_n^{-1} Q_n^* \Omega_n)^{-1}
\]
\[
= Q_{n,\nu}^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n + Q_{n,\nu}^* \Omega_n Q_n D_n^{-1} Q_n^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
\[
= Q_{n,\nu}^* (\mathbb{I} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n
\]
by Remark 2.5.
The operator \((S_nQ_n^{\ast}\Omega_n)^{*}(S_nQ_n^{\ast}\Omega_n)\) maps \(L^2(\mathcal{X}_{\text{crs}})\) to \(L^2(\mathcal{X}_{\text{crs}})\) and has Fourier transform
\[
\sum_{\ell \in B} \tilde{b}_{-k}(-\ell) \tilde{b}_k(\ell) = \sum_{\ell \in B} \hat{D}_n^{-1}(-k - \ell) \hat{D}_n^{-1}(k + \ell) u_n(k + \ell)^{2q} \hat{\Delta}^{(n)}(-k) \hat{\Delta}^{(n)}(k)
\]
For \(k\) real,
\[
\sum_{\ell \in B} \tilde{b}_{-k}(-\ell) \tilde{b}_k(\ell) = \sum_{\ell \in B} \left| \hat{D}_n^{-1}(k + \ell) u_n(k + \ell)^{q} \hat{\Delta}^{(n)}(k) \right|^2 \\
\geq \left| \hat{D}_n^{-1}(k) u_n(k)^{q} \hat{\Delta}^{(n)}(k) \right|^2 \\
\geq \gamma_2^2 \inf_{|k| \leq \pi} |u_n(k)|^{2q} \\
\geq \gamma_2^2 \left( \frac{2}{\pi} \right)^{8q}
\]
by Lemmas 4.2.e and 2.2.f. To show that half this the lower bound extends into a strip along the real axis that has width independent of \(n\) and \(L\), we observe that
- all first order derivatives of \(u_n(k + \ell)\) are uniformly bounded by \(2 \prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi}\) on such a strip by the Cauchy integral formula and Remark 2.1.d and Lemma 2.2.a and
- \(u_n(k + \ell)\) itself is uniformly bounded by \(\prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi}\) on such a strip by Lemma 2.2.a and
- all first order derivatives of \(\hat{\Delta}^{(n)}(k)\) are uniformly bounded on such a strip by Lemma 4.2.c and
- for \(\ell \neq 0\), all first order derivatives of \(\hat{D}_n^{-1}(k + \ell)\) are uniformly bounded on such a strip by parts (c) and (d) of Lemma 3.2 and
- for \(\ell = 0\), all first order derivatives of \(\hat{D}_n^{-1}(k)\hat{\Delta}^{(n)}(k)\) are uniformly bounded on such a strip by Lemma 4.2.f.

The operator \((S_nQ_n^{\ast}\Omega_n)^{*}(S_nQ_n^{\ast}\Omega_n)\) has Fourier transform \(\tilde{b}_{-k}(-\ell)\). So
\[
(S_nQ_n^{\ast}\Omega_n)^{*}(S_nQ_n^{\ast}\Omega_n) = (S_nQ_n^{\ast}\Omega_n)^{*}(S_nQ_n^{\ast}\Omega_n)
\]
The operators \(S_{n,\nu}^{(-)}Q_{n,\nu}^{(-)}\Omega_n = D_n^{-1}Q_{n,\nu}^{(-)}\Delta^{(n)}\) and \(S_{n,\nu}^{(+)}Q_{n,\nu}^{(+)}\Omega_n = D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}\) map \(L^2(\mathcal{X}_{\text{crs}})\) to \(L^2(\mathcal{X}_{\text{fin}})\) and have Fourier transforms
\[
\tilde{c}_n(\ell) = \hat{D}_n^{-1}(\pm k \pm \ell) \zeta_{n,\nu}^{(+)}(k, \ell) u_{n,\nu}(k) u_n(k + \ell)^{q-1} \hat{\Delta}^{(n)}(\pm k)
\]
So the operators \((S_nQ_n^{\ast}^{(-)}Q_n^{(-)}\Omega_n)^{*}(S_nQ_n^{\ast}^{(+)}Q_n^{(+)}\Omega_n)\) both map \(L^2(\mathcal{X}_{\text{crs}})\) to \(L^2(\mathcal{X}_{\text{crs}})\) and have
Fourier transform

\[ \sum_{\ell \in \mathcal{B}} \tilde{c}_{-k}(-\ell) \tilde{c}_k(\ell) = \sum_{\ell \in \mathcal{B}} \hat{D}^{-1}_n(-k - \ell) \hat{D}^{-1}_n(k + \ell) u_n^{(\ell)}(k + \ell)^2 u_n(k + \ell)^2 (q-1) \hat{\Delta}^{(n)}(-k) \hat{\Delta}^{(n)}(k) \]

This is bounded just as \( \sum_{\ell \in \mathcal{B}} \tilde{b}_{-k}(-\ell) \tilde{b}_k(\ell) \) was. The specified bounds, in the special case that \( \mu = 0 \) follow.

By Proposition 5.1, a Neumann expansion gives the desired bounds when \( \mu \) is nonzero.

\( \square \)
6 The Degree One Part of the Critical Field

In [9, Proposition 5.1 and (5.3)] we derive an expansion for the critical fields of the form

\[ \psi(n)(\theta^*, \theta, \mu, \mathcal{V}) = \frac{a}{L_2} C(n)(\mu)^{(s)} Q^*(\theta_{(s)}) + \psi_{(s)n}^{(\geq 3)}(\theta_{(s)}, \theta, \mu, \mathcal{V}) \]

with the \( C(n)(\mu) \) of [7, Proposition 1.15] and with \( \psi_{(s)n}^{(\geq 3)} \) being of degree at least 3 in \( \theta_{(s)} \). In this section we derive bounds on a scaled version of \( C(n)(\mu)^{(s)} Q^*(\theta_{(s)}) \) and some related operators. To do so we use the representation

\[ \frac{a}{L_2} C(n)(\mu)^{(s)} Q^* = \left( \frac{a}{L_2} Q^* Q + \mathcal{Q}_n \right)^{-1} \left\{ \frac{a}{L_2} Q^* + \mathcal{Q}_n Q_n \tilde{S}_{n+1}(\mu)^{(s)} \tilde{Q}_{n+1}^* \tilde{Q}_{n+1} \right\} \]  

(6.1)

of [9, (5.2)]. Here, as in [9, Lemma 2.4 and (5.1)],

\[ \tilde{Q}_{n+1} = L_2 Q_{n+1} L_{n+1}^{-1} : \mathcal{H}_n \to \mathcal{H}_{(n+1)} \]

\[ \tilde{Q}_{n+1}^{-1} = L_2 L_{n+1}^{-1} L_{n+1}^{-1} \]

\[ \tilde{S}_{n+1}(\mu) = L_2 S_{n+1} (L_2^2 \mu L_{n+1}^{-1})^{-1} : \mathcal{H}_n \to \mathcal{H}_{(n+1)} \]

(6.2)

These operators are all translation invariant with respect to \( \mathcal{A}_{(n+1)} \). As

\[ \tilde{S}_{n+1}(\mu) = \tilde{S}_{n+1} + \mu \tilde{\mathcal{S}}_{n+1} \tilde{\mathcal{S}}_{n+1}(\mu) \quad \text{with} \quad \tilde{\mathcal{S}}_{n+1} = \tilde{\mathcal{S}}_{n+1}(0) \]

we have

\[ \frac{a}{L_2} C(n)(\mu)^{(s)} Q^* = \frac{a}{L_2} C(n)(\mu)^{(s)} Q^* + \mu A_{\psi,\phi} \tilde{S}_{n+1}(\mu)^{(s)} \tilde{Q}_{n+1}^* \tilde{Q}_{n+1} \]  

(6.3)

where

\[ A_{\psi,\phi} = (a L^{-2} Q^* Q + \mathcal{Q}_n)^{-1} \mathcal{Q}_n Q_n : \mathcal{H}_n \to \mathcal{H}_{(n)} \]

(6.4)

The operator \( A_{\psi,\phi} \) is also used in the course of bounding \( \psi_{(s)n}^{(\geq 3)} \) in [9, Proposition 5.1].

The main results of this section are

**Proposition 6.1.** There are constants \( m_8 > 0 \) and \( \Gamma_{16}, \Gamma_{17} \)

**Internal Remark**

\( \Gamma_{17} \) depends only on \( q \). \( m_8 \) and \( \Gamma_{16} \) depend only on \( q, h_0 \) and \( a \), and in particular are independent of \( n \) and \( L \).

\[^{5}\text{Recall Convention 1.2.}\]
such that the following holds, for each $L > \Gamma_{17}$ and each $\mu$ obeying $|L^2 \mu| \leq \mu_{up}$.

(a) $\|L_{s}^{-1} A_{\psi, \phi} L_{s}\|_{m=1} \leq \Gamma_{16}$ and $\|L_{s}^{-1} \frac{a}{L^2} C^{(n)}(\mu)(s) Q^* L_{s}\|_{m=8} \leq \Gamma_{16}$

(b) Let $0 \leq \nu \leq 3$. There are operators $A_{\psi, \phi, \nu}$ and $A_{\psi, \theta, \nu}(\mu)$ such that

\[
\partial_{\nu} A_{\psi, \phi} = A_{\psi, \phi, \nu} \partial_{\nu} \text{ and } \partial_{\nu} \frac{a}{L^2} C^{(n)}(\mu)(s) Q^* = A_{\psi, \theta, \nu}(\mu) \partial_{\nu}
\]

and

\[
\|L_{s}^{-1} A_{\psi, \phi, \nu} L_{s}\|_{m=1} \leq \Gamma_{16} \text{ and } \|L_{s}^{-1} A_{\psi, \theta, \nu}(\mu) L_{s}\|_{m=8} \leq \Gamma_{16}
\]

This proposition is proven at the end of this section, after Lemma 6.6. In this proof we write $\frac{a}{L^2} C^{(n)}(s) Q^* = A_{\psi, \theta, \nu}$ so that

\[
A_{\psi, \theta, \nu} = (aL^{-2}Q^* + \Omega_{n})^{-1}\{aL^{-2}Q^* + \Omega_{n}Q_{n} \tilde{S}_{n+1} \tilde{Q}_{n+1} \tilde{\Delta}_{n+1}\} : \mathcal{H}_{-1}^{(n+1)} \to \mathcal{H}_{0}^{(n)}
\]

**Remark 6.2.**

(a) $A_{\psi, \theta, \nu} = \Omega_{n}^{-1} Q^* \tilde{\Delta}_{n+1} + A_{\psi, \phi} D_{n+1}^{-1}(s) \tilde{Q}_{n+1} \tilde{\Delta}_{n+1}(s)$ with

\[
\tilde{\Delta}_{n+1}(s) = \frac{1}{L^2} L_{s} \Delta_{n+1}(s) L_{s}^{-1} = \{I + \tilde{\Delta}_{n+1} \tilde{Q}_{n+1} D_{n+1}^{-1}(s) \tilde{Q}_{n+1}^{*}\}^{-1} \tilde{\Delta}_{n+1}
\]

being a fully translation invariant operator on $\mathcal{H}_{-1}^{(n+1)}$.

(b) Let $0 \leq \nu \leq 3$. We have $\partial_{\nu} A_{\psi, \phi} = A_{\psi, \phi, \nu} \partial_{\nu}$ and $\partial_{\nu} A_{\psi, \theta, \nu}(\mu) = A_{\psi, \theta, \nu}(\mu) \partial_{\nu}$ where

\[
\begin{align*}
A_{\psi, \phi, \nu} & = [I - \Omega_{n}^{-1} Q^{(+)}(\psi, \phi) \tilde{Q}_{n+1} Q^{(-)}(\psi, \phi)] Q_{n, \nu} \\
A_{\psi, \theta, \nu}(\mu) & = \Omega_{n}^{-1} Q^{(+)}(\psi, \theta) \tilde{Q}_{n+1} + A_{\psi, \phi, \nu} L_{s} \Delta_{n+1}(s) Q^{(+)}(\psi, \phi) \tilde{Q}_{n+1} \tilde{\Delta}_{n+1}(s) L_{s}^{-1} \\
A_{\psi, \theta, \nu}(\mu) & = A_{\psi, \theta, \nu} + L^2 \mu A_{\psi, \phi, \nu} L_{s} S^{(+)}(\psi, \phi) S_{n+1, \nu}^{(+)}(\mu) Q_{n+1, \nu} \tilde{\Delta}_{n+1}(s) L_{s}^{-1} \\
A_{\psi, \theta, \nu}(\mu) & = A_{\psi, \theta, \nu} + L^2 \mu A_{\psi, \phi, \nu} L_{s} S^{(-)}(\psi, \phi) S_{n+1, \nu}^{(-)}(\mu) Q_{n+1, \nu} \tilde{\Delta}_{n+1}(s) L_{s}^{-1}
\end{align*}
\]

**Proof.** (a) First observe that, by (5.11),

\[
Q_{n} \tilde{S}_{n+1} \tilde{Q}_{n+1} \tilde{\Delta}_{n+1} = Q_{n} L_{s} D_{n+1}^{-1}(s) \tilde{Q}_{n+1}^{*} \Delta_{n+1}(s) L_{s}^{-1} = (Q_{n} L_{s}^{-1}(s) Q_{n}^{*}) \tilde{Q}_{n+1} \tilde{\Delta}_{n+1}(s) \tilde{Q}_{n+1}^{*} \Delta_{n+1}(s)
\]

(6.5)

**Internal Remark**
Assume that \( \{ \mathbb{1} + \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \tilde{Q}_{n+1}^* \}^{-1} \) and \( \{ \mathbb{1} + \tilde{Q}_{n+1}^* \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \}^{-1} \) exist. Then

\[
\{ \mathbb{1} + \tilde{Q}_{n+1}^* \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \} \tilde{Q}_{n+1} \left\{ \mathbb{1} + \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \tilde{Q}_{n+1}^* \right\}^{-1} = \tilde{Q}_{n+1}^* \left\{ \mathbb{1} + \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \right\}^{-1} \tilde{Q}_{n+1}^*
\]

implies that

\[
\tilde{Q}_{n+1}^* \left\{ \mathbb{1} + \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \right\}^{-1} \tilde{Q}_{n+1} = \left\{ \mathbb{1} + \tilde{\mathcal{Q}}_{n+1} \tilde{Q}_{n+1} D_{n}^{-1(s)} \right\}^{-1} \tilde{Q}_{n+1}^*
\]

Using (6.5), the operator

\[
A_{\psi,\theta} = (a L^{-2} Q^* Q + \mathcal{Q}_{n})^{-1} \left\{ a L^{-2} Q^* + \mathcal{Q}_{n} Q_n D_{n}^{-1(s)} \tilde{Q}_{n+1}^* \Delta (\Omega_n) \right\}
\]

(b) By Remark 2.5,

\[
\partial_{\nu} A_{\psi,\theta} = \partial_{\nu} (a L^{-2} \mathcal{Q}_{n}^{-1} Q^* Q)^{-1} Q_n
\]

Therefore by part (a), (3.1), (6.2) and Remark 2.5,

\[
\partial_{\nu} A_{\psi,\theta} = \partial_{\nu} \left[ \mathcal{Q}_{n}^{-1} Q^* \tilde{\mathcal{Q}}_{n+1} + A_{\psi,\phi} L_s D_{n+1}^{-1(s)} Q_{n+1}^* \Delta (\Omega_{n+1}) \right] = A_{\psi,\theta} \partial_{\nu}
\]

since \( \partial_{\nu} \mathcal{Q} = \frac{1}{L_{\nu}} \mathcal{Q} \partial_{\nu} \) by [7, Remark 2.2.a.b]. To get \( \partial_{\nu} A_{\psi,\theta} (\mu) = A_{\psi,\theta} (\mu) \partial_{\nu} \) when \( \mu \neq 0 \), write, using (6.2),

\[
A_{\psi,\theta} (\mu) = A_{\psi,\theta} (\mu) + \mu A_{\psi,\theta} \mathcal{S}_{n+1} (\mathcal{S}_{n+1} (L^2 \mu)^{s} Q_{n+1}^* \tilde{\mathcal{Q}}_{n+1})
\]

and use (5.1), Remark 2.5 and the fact that \( \mathcal{Q}_{n+1} \) is fully translation invariant. \( \square \)
The operators of principal interest, $A_{\psi^*,\theta^*}$ and $A_{\psi,\theta}$, act from $L^2(\mathcal{X}_{\text{crs}}) = \mathcal{H}^{(n+1)}_{-1}$ to $L^2(\mathcal{X}_{\text{fin}}) = \mathcal{H}^{(n)}_0$ with

$$\mathcal{X}_{\text{fin}} = \mathcal{X}^{(n)}_0 \quad \mathcal{X}_{\text{crs}} = \mathcal{X}^{(n+1)}_{-1} \quad \mathcal{B} = \mathcal{B}^+$$

We now give a bunch of Fourier transforms (in the sense of [5, (7) and (8)] – but we shall suppress the † from the notation). All of the operators above are periodized in the sense of [5, Definition 2]. As before we denote

- momenta dual to the $L$–lattice $L^2\mathbb{Z} \times L\mathbb{Z}^3$ by $\ell \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$,
- momenta dual to the unit lattice $\mathbb{Z} \times \mathbb{Z}^3$ by $k \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$ and decompose $k = \ell + \ell'$ with $\ell$ in a fundamental cell for $(\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$ and $\ell, \ell' \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3) = \hat{B}^+$ and
- momenta dual to the $\varepsilon_j$–lattice $\mathbb{R}^2 \times \mathbb{R}^3$ by $p_j \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$ and decompose $p_j = k + \ell_j$ with $k$ in a fundamental cell for $(\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$ and $\ell_j \in (2\pi \mathbb{Z}/2\varepsilon_j) \times (2\pi \mathbb{Z}^3/2\varepsilon_j) = \hat{B}_j$. Here $1 \leq j \leq n$.

The Fourier transform of $A_{\psi^*,\theta^*}$ is

$$\left( A_{\psi^*,\theta^*} \right)^\dag (\ell) = \sum_{\ell' \in \hat{B}^+} \left( \frac{\partial^2}{\partial \mathbf{y}^2} q^* Q + \Omega_n \right)_{\ell}^{-1} (\ell, \ell') 
\left\{ \frac{\partial^2}{\partial \mathbf{y}^2} q^* Q_{\ell'}(\ell') + \Omega_n (\ell + \ell') (Q_n D_n^{-1(\ast)} Q_n^*) (\ell + \ell') Q_{\ell'} (\ell') \Delta^{(n+1)(\ast)} (\mathbf{y}) \right\}$$

(6.6)

where, by Remark 2.1, Remark 6.2.a and (6.2)

$$(aL^{-2} q^* Q + \Omega_n)_{\ell}(\ell, \ell') = \Omega_n (\ell + \ell') \delta_{\ell, \ell'} + aL^{-2} u_+ (\ell + \ell') q u_+ (\ell + \ell') q$$

and

$$Q_{\ell}^* (\ell) = u_+ (\ell + \ell') q$$

$$\Omega_n (k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in \hat{B}_j} \frac{1}{L^2} u_j (k + \ell_j)^{2q} \right]^{-1}$$

$$(Q_n D_n^{-1(\ast)} Q_n^*) (k) = \sum_{\ell_n \in \hat{B}_n} u_n (k + \ell_n)^{2q} D_n^{1(\ast)} (k + \ell_n)$$

$$\Delta^{(n+1)(\ast)} (\mathbf{y}) = \frac{\tilde{\Omega}_{n+1}(\mathbf{y})}{1 + \tilde{\Omega}_{n+1}(\mathbf{y})} \sum_{\ell_n \in \hat{B}_n} u_n (\ell + \ell + \ell_n)^{2q} u_+ (\ell + \ell)^{2q} D_n^{-1(\ast)} (\ell + \ell + \ell_n)$$

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Lemma 6.3. Let $|\Im t_\nu| \leq \frac{2}{3L^2}$ for each $0 \leq \nu \leq 3$. There is a constant $\Gamma_{17}$, depending only on $q$, such that the following hold for all $L > \Gamma_{17}$.

(a) We have \( \frac{6}{7} aL^2 \leq |\tilde{\mathcal{Q}}_{n+1}(t)| \leq \frac{6}{5} aL^2 \) and \( \Re \tilde{\mathcal{Q}}_{n+1}(t) \geq \frac{a}{2L^2} \).

(b) We have
\[
\left| (aL^{-2}Q^*Q + \mathcal{Q}_n)^{-1}(\ell, \ell') - \mathcal{Q}_n(\ell + \ell')^{-1}\delta_{\ell, \ell'} \right| \\
\leq \frac{2}{aL^2} \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|t_{\nu}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|t_{\nu}'| + \pi} \right)^q \left\{ \prod_{0 \leq \nu \leq 3} L_{\nu}|t_{\nu}| \right\} \text{ for all } \ell \\
\leq \frac{5}{4L^2} \prod_{\nu=0}^{3} \left( \frac{24}{|t_{\nu}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|t_{\nu}'| + \pi} \right)^q \prod_{0 \leq \nu \leq 3} L_{\nu}|t_{\nu}| \text{ if } \ell \neq 0
\]

(c) Let $0 \leq \nu \leq 3$. Then
\[
\left| \left[\mathcal{Q}_n^{-1}Q_{+, \nu}^+\tilde{\mathcal{Q}}_{n+1}Q_{+, \nu}^{-1}\right]_{\ell, \ell'} \right| \leq \frac{3e^4}{2L^4} \left( \frac{24}{\pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|t_{\nu}'| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|t_{\nu}| + \pi} \right)^q
\]

Proof. (a) is proven much as Proposition 2.4.a was.

Internal Remark
If $L > \Gamma_2$, then, by Lemma 2.3.a and Proposition 2.4.a,
\[
\sum_{\ell \in B^+} \frac{a}{L^2} |u_+(t + \ell)2q\mathcal{Q}_n(\ell + \ell')^{-1}| \leq \frac{a}{L^2} \sum_{\ell \in B^+} \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|t_{\nu}| + \pi} \right)^q \\
\leq \frac{5}{4L^2} \sum_{\ell \in B^+} \prod_{\nu=0}^{3} \left( \frac{24}{|t_{\nu}| + \pi} \right)^q \\
\leq \frac{5\Gamma_{17}}{4L^2}
\]
So it suffices to pick $\Gamma_{17} \geq \Gamma_2$ large enough that \( \frac{5\Gamma_{17}}{4L^2} < \frac{1}{6} \).

Internal Internal Remark
\[
\Re \tilde{\mathcal{Q}}_{n+1}(t) \geq \frac{a}{L^2} \left[ 1 + \frac{1}{4} \right] \geq \frac{a}{L^2} \left[ 1 + \frac{1}{4} \right] \geq \frac{5}{2L^2} \geq \frac{a}{2L^2}
\]
(b) The straightforward Neumann expansion gives
\[
\left| \left( \frac{\partial^2}{\partial \ell' \partial \ell} + \Omega_n \right)^{-1} (t, \ell') - \Omega_n (t + \ell)^{-1} \delta_{\ell, \ell'} \right|
\leq \frac{a}{L^2} \left| \Omega_n (t + \ell)^{-1} u_+ (t + \ell) u_+ (t + \ell')^3 \Omega_n (t + \ell')^{-1} \right| \sum_{j=0}^{\infty} \left| \sum_{l \in \mathbb{B}^+} L^2 \frac{u_+ (t+l)^2 a}{\Omega_n (t+l)} \right| j
\leq \frac{a}{L^2} \left( \frac{5}{\ell^2} \right)^2 u_+ (t + \ell) u_+ (t + \ell') \prod_{q=0}^{\infty} \left( \frac{24}{L_\nu |t_\nu'| + \pi} \right)^q \prod_{j=0}^{\infty} \left[ \frac{M_{\nu}'}{24L^2} \right]^j
\leq \frac{2}{aL^2} \prod_{q=0}^{3} \left( \frac{24}{L_\nu |t_\nu'| + \pi} \right)^q \prod_{q=0}^{3} \left( \frac{24}{L_\nu |t_\nu'| + \pi} \right)^q \prod_{0 \leq q \leq 3} L_\nu |t_\nu'| \quad \text{if } \ell \neq 0
\]
by part (a), Lemma 2.3.a,b and Proposition 2.4.a, with \( L \) satisfying the conditions of part (a).

(c) The specified bound follows from (2.12), part (a), Proposition 2.4.a and Lemmas 2.6.b, 2.3.a. \( \square \)

**Lemma 6.4.** For all
\[
\begin{align*}
t &= L^{-1}(k) \in \left( \mathbb{R}/2\pi \mathbb{Z} \right) \times \left( \mathbb{R}^3/2\pi \mathbb{Z}^3 \right) & k & \in \left( \mathbb{R}/2\pi \mathbb{Z} \right) \times \left( \mathbb{R}^3/2\pi \mathbb{Z}^3 \right) \\
p &= L^{-1}(p) \in \left( \mathbb{R}/2\pi \mathbb{Z} \right) \times \left( \mathbb{R}^3/2\pi \mathbb{Z}^3 \right) & p & \in \left( \mathbb{R}/2\pi \mathbb{Z} \right) \times \left( \mathbb{R}^3/2\pi \mathbb{Z}^3 \right) \\
\ell_n \in \left( \mathbb{Z}/n+1 \mathbb{Z} \right) \times \left( \mathbb{Z}/n+1 \mathbb{Z} \right)
\end{align*}
\]
we have

(a) \( u_+ (L^{-1}(k)) = u_1(k) \quad \text{and} \quad u_n (L^{-1}(k)) u_+ (L^{-1}(k)) = u_{n+1}(k) \) for all \( n \in \mathbb{N} \).

(b) \( \mathcal{Q}_{n+1}(L^{-1}(k)) = \frac{1}{L^2} \mathcal{Q}_{n+1}(k) \)

(c) \( D_{n+1}(L^{-1}(p)) = L^2 D_{n+1}(p) \)

(d) \( \tilde{\Delta}(n+1)(k) = \frac{1}{L^2} \Delta(n+1)(k) \)

(e) \( \tilde{\Delta}_{n+1,n} L^{-1}(k) (L^{-1}(\ell_n), L^{-1}(\ell_n')) = L^2 \tilde{\Delta}_{n+1,n} (\ell_n, \ell_n') \)

**Proof.** These all follow from (6.2), Remark 6.2.a and
\( Q_1 = \mathbb{L}_*^{-1}Q\mathbb{L}_* \) by (2.2)
\( D_{n+1} = L^2\mathbb{L}_*^{-1}D_n\mathbb{L}_* \) by (3.1)
and [5, Lemmas 15.b and 16.b].

**Internal Remark**

(a) Directly from the definitions, given just before Lemma A.1,

\[
u_+(\mathbb{L}^{-1}(k)) = \prod_{\nu=0}^{3} \frac{\sin \frac{1}{2}k_\nu}{L_\nu \sin \frac{1}{2\nu}k_\nu} = u_1(k)
\]

\[
u_n(\mathbb{L}^{-1}(k))\nu_+(\mathbb{L}^{-1}(k)) = \frac{1}{\varepsilon_n^3} \prod_{\nu=1}^{3} \frac{\sin \frac{1}{2\nu}k_\nu}{\varepsilon_n^2 \sin \frac{1}{2\nu+1}k_\nu} \prod_{\nu=0}^{3} \frac{\sin \frac{1}{2}k_\nu}{L_\nu \sin \frac{1}{2\nu}k_\nu} = u_{n+1}(k)
\]

(b) Making the change of summation variable \( l_1 = \mathbb{L}(\ell) \),

\[
\mathcal{Q}_{n+1}(\mathbb{L}^{-1}(k))^{-1} = \frac{L^2}{a} \left\{ 1 + \sum_{\ell \in B^+} \frac{a}{L^2} \nu_+(\mathbb{L}^{-1}(k) + \ell)^2 \mathcal{Q}_n(\mathbb{L}^{-1}(k) + \ell)^{-1} \right\}
\]

\[
= \frac{L^2}{a} \left\{ 1 + \sum_{l_1 \in B_1} \frac{1}{L^2} \nu_+((k + l_1))^{2q} \left( 1 + \sum_{j=1}^{n-1} \sum_{l_j \in B_j} \frac{1}{L^2} \nu_j((k + l_1 + \mathbb{L}(l_j))^{2q} \right) \right\}
\]

\[
= \frac{L^2}{a} \left\{ 1 + \sum_{l_1 \in B_1} \frac{1}{L^2} \nu_1(k + l_1)^{2q} + \sum_{j=1}^{n-1} \sum_{l_j \in B_j} \frac{1}{L^2} \nu_{j+1}(k + l_1 + \mathbb{L}(l_j))^{2q} \right\}
\]

\[
= \frac{L^2}{a} \left\{ 1 + \sum_{l_1 \in B_1} \frac{1}{L^2} \nu_1(k + \ell_1)^{2q} + \sum_{j=1}^{n-1} \sum_{\ell_{j+1} \in B_{j+1}} \frac{1}{L^2} \nu_{j+1}(k + \ell_{j+1})^{2q} \right\}
\]

\[
= L^2 \mathcal{Q}_{n+1}(k)^{-1}
\]
(c) Recall that 
\[ \hat{D}_n(p) = \frac{1}{2\varepsilon_n p_0} e^{-\frac{\varepsilon_0(p_n p_0)}{2\varepsilon_n p_0}} \left[ \sin \frac{1}{2\varepsilon_n p_0} + p^2 \frac{1 - e^{-\varepsilon_0(p_n p_0)}}{\varepsilon_n^2 p^2} \right] + i p_0 e^{-\frac{\varepsilon_0(p_n p_0)}{2\varepsilon_n p_0}} \sin \frac{\varepsilon^2 p_0}{\varepsilon_n p_0} \]

Since \(\varepsilon^2_n(\mathbb{L}^{-1}p)_0 = \varepsilon^2_n p_0\) and \(\varepsilon_n(\mathbb{L}^{-1}p)_\nu = \varepsilon_n p_\nu\) for \(1 \leq \nu \leq 3\),
\[ \hat{D}_n(\mathbb{L}^{-1}p) = \frac{1}{L^2} \hat{D}_{n+1}(p) \]

(d) By parts (a)–(c),
\[ \Delta^{(n+1)(\ast)}(\mathbb{L}^{-1}(k)) = \frac{\hat{D}_{n+1}(\mathbb{L}^{-1}(k))}{1 + \hat{D}_{n+1}(\mathbb{L}^{-1}(k)) \sum_{\ell_n \in \mathcal{B}_n \atop \ell_0 \in \mathcal{D}_n} u_n(\mathbb{L}^{-1}(k) + \ell_0 + \ell_n) 2^q u_+(\mathbb{L}^{-1}(k) + \ell)^2 D_n^{-1(\ast)}(\mathbb{L}^{-1}(k) + \ell + \ell_n)} \]
\[ = \frac{1}{L^2} Q_{n+1}(k) \]
\[ = 1 + \frac{1}{L^2} Q_{n+1}(k) \sum_{\ell_n \in \mathcal{B}_n \atop \ell_0 \in \mathcal{D}_n} u_{n+1}(k + \mathbb{L}(\ell + \ell_n))^2 L^2 D_{n+1}^{-1(\ast)}(k + \mathbb{L}(\ell + \ell_n)) \]
\[ = \frac{1}{L^2} Q_{n+1}(k) \sum_{\ell_n \in \mathcal{B}_{n+1}} u_{n+1}(k + \ell_{n+1})^2 D_{n+1}^{-1(\ast)}(k + \ell_{n+1}) \]
\[ = \frac{1}{L^2} \Delta^{(n+1)(\ast)}(k) \]

(e) Using the formula for \(\hat{S}^{-1}_{n+1,\ell}(\ell + \ell_n, \ell' + \ell'_n)\) given just before Lemma 6.3, as well as parts (a), (b) and (c),
\[ \hat{S}^{-1}_{n+1,\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_{n+1}), \mathbb{L}^{-1}(\ell'_{n+1})) = D_n(\mathbb{L}^{-1}(k + \ell_{n+1})) \delta_{\ell_n+1,\ell'_n+1} + u_n(\mathbb{L}^{-1}(k + \ell_{n+1}))^q u_+(\mathbb{L}^{-1}(k + \ell_{n+1}))^q \hat{S}_{n+1}(\mathbb{L}^{-1}(k)) \]
\[ = \frac{1}{L^2} \hat{D}_{n+1}(k + \ell_{n+1}) \delta_{\ell_n+1,\ell'_n+1} + u_{n+1}(k + \ell_{n+1})^q \frac{1}{L^2} Q_{n+1}(k) u_{n+1}(k + \ell_{n+1})^q \]
\[ = \frac{1}{L^2} \hat{S}^{-1}_{n+1,k}(\ell_{n+1}, \ell'_n+1) \]
by (5.8).
Corollary 6.5. There are constants \( m_9 > 0 \) and \( \Gamma_{18} \)

such that, for all \( L > \Gamma_{17} \),

\[
\| L^{-1} S_{n+1} \|_{m_9} \leq L^2 \Gamma_{18} \quad \| L^{-1} \hat{S}_{n+1} \hat{Q}^*_{n+1} \|_{m_9} \leq L^2 \Gamma_{18}
\]

Proof. Set \( m_9 = \min\{m_3, m_6\} \). Just combine [5, Lemmas 15.b and 16.b] and parts (a) and (e) of Lemma 6.4 to yield

\[
\| L^{-1} S_{n+1} \|_{m_9} = L^2 \| S_{n+1} \|_{m_9} \quad \| L^{-1} \hat{S}_{n+1} \hat{Q}^*_{n+1} \|_{m_9} = L^2 \| S_{n+1} Q_{n+1} \|_{m_9}
\]

and then apply Proposition 5.1 and Lemma 5.6.

Internal Remark

If \( \theta(y) = \psi(L^{-1}(y)) \) and \( x = L(u) \in X_0^{(n)} \), then

\[
(A_{\psi, \theta})(L(u)) = \text{vol}_- \sum_{y \in X_0^{(n+1)}} A_{\psi, \theta}(L(u), y) \psi(L^{-1}(y)) \]

\[
= \text{vol}_- \sum_{x' \in X_0^{(n+1)}} A_{\psi, \theta}(L(u), L(x')) \psi(x')
\]

The operator \( A_{\psi, \theta} \) is the periodization of an operator that we denote \( a_{\psi, \theta} \). So the “scaled” operator \( A^{(s)}_{\psi, \theta} : H_0^{(n+1)} \rightarrow H_1^{(n)} \) with kernel

\[
A^{(s)}_{\psi, \theta}(u, x) = \text{vol}_- A_{\psi, \theta}(L(u), L(x))
\]

is the periodization of

\[
a^{(s)}_{\psi, \theta}(u, x) = \text{vol}_- a_{\psi, \theta}(L(u), L(x))
\]

\[
= \sum_{\ell \in \mathbb{B}^+} \text{vol}_- \int_{\mathbb{Z}_-} e^{i \ell \cdot L(u)} \hat{a}_{\psi, \theta, \ell}(\ell) e^{i \ell \cdot L(u) - L(x)} \frac{d^4 \ell}{(2\pi)^4}
\]

\[
= \sum_{\ell \in \mathbb{B}^+} \text{vol}_- \int_{\mathbb{Z}_-} e^{i \ell \cdot u} \hat{a}_{\psi, \theta, \ell}(\ell) e^{i \ell \cdot (u - x)} \frac{d^4 \ell}{(2\pi)^4}
\]

\[
= \sum_{\ell_1 \in \mathbb{B}_1} \int_{\mathbb{Z}_0} e^{i \ell_1 \cdot u} \hat{a}_{\psi, \theta, L^{-1}(\ell_1)}(L^{-1}(\ell_1)) e^{i \ell_1 \cdot (u - x)} \frac{d^4 \ell}{(2\pi)^4}
\]
where $\hat{Z}_{-1} = (\mathbb{R}/\frac{2\pi}{L} \mathbb{Z}) \times (\mathbb{R}^3/\frac{2\pi}{L} \mathbb{Z}^3)$ and $\hat{Z}_0 = (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3)$ and we have made the changes of variables $k = \mathbb{L}(\xi)$ and $\ell = \mathbb{L}(\ell)$. Thus

$$\hat{a}_{\psi, \theta, k}^{(s)}(\ell_1) = \hat{a}_{\psi, \theta, \mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1))$$

$$\left(\frac{\partial a_{\psi, \theta, a}^{(s)}}{\partial u_{\nu}}\right)_k(\ell_1) = L_{\nu} \left(e^{i(k + \ell_1)\nu/L_{\nu}} - 1\right) \hat{a}_{\psi, \theta, \mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1))$$

$$= 2iL_{\nu} e^{ik + \ell_1\nu/(2L_{\nu})} \sin \frac{k_{\nu} + \ell_1\nu}{2L_{\nu} \mathbb{L}} \hat{a}_{\psi, \theta, \mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1))$$

**Lemma 6.6.** Assume that $L > \Gamma_{17}$, the constant of Lemma 6.3. There are constants $m_{10} > 0$ and $\Gamma_{19}$ such that the following hold for all $k \in \mathbb{C}^4$ with $|\text{Im } k| \leq m_{10}$.

(a) $\left| (\mathbb{L}^{-1}_s A_{\psi, \theta}^{(s)} \mathbb{L}_s)_k(\ell_1) \right| = \left| (A_{\psi, \theta}^{(s)} \mathbb{L}^{-1}_s)_k \mathbb{L}^{-1}(\ell_1) \right| \leq \Gamma_{19} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1, \nu}| + \pi} \right)^q$

(b) If $\ell_1 \neq 0$, then $\left| (\mathbb{L}^{-1}_s A_{\psi, \theta}^{(s)} \mathbb{L}_s)_k(\ell_1) \right| \leq \Gamma_{19} |k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1, \nu}| + \pi} \right)^q$

**Internal Remark**

They depend only on $q$, $\mathbb{h}_0$ and $a$, and in particular are independent of $n$ and $L$.

(c) Let $q > 2$. There are $D_{(s)\bar{\nu}} : \mathcal{H}_0^{(n+1)} \rightarrow \mathcal{H}_1^{(n)}$, $0 \leq \bar{\nu} \leq 3$, with $\| D_{(s)\bar{\nu}} \|_{m_{10}} \leq \Gamma_{19}$ such that

$$L_{\nu} e^{ik + \ell_1\nu/(2L_{\nu})} \sin \frac{k_{\nu} + \ell_1\nu}{2L_{\nu} \mathbb{L}} \left( \mathbb{L}^{-1}_s A_{\psi, \theta}^{(s)} \mathbb{L}_s \right)_k(\ell_1) = \sum_{\bar{\nu}=0}^{3} \left( D_{(s)\bar{\nu}} \right)_k(\ell_1) e^{ik_{\bar{\nu}}/2} \sin \frac{k_{\bar{\nu}}}{2}$$

and

$$\left| (D_{(s)\nu})_k(\ell_1) \right| \leq \Gamma_{19} \left( \frac{24}{|\ell_{1, \nu}| + \pi} \right)^q \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \left( \frac{24}{|\ell_{1, \nu'}| + \pi} \right)^q$$

for all $0 \leq \nu \leq 3$ and $|\text{Re } k_{\nu}| \leq \pi$.  

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Proof. By (6.6), Lemma 6.4 and Remark 2.1.e,
\[
(A_{\psi_{(\ast)}^{(\ast)}(\ast)}^{\ast})_{L-1(k)}(L^{-1}(\ell_1)) \\
= \sum_{\ell'_1 \in B_1} \left( \frac{a}{\ell'} Q^* Q + \Omega_n \right)_{L-1(k)}^{-1}(L^{-1}(\ell_1), L^{-1}(\ell'_1)) \left\{ \frac{a}{\ell'} Q^*_{L-1(k)}(L^{-1}(\ell'_1)) \\
+ \Omega_n(L^{-1}(k + \ell'_1)) \left( Q_n D_{n+1}^{1(*)} Q_n^* (L^{-1}(k + \ell'_1)) \\
Q^*_{L-1(k)}(L^{-1}(\ell'_1)) \Delta^{(n+1)(*)}(L^{-1}(k)) \right\} \\
= \sum_{\ell'_1 \in B_1} \left( \frac{a}{\ell'} Q^* Q + \Omega_n \right)_{L-1(k)}^{-1}(L^{-1}(\ell_1), L^{-1}(\ell'_1)) B_1(k, \ell'_1) \\
= \Omega_n(L^{-1}(k + \ell_1))^{-1} B_1(k, \ell_1) + \sum_{\ell'_1 \in B_1} C(k, \ell_1, \ell'_1) B_1(k, \ell'_1)
\]
where
\[
B_1(k, \ell'_1) = u_1(k + \ell'_1) \left\{ \frac{a}{\ell'} + \Omega_n(L^{-1}(k + \ell'_1)) \sum_{\ell_n \in B_n} B_2(k, \ell'_1, \ell_n) \right\}
\]
\[
B_2(k, \ell'_1, \ell_n) = u_n(L^{-1}(k + \ell'_1 + \ell_n)^2 \quad D_{n+1}^{1(*)} (k + \ell'_1 + L(\ell_n)) \quad \Delta^{(n+1)(*)}(k)
\]
\[
C(k, \ell_1, \ell'_1) = \left( \frac{a}{\ell'} Q^* Q + \Omega_n \right)_{L-1(k)}^{-1}(L^{-1}(\ell_1), L^{-1}(\ell'_1)) - \Omega_n(L^{-1}(k + \ell_1))^{-1} \delta_{\ell_1, \ell'_1}
\]

(a) Choose \( m_{10} = \min\{2, m_1, \tilde{m}(\pi)\} \). Then
\[
\left| \left( \frac{a}{\ell'} Q^* Q + \Omega_n \right)_{L-1(k)}^{-1}(L^{-1}(\ell_1), L^{-1}(\ell'_1)) \right| \leq \frac{6}{5} a \delta_{\ell_1, \ell'_1} + \frac{2}{a \ell^2} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1,v}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1,v}| + \pi} \right)^q
\]
(by Lemma 6.3.b, Proposition 2.4.a)
\[
|u_1(k + \ell'_1)|^q \leq \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1,v}| + \pi} \right)^q
\]
(by Lemma 2.2.a)
\[
|\Omega_n(L^{-1}(k + \ell'_1))| \leq \frac{6}{5} a
\]
(by Proposition 2.4.a)
\[
|u_n(L^{-1}(k + \ell'_1 + \ell_n)|^2) \leq \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{n,v}| + \pi} \right)^{2q}
\]
(by Lemma 2.2.a)
\[
|D_{n+1}^{1(*)} (k + \ell'_1 + L(\ell_n))| \leq \frac{1}{\gamma_1 \pi} \text{ if } (\ell'_1, \ell_n) \neq (0, 0)
\]
(by Lemma 3.2.d)
\[
|\Delta^{(n+1)(*)}(k)| \leq 2a
\]
(by Lemma 4.2.c)
\[
|D_{n+1}^{1(*)} (k) \Delta^{(n+1)(*)}(k)| \leq \Gamma_6
\]
(by Lemma 4.2.f)
Internal Remark

We restrict

\[ |\text{Re } k_\nu| \leq \pi \quad |\ell'_{1,\nu}| \leq L_\nu \pi \quad \ell'_{1,\nu} \in 2\pi \mathbb{Z} \quad |\ell_{n,\nu}| \leq L_n \pi \quad \ell_{n,\nu} \in 2\pi \mathbb{Z} \]

Recalling that \( \ell'_{1,\nu} \) and \( \ell_{n,\nu} \) are even multiples of \( \pi \) and that \( L_\nu \) is odd,

\[ |(\ell'_{1} + L(\ell_{n}))_\nu| \leq L^{n+1}_\nu \pi \quad (\ell'_{1} + L(\ell_{n}))_\nu \in 2\pi \mathbb{Z} \]

If \( \ell'_{1,\nu} \) and \( \ell_{n,\nu} \) are not both zero and \( |\text{Re } k_\nu| \leq \pi \), then

\[ |(\ell'_{1} + L(\ell_{n}))_\nu| \geq 2\pi \quad |(k + \ell'_{1} + L(\ell_{n}))_\nu| \geq \pi \]

So

\[ \left| (A_{\psi(\nu)} \theta_{\nu}) (L^{-1}(k)) (L^{-1}(\ell_{1})) \right| \leq \sum_{\ell'_{1} \in \mathcal{B}_{1}} \left\{ \frac{6}{5} a \delta_{\ell_{1},\ell'_{1}} + \frac{2}{a^2} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{1,\nu}|+\pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1,\nu}|+\pi} \right)^q \right\} \leq \text{const} \sum_{\ell'_{1} \in \mathcal{B}_{1}} \left\{ \frac{6}{5} a \delta_{\ell_{1},\ell'_{1}} + \frac{2}{a^2} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{1,\nu}|+\pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{1,\nu}|+\pi} \right)^q \right\} \leq \Gamma_{19} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{1,\nu}|+\pi} \right)^q \]

(b) Using the bounds of part (a) together with

\[ |u_{1}(k + \ell'_{1})|^q \leq |k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{1,\nu}|+\pi} \right)^q \quad \text{if } \ell'_{1} \neq 0 \]  

(by Lemma 2.2.b)

\[ |u_{n}(L^{-1}(k + \ell'_{1}) + \ell_{n})|^q \leq |L^{-1}(k + \ell'_{1})| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{n}|+\pi} \right)^q \quad \text{if } \ell_{n} \neq 0 \]  

(by Lemma 2.2.b)

we have, if \( \ell'_{1} \neq 0 \),

\[ |B_{1}(k, \ell'_{1})| \leq |k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{1,\nu}|+\pi} \right)^q \left\{ \frac{a}{L_\pi} + \frac{6}{5} a \sum_{\ell_{n} \in \mathcal{B}_{n}} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{n}|+\pi} \right)^q \max \{ \Gamma_{6}, \frac{2a}{\gamma_{11} \pi} \} \right\} \]

\[ \leq \text{const} |k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{1,\nu}|+\pi} \right)^q \]

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and
\[
|B_1(k, 0)| \leq \frac{3}{5} \prod_{\nu=0}^{3} \left( \frac{24}{\pi} \right)^{q} \left\{ \frac{a}{\ell} + \frac{a}{5} \sum_{\ell_n \in B_n} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_n| + \pi} \right)^{2q} \max \{ \Gamma_6, \frac{2a}{3\gamma_1 \pi} \} \right\} \leq \text{const}
\]

**Internal Remark**

and if \( \ell_1' = 0, \ell_n \neq 0, \)
\[
|B_2(k, \ell_1', \ell_n)| \leq |L^{-1}k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_n| + \pi} \right)^{2q} \max \{ \Gamma_6, \frac{2a}{3\gamma_1 \pi} \}
\]
so that
\[
\left| B_1(k, 0) - u_1(k)^{q} \left\{ \frac{a}{\ell} + \Omega_n(\mathbb{L}^{-1}k) u_n(\mathbb{L}^{-1}k)^{2q} \hat{\Delta}_{n+1}^{-1}(k) \Delta^{(n+1)}(k) \right\} \right|
\]
\[
\leq \frac{6}{5} a \prod_{\nu=0}^{3} \left( \frac{24}{\pi} \right)^{2q} \sum_{\ell_n \in B_n} |L^{-1}k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_n| + \pi} \right)^{2q} \max \{ \Gamma_6, \frac{2a}{3\gamma_1 \pi} \}
\]
\[
\leq \text{const}|L^{-1}k|
\]

Using these bounds, the first bound of part (a) and Lemma 6.3.b, and assuming that \( \ell_1 \neq 0, \)
\[
(A_{\psi(k), \theta(k)})_{L^{-1}k} = \sum_{\ell_1' \in \mathbb{B}_1} \left( \frac{a}{\ell} Q^* Q + \Omega_n \right)^{-1} (L^{-1}(\ell_1'), L^{-1}(\ell_1')) B_1(k, \ell_1')
\]
\[
= \left( \frac{a}{\ell} Q^* Q + \Omega_n \right)^{-1} (L^{-1}(\ell_1), 0)) B_1(k, 0) + O\left(|k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_1'| + \pi} \right)^{q} \right)
\]
\[
= \frac{2}{a \ell^2} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_1'| + \pi} \right)^{q} \prod_{\nu=0}^{3} \left( \frac{24}{\pi} \right)^{q} \prod_{0 \leq \nu \leq 3 \ell_1' \neq 0} |k_\nu| B_1(k, 0) + O\left(|k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_1'| + \pi} \right)^{q} \right)
\]
\[
= O\left(|k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_1'| + \pi} \right)^{q} \right)
\]

**Internal Remark**

(c) Partition \( \mathbb{B}_1 = \{0\} \cup \bigcup_{\nu=0}^{3} \mathbb{B}_{1\nu} \) into 5 disjoint subsets with, for each \( 0 \leq \nu \leq 3, \) every element \( \ell_1 \) of \( \mathbb{B}_{1\nu} \) obeying \( \ell_1, \nu \neq 0. \) Observe that
- if \( \ell_1' \in \mathbb{B}_{1\nu}, \) then, using Lemma 2.2.d,
\[
L_\nu \sin \frac{k_\nu + \ell_1}{2L_\nu} B_1(k, \ell_1') = \sin \left( \frac{\theta}{2} k_\nu \right) \hat{B}_{1\nu}(k, \ell_1')
\]
with
\[
\tilde{B}_{1,\nu}(k, \ell') = L_\nu \sin \frac{k_\nu + \ell_1,\nu}{2L_\nu} v_{1,\nu}(k + \ell'_1) u_1(k + \ell'_1)^q - 1 \\
\left\{ \frac{a}{L^2} + \Omega_n \left( \mathcal{L}^{-1}(k + \ell'_1) \right) \sum_{\ell_n \in \mathcal{B}_n} B_2(k, \ell'_1, \ell_n) \right\}
\]

obeying, as in part (b), \(|\tilde{B}_{1,\nu}(k, \ell'_1)| \leq \text{const} (\pi + |\ell_{1,\nu}|) \prod_{\nu' = 0}^{q} \left( \frac{24}{|\nu'| + \pi} \right)^q \).

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\[
|\tilde{B}_{1,\nu}(k, \ell'_1)|
\leq L_\nu \sin \frac{k_\nu + \ell_{1,\nu}}{2L_\nu} \prod_{\nu' = 0}^{3} \left( \frac{24}{|\nu'| + \pi} \right)^q \left\{ \frac{a}{L^2} + \frac{3}{4} a \sum_{\ell_n \in \mathcal{B}_n} \prod_{\nu' = 0}^{3} \left( \frac{24}{|\nu'| + \pi} \right)^2 \max \left\{ \Gamma_6, \frac{24}{\gamma_1 \pi} \right\} \right\}
\leq |k_\nu + \ell_{1,\nu}| \prod_{\nu' = 0}^{3} \left( \frac{24}{|\nu'| + \pi} \right)^q \left\{ \frac{a}{L^2} + \text{const} \right\}
\]

- if \(\ell_1 \in \hat{B}_{1,\nu}\), then, using Lemma 6.3.c,

\[
L_\nu \sin \frac{k_\nu + \ell_{1,\nu}}{2L_\nu} C(k, \ell_1, 0) = \sin \left( \frac{1}{2} k_\nu \right) \tilde{C}_{\nu}(k, \ell_1, 0)
\]

with
\[
\tilde{C}_{\nu}(k, \ell_1, 0) = L_\nu \sin \frac{k_\nu + \ell_{1,\nu}}{2L_\nu} B_n,\tilde{\nu}(\mathcal{L}^{-1}(\ell_1), 0)
\]

obeying \(|\tilde{C}_{\nu}(k, \ell_1, 0)| \leq \frac{4}{3L^2} (\pi + |\ell_{1,\nu}|) \prod_{\nu' = 0}^{3} \left( \frac{24}{|\nu'| + \pi} \right)^q \prod_{\nu' = 0}^{3} \left( \frac{24}{\pi} \right)^q .
\]

- if \(\ell_1 = \ell'_1 = 0\), then, since

\[
L_\nu \sin \left( \frac{k_\nu}{2L_\nu} \right) u_1(k) = \sin \frac{1}{2} k_\nu \prod_{0 \leq \rho \leq 3} \frac{\sin \frac{1}{2} k_\nu}{L_\rho \sin \frac{1}{2} k_\nu}
\]

we have \(L_\nu \sin \frac{k_\nu}{2L_\nu} B_1(k, 0) = \sin \left( \frac{1}{2} k_\nu \right) \tilde{B}_1(k, 0)\) with

\[
\tilde{B}_1(k, 0) = \left( \prod_{0 \leq \rho \leq 3} \frac{\sin \frac{1}{2} k_\nu}{L_\rho \sin \frac{1}{2} k_\nu} \right) u_1(k)^q - 1 \left\{ \frac{a}{L^2} + \Omega_n \left( \mathcal{L}^{-1}(k) \right) \sum_{\ell_n \in \mathcal{B}_n} B_2(k, 0, \ell_n) \right\}
\]

obeying, as in part (b), \(|\tilde{B}_1(k, 0)| \leq \text{const} .
\]
So the claim follows with \((D_{(s)})\) \(_{k}^{\ell_{l}}(\ell_{1}) = e^{i(k+\ell_{1})\nu/(2L_{\nu})}e^{-ik_{\nu}/2}\tilde{D}_{\nu,k}(\ell_{1})\) where
\[
\tilde{D}_{\nu,k}(\ell_{1}) = \sum_{\ell'_{1} \in \mathcal{B}_{\nu}} C(k, \ell_{1}, \ell'_{1})\tilde{B}_{1,\nu}(k, \ell'_{1})
\]
\[
\begin{cases}
\Omega_{\ell}^{\nu} \left( \mathbb{L}^{-1}(k + \ell_{1}) \right)^{-1} \tilde{B}_{1,\nu}(k, \ell_{1}) + \tilde{C}_{\nu}(k, \ell_{1}, 0)B_{1}(k, 0) & \text{if } \ell_{1} \in \mathcal{B}_{1,\nu} \\
\left[ \Omega_{\ell}^{\nu} \left( \mathbb{L}^{-1}(k) \right)^{-1} + C(k, 0, 0) \right] \tilde{B}_{1}(k, 0) & \text{if } \ell_{1} = 0, \ \bar{\nu} = \nu \\
0 & \text{if } \ell_{1} = 0, \ \bar{\nu} \neq \nu
\end{cases}
\]

**Internal Remark**

\[
L_{\nu}e^{i(k+\ell_{1})\nu/(2L_{\nu})} \sin \frac{k_{\nu}+\ell_{1}\nu}{2L_{\nu}} \left( \mathbb{L}^{-1}_{\nu} A_{\psi}(\nu) \mathbb{L}_{\nu} \right)_{k}^{\ell_{l}}(\ell_{1})
\]
\[
= L_{\nu}e^{i(k+\ell_{1})\nu/(2L_{\nu})} \sin \frac{k_{\nu}+\ell_{1}\nu}{2L_{\nu}} \left\{ \Omega_{\ell}^{\nu} \left( \mathbb{L}^{-1}(k + \ell_{1}) \right)^{-1}B_{1}(k, \ell_{1}) \right. \\
\left. + \sum_{\ell'_{1} \in \mathcal{B}_{1}} C(k, \ell_{1}, \ell'_{1})B_{1}(k, \ell'_{1}) \right\}
\]
\[
= \sum_{\bar{\nu}=0}^{3} e^{ik_{\nu}/2} \sum_{\ell'_{1} \in \mathcal{B}_{1,\nu}} \left\{ e^{i(k+\ell_{1})\nu/(2L_{\nu})}e^{-ik_{\nu}/2} \right\}C(k, \ell_{1}, \ell'_{1}) \left\{ L_{\nu} \sin \frac{k_{\nu}+\ell_{1}\nu}{2L_{\nu}}B_{1}(k, \ell'_{1}) \right\}
\]
\[
+ L_{\nu}e^{i(k+\ell_{1})\nu/(2L_{\nu})} \sin \frac{k_{\nu}+\ell_{1}\nu}{2L_{\nu}} \left\{ \Omega_{\ell}^{\nu} \left( \mathbb{L}^{-1}(k + \ell_{1}) \right)^{-1}B_{1}(k, \ell_{1}) + C(k, \ell_{1}, 0)B_{1}(k, 0) \right\}
\]
\[
= \sum_{\bar{\nu}=0}^{3} \left( D_{(s)} \right)_{k}^{\ell_{l}}(\ell_{1}) e^{ik_{\nu}/2} \sin \frac{k_{\nu}}{2}
\]

\[
\left| \tilde{D}_{\nu,k}(\ell_{1}) \right| \leq \text{const} \sum_{\ell'_{1} \in \mathcal{B}_{1,\nu}} C(2_{a}, \nu)^{q} \prod_{\nu'_{1}=0}^{3} \left( \frac{24}{|\ell_{1,\nu'}|+\pi} \right)^{q} \prod_{\nu'_{1}=0}^{3} \left( \frac{24}{|ar{\ell}_{1,\nu'}|+\pi} \right)^{q} \times \left( \pi + |\ell_{1,\nu'}| \right) \prod_{\nu'_{1}=0}^{3} \left( \frac{24}{|ar{\ell}_{1,\nu'}|+\pi} \right)^{q}
\]
\[
\begin{cases}
\text{const} \frac{6}{5} a \left( \pi + |\ell_{1,\nu'}| \right) \prod_{\nu'_{1}=0}^{3} \left( \frac{24}{|\ell_{1,\nu'}|+\pi} \right)^{q} \\
+ \text{const} \left( \pi + |\ell_{1,\nu'}| \right) \prod_{\nu'_{1}=0}^{3} \left( \frac{24}{|\ell_{1,\nu'}|+\pi} \right)^{q} \text{const} & \text{if } \ell_{1} \in \mathcal{B}_{1,\nu} \\
\left[ \frac{6}{5} a + \text{const} \right] \text{const} & \text{if } \ell_{1} = 0, \ \bar{\nu} = \nu \\
0 & \text{if } \ell_{1} = 0, \ \bar{\nu} \neq \nu
\end{cases}
\]
\[
\leq \text{const} \left( \frac{24}{|\ell_{1,\nu'}|+\pi} \right)^{q-1} \prod_{\nu'_{1}=0}^{3} \left( \frac{24}{|ar{\ell}_{1,\nu'}|+\pi} \right)^{q}
\]
The bound on $\|D(\nu)\|_{m_{10}}$ follows from the bound on $|\langle D(\nu) \rangle_k(\ell_1)|$ by [5, Lemma 12.c].

\[\]

\textbf{Proof of Proposition 6.1.}

\textbf{Bound on $\|L^{-1}A_{\psi,\nu}L^*\|_{m=1}$:} By [5, Lemma 15.b] and Lemma 6.3.b, if $|\text{Im} k_{\nu'}| \leq 2$ for each $0 \leq \nu' \leq 3$ then

\[
\left| \left[ L^{-1}_s \left\{ (aL^{-2}Q^* + \Omega_n)^{-1} - \Omega^{-1}_n \right\} \right]_k (\ell, \ell') \right| = \left| \left[ (aL^{-2}Q^* + \Omega_n)^{-1} - \Omega^{-1}_n \right]_{L^{-1}k} \right| (L^{-1}_\ell, L^{-1}_\ell') \right| \leq \frac{2}{2L^2} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{\nu}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{\nu'}| + \pi} \right)^q \]

So, by [5, Lemma 12.b],

\[
\|L^{-1}_s \left\{ (aL^{-2}Q^* + \Omega_n)^{-1} - \Omega^{-1}_n \right\} \|_{L^*} \leq \text{const}_q
\]

By Proposition 2.4.a, Lemma 2.2.a and [5, Lemmas 12.b,c],

\[
\|\Omega_n\|_{m=1}, \|\Omega^{-1}_n\|_{m=1}, \|Q_n\|_{m=1} \leq \text{const}_q
\]

too. Now just apply [5, Lemmas 15.c and 16.c].

\textbf{Bound on $\|L^{-1}_s A_{\psi,\phi,\nu}L^*\|_{m=1}$:} By [5, Lemma 15.b] and Lemma 6.3.c, if $|\text{Im} k_{\nu'}| \leq 2$ for each $0 \leq \nu' \leq 3$ then

\[
\left| \left[ L^{-1}_s \left\{ (aL^{-2}Q^* + \Omega_n)^{-1} - \Omega^{-1}_n \right\} \right]_k (\ell, \ell') \right| = \left| \left[ (aL^{-2}Q^* + \Omega_n)^{-1} - \Omega^{-1}_n \right]_{L^{-1}k} \right| (L^{-1}_\ell, L^{-1}_\ell') \right| \leq \frac{3e^4}{2L^2} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{\nu}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{\nu'}| + \pi} \right)^q \]

As $q > 2$, [5, Lemma 12.b] yields

\[
\|L^{-1}_s \left\{ (aL^{-2}Q^* + \Omega_n)^{-1} - \Omega^{-1}_n \right\} \|_{L^*} \leq \text{const}_q
\]

By Lemma 2.6.b and [5, Lemma 12.c], $\|Q_{\nu'}\|_{m=1} \leq \text{const}_q$, too, since $q > 2$. Now just apply [5, Lemma 16.c].

\textbf{Bound on $\|L^{-1}_s A_{\psi,\phi,\nu}L^*\|_{m_8}$:} This follows from Lemma 6.6.a by [5, Lemma 12.c].
Bound on \( \| L_s^{-1} \frac{\partial}{\partial x} C^{(n)}(\mu)^{(s)} \|_{m_s} \): By (6.3)

\[
L_s^{-1} \frac{\partial}{\partial x} C^{(n)}(\mu)^{(s)} \|_{m_s} = L_s^{-1} A_{\psi_{(s)} \theta_{(s)}} L_s + L^2 \mu L_s^{-1} A_{\psi_\phi \theta_\phi} L_s \delta_{n+1} S_{n+1} (L^2 \mu)^{(s)} Q_{n+1} \Omega_{n+1}
\]

Now just apply Proposition 5.1, Lemma 2.2 and Proposition 2.4.c.

Bound on \( \| L_s^{-1} A_{\psi_{(s)} \theta_{(s)} \nu} L_s \|_{m_s} \): It suffices to bound

- \( L_s^{-1} A_{\psi_\phi \nu} L_s \) as above,
- bound \( L_s^{-1} \Omega_{n+1} Q_{n+1} \Omega_{n+1} L_s \) using

\[
\left| \left[ L_s^{-1} (\Omega_{n+1} Q_{n+1} \Omega_{n+1}) L_s \right] \right| \leq 2 \pi \left( \frac{24}{\pi} \right)^3 \prod_{\nu=0}^{3} \left( \frac{24}{|\nu|+\pi} \right)^{q-1}
\]

(by [5, Lemma 16.b], (2.12), Proposition 2.4.a and Lemmas 2.6.b, 2.3.a, 6.3.a) and

- bound \( D_{n+1}^{-1} Q_{n+1 \nu} \Delta^{(n+1)(s)} \) using

\[
\left| (D_{n+1}^{-1} Q_{n+1 \nu} \Delta^{(n+1)(s)})_{k}(\ell_{n+1}) \right| \leq 2 \pi \left( \frac{24}{\pi} \right)^3 \prod_{\nu=0}^{3} \left( \frac{24}{|\nu|+\pi} \right)^{q-1}
\]

(by (2.11)) and

\[
\left| \zeta_{n+1 \nu}(k, \ell_{n+1}) u_{n+1 \nu}(k, \ell_{n+1}) \right| \leq e^2 \left( \frac{24}{\pi} \right)^3 \quad \text{(by Lemma 2.6.b)}
\]

\[
\left| u_{n+1}(k, \ell_{n+1}) \right| \leq \frac{3}{\pi} \left( \frac{24}{|\ell_{n+1}|+\pi} \right)^{q-1} \quad \text{(by Lemma 2.2.a)}
\]

\[
\left| \tilde{D}_{n+1}^{-1}(k+\ell_{n+1}) \right| \leq \frac{1}{\pi} \text{ if } \ell_{n+1} \neq 0 \quad \text{(by Lemma 3.2.d)}
\]

\[
\left| \tilde{\Delta}^{(n+1)(s)}(k) \right| \leq 2a \quad \text{(by Lemma 4.2.c)}
\]

and [5, Lemma 12.c].

Bound on \( \| L_s^{-1} A_{\psi_{(s)} \theta_{(s)} \nu}(\mu) L_s \|_{m_s} \): This follows from the previous bounds of this Proposition, Remark 6.2.b, Proposition 5.1, Lemma 2.6.c and Proposition 2.4.c.

\[\square\]
A  Trigonometric Inequalities

Lemma A.1.

(a) For $x, y$ real with $|x| \leq \frac{\pi}{2}$,

$$|\sin(x + iy)| \geq \sqrt{2} \left| \sin \frac{x}{2} \right|$$

(b) For $x, y$ real with $|y| \leq 1$,

$$\left| \frac{\sin(x + iy)}{|x + iy|} \right| \leq 2 \min \left\{ 1, \frac{1}{|x + iy|} \right\}$$

$$\left| \frac{\sin \left( \frac{1}{2}(x + iy) \right)}{\frac{1}{2} \sin \left( \frac{1}{2} \epsilon(x + iy) \right)} \right| \leq 4 \min \left\{ 1, \frac{2}{|x|} \right\}$$

$$\left| \frac{\sin \left( \frac{1}{2}(x + iy) \right)}{\frac{1}{2} \epsilon \sin \left( \frac{1}{2} \epsilon(x + iy) \right)} \right| \leq 6 \min \left\{ 1, \frac{8}{|x|} \right\}$$

(c) For $0 < \epsilon \leq 1$ and $x, y$ real with $|\epsilon x| \leq \pi$, $|y| \leq 2$

$$\left| \frac{\sin \left( \frac{1}{2}(x + iy) \right)}{\frac{1}{2} \sin \left( \frac{1}{2} \epsilon(x + iy) \right)} \right| \leq 4 \min \left\{ 1, \frac{2}{|x|} \right\}$$

$$\left| \frac{\sin \left( \frac{1}{2}(x + iy) \right)}{\frac{1}{2} \epsilon \sin \left( \frac{1}{2} \epsilon(x + iy) \right)} \right| \leq 6 \min \left\{ 1, \frac{8}{|x|} \right\}$$

(d) For $x$ real with $|x| \leq \frac{\pi}{2}$,

$$2 \pi \leq \frac{\sin x}{x} \leq 1$$

(e) For any complex number $z$ obeying $|z| \leq 2$,

$$\left| \frac{\sin z}{z} - 1 \right| \leq \frac{1}{2} |z|^2$$

Proof. By the standard trig identity

$$\sin(x + iy) = \sin(x) \cos(iy) + \cos(x) \sin(iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

(a) For $x, y$ real with $|x| \leq \frac{\pi}{2}$

$$|\text{Re} \sin(x + iy)| = |\sin(x) \cosh(y)| \geq |\sin(x)| \geq \frac{2}{\pi} |x|$$

$$|\text{Re} \sin(x + iy)| = |\sin(x) \cosh(y)| \geq |\sin(x) \sinh(y)| \geq |\sin(x)| |y|$$

$$|\text{Im} \sin(x + iy)| = |\cos(x) \sinh(y)| \geq |\cos(x)| |y|$$

so that

$$|\sin(x + iy)| \geq \max \left\{ \frac{2}{\pi} |x|, |y| \right\} \geq \frac{\sqrt{2}}{\pi} |x + iy|$$

(b) For $x, y$ real with $|y| \leq 1$,

$$|\sin(x + iy)| = |\sin(x) \cosh(y) + i \cos(x) \sinh(y)| \leq \cosh(1) |\sin(x) + i \cos(x)|$$

$$= \cosh(1)$$
and, since  
\[ |\sinh(y)| = \left| \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} \right| \leq |y| \sum_{n=0}^{\infty} \frac{|y|^{2n}}{(2n)!} = |y| \cosh(y) \leq \cosh(1)|y|, \]

\[ |\sin(x + iy)| = |\sin(x) \cosh(y) + i \cos(x) \sinh(y)| \leq \cosh(1) |x| + i |y| = \cosh(1)|x + iy| \]

Thus
\[ \frac{|\sin(x + iy)|}{|x + iy|} \leq \cosh(1) \min \{1, \frac{1}{|x + iy|} \} \leq 2 \min \{1, \frac{1}{|x + iy|} \} \]
giving the first bound.

For the second bound
\[ \left| \text{Im} \frac{\sin(x + iy)}{x + iy} \right| = \left| \frac{x \text{Im} \sin(x + iy) - y \text{Re} \sin(x + iy)}{x^2 + y^2} \right| = \left| \frac{x \cos(x) \sinh(y) - y \sin(x) \cosh(y)}{x^2 + y^2} \right| \]

Using
\[ \cos x - 1 = -\int_0^x dt \sin t = -\int_0^x dt \int_0^t ds \cos s \]
\[ \sin x - x = \int_0^x dt \cos t - 1 = -\int_0^x dt \int_0^t ds \int_0^s du \cos u \]
\[ \cosh y - 1 = \int_0^y dt \sinh t = \int_0^y dt \int_0^t ds \cosh u \]
\[ \sinh y - y = \int_0^y dt \cos t - 1 = \int_0^y dt \int_0^t ds \int_0^s du \cos u \]

and \( \cosh(1) < 2 \), we have
\[ \cos x = 1 + \alpha(x)\frac{x^2}{2} \quad \sin x = x + \beta(x)\frac{x^3}{6} \]
\[ \cosh y = 1 + \gamma(y)y^2 \quad \sinh y = y + \delta(y)\frac{y^3}{3} \]

with, for \( |y| \leq 1 \), \( |\alpha(x)|, |\beta(x)|, |\gamma(y)|, |\delta(y)| \leq 1 \). Consequently,
\[ \left| \text{Im} \frac{\sin(x + iy)}{x + iy} \right| \leq 2|xy| \]

**Internal Remark**

\[ x \cos(x) \sinh(y) - y \sin(x) \cosh(y) \]
\[ = xy\left\{ \left[ 1 + \alpha(x)\frac{x^2}{2} \right] \left[ 1 + \delta(y)\frac{y^2}{3} \right] - \left[ 1 + \beta(x)\frac{x^3}{6} \right] \left[ 1 + \gamma(y)y^2 \right] \right\} \]
\[ = xy\left\{ x^2\left[ \frac{1}{2}\alpha(x)^2 + \alpha(x)\delta(y)\frac{y^2}{6} - \frac{1}{6}\beta(x) - \frac{1}{6}\beta(x)y^2\gamma(y) \right] + y^2\left[ \frac{1}{3}\delta(y) - \gamma(y) \right] \right\} \]

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Now just use that, for $|y| \leq 1$,
\[
\left| \frac{1}{2} \alpha(x)^2 + \alpha(x)\delta(y) \frac{y^2}{\delta} - \frac{1}{6} \beta(x) - \frac{1}{6} \beta(x)y^2 \gamma(y) \right| \leq \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \leq 1
\]
\[
\left| \frac{1}{3} \delta(y) - \gamma(y) \right| \leq \frac{1}{3} + 1 \leq 2
\]

Alternatively, using $|\sin(x)| \leq |x|$, $|\sinh(y)| \leq 2|y|$ and $|\cosh(y)| \leq 2$,
\[
\left| \text{Im} \frac{\sin(x+iy)}{x+iy} \right| \leq \frac{4|x|}{x^2+y^2} \leq \frac{4|y|}{\sqrt{x^2+y^2}}
\]

(c) For $0 < \varepsilon \leq 1$, $x, y$ real and $|\varepsilon x| \leq \pi$, $|y| \leq 2$,
\[
\left| \frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{1}{\varepsilon} \sin \frac{1}{\varepsilon}(x+iy)} \right| = \left| \frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{1}{\varepsilon} \sin \frac{1}{\varepsilon}(x+iy)}} \right| \leq \frac{\varepsilon}{\sqrt{2}} \cosh(1) \min \left\{ 1, \frac{2}{|x+iy|} \right\} \leq 4 \min \left\{ 1, \frac{2}{|x|} \right\}
\]

and
\[
\left| \text{Im} \frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{1}{\varepsilon} \sin \frac{1}{\varepsilon}(x+iy)} \right| = \left| \text{Im} \frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{1}{\varepsilon} \sin \frac{1}{\varepsilon}(x+iy)}} \right| \leq \frac{|y|}{\varepsilon} \min \left\{ \frac{1}{2} |x|, \frac{4}{|x+iy|} \right\} + 4 \min \left\{ 1, \frac{2}{|x|} \right\} \left| \text{Im} \frac{\sin \frac{1}{\varepsilon}(x+iy)}{\frac{1}{\varepsilon} \sin \frac{1}{\varepsilon}(x+iy)} \right| \leq \frac{|y|}{\varepsilon^2} \min \left\{ \frac{1}{2} |x|, \frac{4}{|x+iy|} \right\} + 4 \min \left\{ 1, \frac{2}{|x|} \right\} |y| \min \left\{ \frac{1}{2} |x|, \frac{4}{|x+iy|} \right\} \leq \frac{5\sqrt{2}}{\pi} |y| \min \left\{ \left( \frac{1}{2} + 2\varepsilon^2 \right) |x|, \frac{4+16}{|x+iy|} \right\} \leq 6 |y| \min \left\{ |x|, \frac{8}{|x+iy|} \right\}
\]

(d) and (e) are standard.

**Internal Remark**

(d) This follows from the observation that, for $0 \leq x \leq \frac{\pi}{2}$, $\sin x$ is monotone decreasing. To see this, observe that
\[
\frac{d}{dx} \sin x = \frac{1}{x} \left( \cos x - \frac{\sin x}{x} \right)
\]
and
\[
\cos x - \frac{\sin x}{x} = \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right) x^{2n} = \sum_{n=1}^{\infty} (-1)^n \frac{2n}{(2n+1)!} x^{2n}
\]

This is an alternating series with the first term being negative and with the ratio between term \(n + 1\) and term \(n\) having magnitude
\[
\frac{(2n+1)!}{2n(2n+3)!} \leq \frac{x^2}{10}
\]
which is less than 1 for all \(|x| \leq \sqrt{10} = 3.162\).

(e) We have
\[
\left| \frac{\sin z}{z} - 1 \right| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \right| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{z^{2(n+1)}}{(2n+3)!} \right| = |z|^2 \left| \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+3)!} \right|
\]

As
\[
(2n + 3)! = n! (n + 1) \cdots (2n + 3) \geq n! 3^n
\]
we have
\[
\left| \frac{\sin z}{z} - 1 \right| \leq \frac{|z|^2}{5} e^{|z|^2}/5
\]
When \(|z| \leq 2\)
\[
e^{|z|^2}/5 \leq e \implies \left| \frac{\sin z}{z} - 1 \right| \leq \frac{1}{2} |z|^2
\]

\[\Box\]
B Lattice and Operator Summary

The following table gives, for most of the operators considered in this paper,
- the definition of the operator
- a reference to where in [7, 8], the operator is introduced and
- the translation invariance properties of the operator.

A later table will specify where, in this paper, bounds on the operators are proven.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Definition</th>
<th>Tiwrt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0 = -e^{-h_0} \partial_0 + \left[1 - e^{-h_0}\right]$</td>
<td>$\mathcal{H}_0 \to \mathcal{H}_0$</td>
<td>§1.5 $\mathcal{X}_0$</td>
</tr>
<tr>
<td>$D_n = C^\ast \sum_{k=0}^{n-1} D_0 Q_k$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Def 1.5.a $\mathcal{X}_n$</td>
</tr>
<tr>
<td>$\mathcal{Q}<em>n = a\left(1 + \sum</em>{j=1}^{n-1} \frac{1}{\mathcal{L}_j \mathcal{Q}_j} \right)^{-1}$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>Def 1.5.b $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$\Delta^{(0)} = D_0$</td>
<td>$\mathcal{H}_0 \to \mathcal{H}_0$</td>
<td>(1.14) $\mathcal{X}_0$</td>
</tr>
<tr>
<td>$\Delta^{(n)} = \left(1 + \mathcal{Q}_n D_n^{-1} \mathcal{Q}_n^\ast \right)^{-1}$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>(1.14) $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$C^{(n)} = \left(\frac{\mathcal{Q}_n}{\mathcal{L}_n} \mathcal{Q}_n + \Delta^{(n)} \right)^{-1}$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>(1.15) $\mathcal{X}_0^{(n+1)}$</td>
</tr>
<tr>
<td>$S_n^{-1} = D_0 + \mathcal{Q}_n \mathcal{Q}_n$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Thm 1.13 $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$S_n \left(\mu^{-1}\right) = D_0 + \mathcal{Q}_n \mathcal{Q}_n - \mu$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Thm 1.13 $\mathcal{X}_0^{(n)}$</td>
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<tr>
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<td>Thm 1.13 $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$\mathcal{S}_n \left(\mu^{-1}\right) = \left(D_0 + \mathcal{Q}_n \mathcal{Q}_n \mathcal{Q}_n - \mu \right)^{-1}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>(5.1) $\mathcal{X}_0^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi, \phi}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_0^{(n)}$</td>
<td>Prop 5.1 $\mathcal{X}_0^{(n+1)}$</td>
</tr>
</tbody>
</table>

The references in the above table are to [7, 8] and “Tiwrt” stands for “translation invariant with respect to”.

**Internal Remark**

To verify that $S_n Q_n^\ast = D_n^{-1} Q_n^\ast \left(1 + \mathcal{Q}_n D_n \mathcal{Q}_n^\ast \right)^{-1}$ we compute

\[
S_n^{-1} D_n^{-1} Q_n^\ast \left(1 + \mathcal{Q}_n D_n \mathcal{Q}_n^\ast \right)^{-1} = (D + Q_n^\ast \mathcal{Q}_n D_n^{-1} Q_n^\ast ) D_n^{-1} Q_n^\ast \left(1 + \mathcal{Q}_n D_n \mathcal{Q}_n^\ast \right)^{-1}
\]

\[
= Q_n^\ast \left(1 + \mathcal{Q}_n D_n \mathcal{Q}_n^\ast \right)^{-1} + Q_n^\ast \mathcal{Q}_n D_n^{-1} Q_n^\ast \left(1 + \mathcal{Q}_n D_n \mathcal{Q}_n^\ast \right)^{-1}
\]

\[
= Q_n^\ast \left(1 + \mathcal{Q}_n D_n \mathcal{Q}_n^\ast \right)^{-1}
\]

\[
= Q_n^\ast
\]
<table>
<thead>
<tr>
<th>Operator</th>
<th>Definition</th>
<th>Tiwrt</th>
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<tbody>
<tr>
<td>$S_n Q_n^* = D_n^{-1} Q_n^* (\mathbb{1} + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} : \mathcal{H}_0^{(n)} \to \mathcal{H}_n$</td>
<td>Lemma 5.6</td>
<td>$\mathcal{X}_0^{(n)}$</td>
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<tr>
<td>$\tilde{S}<em>{n+1} = \tilde{S}</em>{n+1}(0) : \mathcal{H}_n \to \mathcal{H}_n$</td>
<td>after (6.2)</td>
<td>$\mathcal{X}_{n+1}^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi} : \mathcal{H}_n \to \mathcal{H}_0^{(n)}$</td>
<td>(6.4)</td>
<td>$\mathcal{X}_{n+1}^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi,\nu} : \mathcal{H}_n \to \mathcal{H}_0^{(n)}$</td>
<td>Remark 6.2</td>
<td>$\mathcal{X}_{n+1}^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi,\nu} = \frac{a}{L^2} C^{(n)}(\nu) Q^* : \mathcal{H}_0^{(n+1)} \to \mathcal{H}_0^{(n)}$</td>
<td>before Rmk 6.2</td>
<td>$\mathcal{X}_{n+1}^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi,\nu}^<em>, A_{\psi,\phi,\nu}^</em> (\mu), A_{\psi,\phi,\nu} (\mu) : \mathcal{H}_0^{(n+1)} \to \mathcal{H}_0^{(n)}$</td>
<td>Remark 6.2</td>
<td>$\mathcal{X}_{n+1}^{(n+1)}$</td>
</tr>
</tbody>
</table>

The lattices involved are

\[
\mathcal{X}_n = \left( \frac{\varepsilon_n^2 \mathbb{Z}}{\varepsilon_n^2 L_{\text{tp}} \mathbb{Z}} \right) \times \left( \frac{\varepsilon_n^3 \mathbb{Z}^3}{\varepsilon_n L_{\text{sp}} \mathbb{Z}^3} \right) \\
\mathcal{X}_0^{(n)} = \left( \frac{\varepsilon_n^2 \mathbb{Z}}{\varepsilon_n^2 L_{\text{tp}} \mathbb{Z}} \right) \times \left( \frac{\varepsilon_n^3 \mathbb{Z}^3}{\varepsilon_n L_{\text{sp}} \mathbb{Z}^3} \right) \\
\mathcal{X}_{-1}^{(n+1)} = \left( \frac{L^2 \mathbb{Z}}{\varepsilon_n^2 L_{\text{tp}} \mathbb{Z}} \right) \times \left( \frac{L \mathbb{Z}^3}{\varepsilon_n L_{\text{sp}} \mathbb{Z}^3} \right) \\
\mathcal{X}_{-1}^{(n+1)} = \left( \frac{2\pi \mathbb{Z}}{\varepsilon_n L_{\text{tp}} \mathbb{Z}} \right) \times \left( \frac{\varepsilon_n^3 \mathbb{Z}^3}{2\pi \mathbb{Z}^3} \right)
\]

where $\varepsilon_n = \frac{1}{L_n}$. The “single period” lattices are

\[
\mathcal{B}_n = \left( \frac{\varepsilon_n^2 \mathbb{Z}}{\varepsilon_n \mathbb{Z}} \right) \times \left( \frac{\varepsilon_n \mathbb{Z}^3}{\mathbb{Z}} \right) \\
\mathcal{B}_n = \left( \frac{2\pi \mathbb{Z}}{\varepsilon_n \mathbb{Z}} \right) \times \left( \frac{2\pi \mathbb{Z}^3}{\varepsilon_n \mathbb{Z}^3} \right) \\
\mathcal{B}^+ = \left( \frac{\mathbb{Z}}{L^2 \mathbb{Z}} \right) \times \left( \frac{\mathbb{Z}^3}{L \mathbb{Z}^3} \right) \\
\mathcal{B}^+ = \left( \frac{2\pi \mathbb{Z}}{L^2 \mathbb{Z}} \right) \times \left( \frac{2\pi \mathbb{Z}^3}{L \mathbb{Z}^3} \right)
\]

The following table specifies where, in this paper, bounds on the various operators are proven.
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<tr>
<td>$S_{n,\nu}^{(\pm)}(\mu), S_{n,\nu}^{(\pm)}$</td>
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<td>$A_{\psi,\phi}$</td>
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<tr>
<td>$A_{\psi,\phi,\nu,\nu}(\mu)$</td>
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<tr>
<td>$\delta_{n+1}$</td>
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References


