Operators for Block Spin Transformations

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I. Introduction

In [parabolic-all.tex], we exhibit, for a many particle system of weakly interacting Bosons in three space dimensions, the formation of a potential well of the type that typically leads to symmetry breaking in the thermodynamic limit. To do so, we use the block spin renormalization group approach. In previous papers [BFKT4,BFKT1,BFKT2,BFKT3] (followed by a simple change of variables) we have written the partition function of such a system on a discrete torus* in terms of a functional integral on a 1 + 3 dimensional space

\[ X_0 = \left( \mathbb{Z}/L_{tp} \mathbb{Z} \right) \times \left( \mathbb{Z}^3/L_{sp} \mathbb{Z}^3 \right) \]

with positive integers \( L_{tp}, L_{sp} \). Up to corrections which are exponentially small in the coupling constant, and up to a multiplicative normalization factor, this representation is of the form

\[
\int \left[ \prod_{x \in X_0} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \right] e^{A_0(\psi^*, \psi)} \chi_0(\psi)
\]

with an action \( A_0 \) of the form

\[
A_0(\psi^*, \psi) = -\langle \psi^*, D_0 \psi \rangle_0 - V_0(\psi^*, \psi) + \mu_0 \langle \psi^*, \psi \rangle_0 + E'_0(\psi^*, \psi)
\]

Here
- \( D_0 = 1 - e^{-h_0} - e^{-h_0} \partial_0 \), where \( \partial_0 \) the forward time derivative (see (II.8) below) and \( h_0 \) is – up to a scaling – the single particle Hamiltonian.
- \( V_0(\psi^*, \psi) \) is a quartic monomial that describes the coupling between the particles
- \( \mu_0 \) is related to the chemical potential of the system
- \( E'_0(\psi^*, \psi) \) is perturbatively small
- \( \chi_0(\psi) \) is a “small field cut off function”.

See [parabolic-all.tex, (I.3), (I.4)].

For the block spin renormalization group action, we pick a “block rectangle” of length \( L_{tp} \) in the “time direction” and \( L \) in “space directions”, where \( L \) is a sufficiently large odd positive integer, and a corresponding nonnegative, compactly supported function \( q(x) \) on \( \mathbb{Z} \times \mathbb{Z}^3 \) (the averaging profile). The choice of this kind of rectangle is characteristic of the “parabolic scaling”, see [parabolic-all.tex, Definition I.3, Remark I.4, Definition I.13.d]. For simplicity we assume that \( L_{sp} \) and \( L_{tp} \) are powers of \( L \).

The block spin averaging operator, which we denote \( Q \), maps functions on the lattice \( X_0 \) to functions on the “coarse” lattice \( X_0^{(1)} = \left( L^2 \mathbb{Z}/L_{tp} \mathbb{Z} \right) \times \left( L \mathbb{Z}^3/L_{sp} \mathbb{Z}^3 \right) \). After each

* All bounds achieved so far are uniform in the volume of this torus.
renormalization group step we scale, to again give functions on a unit lattice. After the first
RG step this unit lattice is \( \mathcal{X}_0^{(1)} = \left( \mathbb{Z}/\frac{1}{L_{tp}} \mathbb{Z} \right) \times \left( \mathbb{Z}^3/\frac{1}{L_{sp}} \mathbb{Z}^3 \right) \). The “scaled” block spin
averaging operator maps functions on the lattice \( \mathcal{X}_1 = \left( \frac{1}{L_{tp}} \mathbb{Z}/\frac{1}{L_{tp}} \mathbb{Z} \right) \times \left( \frac{1}{L_{sp}} \mathbb{Z}^3/\frac{1}{L_{sp}} \mathbb{Z}^3 \right) \) to functions on the unit lattice \( \mathcal{X}_0^{(1)} \).

In the \( n \)th renormalization group step, we end up considering functions on the chain
of lattices
\[
\mathcal{X}_{-1}^{(n+1)} \subset \mathcal{X}_0^{(n)} \subset \mathcal{X}_1^{(n-1)} \subset \cdots \subset \mathcal{X}_{n-1}^{(1)} \subset \mathcal{X}_0^{(0)}
\]
where, for integers \( j \geq -1 \) and \( n \geq 0 \),
\[
\mathcal{X}_j^{(n)} = \left( \varepsilon_j^2 \mathbb{Z}/\varepsilon_{n+j}^2 L_{tp} \mathbb{Z} \right) \times \left( \varepsilon_j^3/\varepsilon_{n+j} L_{sp} \mathbb{Z}^3 \right)
\]
The subscript in \( \mathcal{X}_j^{(n)} \) determines the “coarseness” of the lattice — nearest neighbour
points are a distance \( \varepsilon_{2j} = \frac{1}{L} \) apart in the time direction and a distance \( \varepsilon_j = \frac{1}{L} \) apart in
spatial directions. The superscript in \( \mathcal{X}_j^{(n)} \) determines the number of points in the lattice — \( |\mathcal{X}_j^{(n)}| = |\mathcal{X}_0|/L^{5n} \) for all \( j \). We usually write \( \mathcal{X}_n^{(0)} = \mathcal{X}_n \). See [parabolic-all.tex, Definition I.5.a].

The \( (n+1) \)st block spin transformation involves the passage from \( \mathcal{X}_0^{(n)} \) to its sublattice
\( \mathcal{X}_{-1}^{(n+1)} \). The averaging operations determine linear maps
\[
Q : \mathcal{H}_0^{(n)} \mapsto \mathcal{H}_{-1}^{(n+1)} \quad \text{and} \quad Q_n : \mathcal{H}_n = \mathcal{H}_0^{(0)} \mapsto \mathcal{H}_0^{(n)}
\]
where \( \mathcal{H}_j^{(n)} = L^2(\mathcal{X}_j^{(n)}) \) denotes the the (finite dimensional) Hilbert space of functions on
\( \mathcal{X}_j^{(n)} \) with integral \( \int_{\mathcal{X}_j^{(n)}} du = \varepsilon_j^3 \sum_{u \in \mathcal{X}_j^{(n)}} \) and the *real* inner product
\[
\langle \alpha_1, \alpha_2 \rangle_j = \int_{\mathcal{X}_j^{(n)}} \alpha_1(u) \alpha_2(u) \, du
\]
Again see [parabolic-all.tex, Definition I.5.a]. In §II we pick a specific averaging profile
and give bounds on the operators \( Q, Q_n, \) their Fourier transforms, and related operators.

Scaling is performed by the linear isomorphisms
\[
\mathbb{L} : \mathcal{X}_j^{(n)} \to \mathcal{X}_{j-1}^{(n)} \quad (u_0, u) \mapsto (L^2 u_0, L u)
\]
For a function \( \alpha \in \mathcal{H}_j^{(n)} \), define the function \( \mathbb{L}_\ast(\alpha) \in \mathcal{H}_{j-1}^{(n)} \) by \( \mathbb{L}_\ast(\alpha)(L u) = \alpha(u) \). In
particular, after rescaling and multiplication with the “scaling factor” \( L^{2n} \), the differential
operator \( D_0 \) in (I.2) becomes the operator
\[
D_n = L^{2n} \mathbb{L}_\ast^{-n} \left( \mathbb{I} - e^{-h_0} - e^{-h_0} \partial_0 \right) \mathbb{L}_\ast^{n}
\]
on $\mathcal{H}_n$. This operator is discussed in §III.

As mentioned above, the passage from a functional integral on $\mathcal{X}_0^{(n)}$ to a functional integral on $\mathcal{X}_1^{(n+1)}$ is an averaging procedure over, roughly speaking, a rectangle of size $L^2$ in the time direction and size $L$ in the spatial directions. This passage is analyzed using stationary phase techniques that involve

- the determination of critical fields on $\mathcal{X}_0^{(n)}$ (that are functions of external fields on $\mathcal{X}_1^{(n+1)}$) for an appropriate action, and
- a functional integral over “fluctuation fields” around the critical field.

The covariance for the integral over the fluctuation fields has been identified in [parabolic-all.tex, (I.15)] and is bounded in §IV.

The composition of the critical fields of $n$ renormalization group steps is – after rescaling – a field on $\mathcal{X}_n$, called the “background field”, that is a function of an external field on $\mathcal{X}_0^{(n)}$. It is crucial in our representation of the partition function. See [parabolic-all.tex, Theorem I.19]. The “leading order” part of the background field is linear in the external field and has been identified in [parabolic-all.tex, Proposition I.16]. It is the composition of an operator, from $\mathcal{H}_0^{(n)}$ to $\mathcal{H}_n$ determined by the averaging profile $q$, and an operator $S_n$ on $\mathcal{H}_n$ which can be viewed as a Green’s function for the differential operator $D_n$ (plus a mass term). This operator, $S_n$, is discussed in §V.

To get bounds on the critical fields in the fluctuation integral at step $n + 1$, we use a well known algebraic relation between these critical fields and the background fields at step $n + 1$ given in [blockspinRGalg.tex, Proposition B.8], [parabolic-all.tex, Proposition IV.4.a]. The operators in the linearization of this relation, and various other linearizations, are studied in §VI.

By construction, many of the operators discussed in this paper are linear operators defined on the Hilbert space of functions on a lattice that are invariant under translations with respect to a sublattice. It is natural to use Bloch decompositions and Fourier transforms for an analysis of such operators. Abstract basic results about Bloch decompositions and their connections to Fourier transform are summarized in Appendix A, giving special emphasis to averaging operators.

Most operator estimates we obtain in this paper are with respect to a norm of the following kind.

**Definition I.1** For any operator $A : \mathcal{H}_j^{(n-j)} \to \mathcal{H}_k^{(n-k)}$, with kernel $A(u, u')$, and for any mass $m \geq 0$, we define the norm

$$
\|A\|_m = \max \left\{ \sup_{u \in \mathcal{X}_k^{(n-k)}} \int_{\mathcal{X}_j^{(n-j)}} du' \ e^{m |u-u'|} |A(u, u')|, \sup_{u' \in \mathcal{X}_j^{(n-j)}} \int_{\mathcal{X}_k^{(n-k)}} du \ e^{m |u-u'|} |A(u, u')| \right\}
$$
In the special case that \( m = 0 \), this is just the usual \( \ell^1-\ell^\infty \) norm of the kernel.

We see in Lemmas A.11 and A.12 that this norm is related to the analyticity properties of the Fourier transform. In this paper we use the following Fourier transform conventions.

The dual lattice of \( \mathcal{X}_j^{(n)} \) is

\[
\hat{\mathcal{X}}_j^{(n)} = \left( \frac{2\pi}{\epsilon_{n+j} L_{tp}} \mathbb{Z} / \frac{2\pi}{\epsilon_j} \mathbb{Z} \right) \times \left( \frac{2\pi}{\epsilon_{n+j} L_{sp}} \mathbb{Z}^3 / \frac{2\pi}{\epsilon_j} \mathbb{Z}^3 \right)
\]

For a function \( \alpha \in \mathcal{H}_j^{(n)} \)

\[
\hat{\alpha}(p) = \int_{\mathcal{X}_j^{(n)}} \alpha(u) e^{-ip \cdot u} \, du \quad \alpha(u) = \int_{\hat{\mathcal{X}}_j^{(n)}} \hat{\alpha}(p) e^{iu \cdot p} \, \frac{dp}{(2\pi)^4}
\]

where \( \int_{\mathcal{X}_j^{(n)}} \frac{dp}{(2\pi)^4} = \frac{1}{\epsilon_{n+j} L_{tp} L_{sp}} \sum_{p \in \hat{\mathcal{X}}_j^{(n)}} \). The maps

\[
\mathbb{L} : \hat{\mathcal{X}}_{j-1}^{(n)} \to \hat{\mathcal{X}}_j^{(n)} \quad (q_0, q) \mapsto (L^2 q_0, Lq)
\]

are again linear isomorphisms, and, for a function \( \alpha \in \mathcal{H}_j^{(n)} \),

\[
\overline{\mathbb{L} \ast (\alpha)}(q) = L^5 \hat{\alpha}(\mathbb{L} q)
\]

The quotient map dual to the inclusion \( \mathcal{X}_j^{(n)} \subset \mathcal{X}_{j+k}^{(n-k)} \) is

\[
\tilde{\pi}^{(j+k,j)} : \hat{\mathcal{X}}_{j+k}^{(n-k)} \to \hat{\mathcal{X}}_j^{(n)}
\]

When the indices are clear from the context we suppress them and write \( \hat{\pi} \).

The estimates of this paper are used in [parabolic-all.tex]. In particular, the construction of the background fields and the critical fields in [parabolic-all.tex, Appendix G] uses a contraction mapping argument around the linearizations of §VI and §V in this paper.

For the readers’ convenience we have included, in Appendix C, a list of most of the operators and lattices that appear in this paper.

**Convention I.2** Most estimates in this paper are bounds on norms of operators as in Definition I.1. The (finite number of) constants that appear in these bounds are consecutively labelled \( \Gamma_1, \Gamma_2, \cdots, \gamma_1, \gamma_2, \cdots, m_1, m_2, \cdots \). All of these constants \( \Gamma_j, \gamma_j, m_j \) are independent of \( L \) and the scale index \( n \). We define \( \Gamma_{op} \) to be the maximum of the \( \Gamma_j \)’s, and, in [parabolic-all.tex], refer to the estimates using this constant \( \Gamma_{op} \).
II. Block Spin Operators

In this chapter, we analyze the block spin “averaging operators” \( Q \) of [parabolic-all.tex, Definitions I.1.a and I.13.d] and \( Q_n \) of [parabolic-all.tex, Definition I.13.d] as well as the operator \( \mathcal{Q}_n \) of [parabolic-all.tex, Definition I.5.b]. Recall that \( Q : \mathcal{H}_{0}^{(n)} \to \mathcal{H}_{-1}^{(n+1)} \) is defined by

\[
(Q\psi)(y) = \sum_{x \in \mathbb{Z} \times \mathbb{Z}^3} q(y - x) \psi([x]) \tag{II.1}
\]

where \([x]\) denotes the class of \( x \in \mathbb{Z} \times \mathbb{Z}^3 \) in the quotient space \( \mathcal{X}_0^{(n)} \). The averaging profile \( q \) is the \( q \)-fold convolution of the characteristic function, \( 1_{\mathbb{Z}}(x) \), of the rectangle \([-L^{-1}, L^{-1}] \times [-L^{-1}, L^{-1}]^3 \), normalized to have integral one. That is,

\[
q = \frac{1}{L^2 q} 1_{\mathbb{Z}} * 1_{\mathbb{Z}} * \cdots * 1_{\mathbb{Z}}
\]

See Example A.8 and Remark A.10. Here \( q \geq 4 \) is a fixed even natural number.\(^{(1)}\)

The operator \( Q_n = Q^{(1)} \cdots Q^{(n)} = (\mathbb{L}_s^{-1} Q)^n \mathbb{L}_s^n : \mathcal{H}_n = \mathcal{H}_n^{(0)} \to \mathcal{H}_0^{(n)} \) \( (II.2) \)

where \( Q^{(j)} = \mathbb{L}_s^{-j} Q \mathbb{L}_s^j : \mathcal{H}_j^{(n-j)} \to \mathcal{H}_{j-1}^{(n-j+1)} \). The operator

\[
\mathcal{Q}_n = a(1 + \sum_{j=1}^{n-1} \frac{1}{L^{2j}} Q_j Q_j^*)^{-1}
\]

The Fourier transform of the characteristic function \( 1_{\mathbb{Z}} \) is \( \frac{\sigma(L k)}{\sigma(k)} \) with \( k \in \hat{\mathcal{X}}_0^{(n)} \) and with

\[
\sigma(k) = \sin \left( \frac{1}{2} k_0 \right) \prod_{\nu=1}^{3} \sin \left( \frac{1}{2} k_\nu \right) \tag{II.3}
\]

Therefore

\[
\hat{q}(k) = u_+(k)^q \quad \text{with} \quad u_+(k) = \frac{\sigma(\mathbb{L}_s k)}{L^5 \sigma(k)} \tag{II.4}
\]

and, by Lemma A.9.a

\[
\hat{(Q\psi)}(\hat{\mathcal{t}}) = \sum_{k \in \hat{\mathcal{X}}_0^{(n)} : \hat{\theta}(k) = \hat{\mathcal{t}}} \hat{q}(k) \hat{\psi}(k) \tag{II.5}
\]

for all \( \psi \in \mathcal{H}_0^{(n)} \) and \( \mathcal{t} \in \hat{\mathcal{X}}_{-1}^{(n+1)} \).

\(^{(1)}\) See Remark II.7 for a discussion of the condition \( q > 2 \). The condition that \( q \) be even is imposed purely for convenience.
Remark II.1

(a) Since \( q \) is even, \( \sigma(k)^q \) is an entire function of \( k \in \mathbb{C} \times \mathbb{C}^3 \) that is periodic with respect to the lattice \( 2\pi(\mathbb{Z} \times \mathbb{Z}^3) \). Also

\[
\sigma(p_j)^q = \sigma(\hat{\pi}^{(j,0)}(p_j))^q \quad \text{for all } p_j \in \hat{X}_j^{(n-j)}
\]

(b) For all \( \phi \in \mathcal{H}_n \) and \( k \in \hat{X}_0^{(n)} \),

\[
(\hat{Q}_n \phi)(k) = \sum_{p \in \hat{X}_n, \hat{n}(p) = k} u_n(p)^q \phi(p) \quad \text{with} \quad u_n(p) = \varepsilon_n^5 \frac{\sigma(p)}{\sigma(\varepsilon_n^{-n} p)}
\]

(c) For all \( \psi \in \mathcal{H}_0^{(n)} \) and \( k \in \hat{X}_0^{(n)} \), \( \hat{\Omega}_n \psi(k) = \hat{\Omega}(k) \hat{\psi}(k) \) where

\[
\hat{\Omega}_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{p_j \in \hat{X}_j^{(n-j)}} \frac{1}{L_{p_j}} u_j(p_j)^{2q} \right]^{-1}
\]

(d) The functions \( u_n(p) \) and \( u_+(p) \) are entire in \( p \) and are invariant under \( p_\nu \to -p_\nu \) for each \( 0 \leq \nu \leq 3 \).

(e) Set, with the notation of (I.4), the “single period” lattices and their duals

\[
\mathcal{B}^+ = (\mathbb{Z} / L^2 \mathbb{Z}) \times (\mathbb{Z}^3 / L \mathbb{Z}^3) \quad \hat{\mathcal{B}}^+ = (\varepsilon_1^2 \mathbb{Z} / 2\pi \mathbb{Z}) \times (\varepsilon_1^3 \mathbb{Z}^3 / 2\pi \mathbb{Z}^3) = \ker \hat{\pi}^{(0,-1)}
\]

\[
\mathcal{B}_j = (\varepsilon_j^2 \mathbb{Z} / \mathbb{Z}) \times (\varepsilon_j \mathbb{Z}^3 / \mathbb{Z}^3) \quad \hat{\mathcal{B}}_j = (2\pi \mathbb{Z} / \varepsilon_j \mathbb{Z}) \times (2\pi \mathbb{Z}^3 / \varepsilon_j \mathbb{Z}^3) = \ker \hat{\pi}^{(j,0)}
\]

for each integer \( j \geq 0 \). In this notation, the representations of \( Q, Q_n \) and \( \Omega_n \) of (II.5) and parts (b) and (c) are

\[
(\hat{Q} \psi)(\ell) = \sum_{\ell \in \bar{\mathcal{B}}^+} u_+(\ell + \ell)^q \hat{\psi}(\ell + \ell)
\]

\[
(\hat{Q}_n \phi)(k) = \sum_{\ell \in \mathcal{B}_n} u_n(k + \ell)^q \hat{\phi}(k + \ell)
\]

\[
\hat{\Omega}_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in \bar{\mathcal{B}}_j} \frac{1}{L_{\ell_j}} u_j(k + \ell_j)^{2q} \right]^{-1}
\]

Here in \( (\hat{Q} \psi)(\ell) = \sum_{\ell \in \bar{\mathcal{B}}^+} u_+(\ell + \ell)^q \hat{\psi}(\ell + \ell) \), for example, \( \ell \in \hat{X}_{-1}^{(n+1)} \) is represented by the element of \( \varepsilon_1^{-2} \mathbb{Z} / L_{\ell_0} \mathbb{Z} \times \varepsilon_1^{-3} \mathbb{Z}^3 / L_{\ell_0} \mathbb{Z}^3 \) having minimal components and \( \ell \) is represented by the element of \( \varepsilon_1^{-2} \mathbb{Z} \times \varepsilon_1^{-3} \mathbb{Z}^3 \) having minimal components. Similarly

\[
(\hat{Q}^* \hat{\theta})(\ell + \ell) = u_+(\ell + \ell)^q \hat{\theta}(\ell)
\]

\[
(\hat{Q}_n^* \psi)(k + \ell_n) = u_n(k + \ell_n)^q \hat{\psi}(k)
\]
Proof: (a) Any two points of \( \hat{X}^{(n-j)}_j \) with the same image in \( \hat{X}^{(n)}_0 \) under \( \hat{\pi}^{(j,0)}_n \) differ by \( 2\pi \) times an integer vector. The formula follows.

(b) By (I.3) and (II.5), we have, for \( \alpha \in H^{(n-j)}_j \) and \( p_{j-1} \in \hat{X}^{(n-j+1)}_{j-1} \)

\[
(\hat{Q}^{(j)}\alpha)(p_{j-1}) = \frac{1}{L^{5q}}(QL^j_\alpha)(L^{-j}p_{j-1}) \\
= \frac{1}{L^{5q}} \sum_{k \in \hat{X}^{(n-j)}_0 \atop \#(k) = L^{-j}p_{j-1}} \hat{q}(k)(L^j_\alpha)(k) \\
= \frac{1}{L^{5q}} \sum_{k \in \hat{X}^{(n-j)}_0 \atop \#(k) = L^{-j}p_{j-1}} \frac{\sigma(L^j k)}{\sigma(k)^q} \hat{\alpha}(L^j k) \\
= \frac{1}{L^{5q}} \sum_{p_{j-1} \in \hat{X}^{(n-j)}_j \atop \#(p_{j-1}) = p_{j-1}} \frac{\sigma(L^{-j+1}p_{j})}{\sigma(L^{-j}p_{j})^q} \hat{\alpha}(p_{j})
\]

so that, by part (a),

\[
(\hat{Q}_n \hat{\phi})(p_0) = \frac{1}{L^{5qn}} \sum_{p_{j-1} \in \hat{X}^{(n-j)}_j \atop \#(p_{j-1}) = p_{j-1}} \frac{\sigma(p_1)^q}{\sigma(L^{-1}p_1)^q} \frac{\sigma(L^{-2}p_2)^q}{\sigma(L^{-2}p_2)^q} \ldots \frac{\sigma(L^{-n+1}p_n)^q}{\sigma(L^{-n}p_n)^q} \hat{\phi}(p_n) \\
= \varepsilon_n^{5q} \sum_{p_{j-1} \in \hat{X}^{(n-j)}_j \atop \#(p_{j-1}) = p_{j-1}} \frac{\sigma(p_1)^q}{\sigma(L^{-1}p_1)^q} \frac{\sigma(L^{-1}p_2)^q}{\sigma(L^{-2}p_2)^q} \ldots \frac{\sigma(L^{-n+1}p_n)^q}{\sigma(L^{-n}p_n)^q} \hat{\phi}(p_n) \\
= \varepsilon_n^{5q} \sum_{p_n \in \hat{X}_n \atop \#(p_n) = p_0} \frac{\sigma(p_n)^q}{\sigma(L^{-n}p_n)^q} \hat{\phi}(p_n)
\]

(c) follows from part (b) and Lemma A.9.a.

(d) is obvious since \( \frac{\sin z}{\sin \frac{z}{m}} \) is even and entire for any nonzero integer \( m \).

In Lemmas II.2 and II.3 we derive a number of bounds on the kernels \( u_n \) and \( u_\perp \) that appear in the representations for \( Q_n \) and \( Q \) of Remark II.1.e. In Proposition II.4 we analyze the operator \( \hat{Q}_n \). Then in Remark II.5 and Lemma II.6 we study how to move derivatives past \( Q \) and \( Q_n \).
When dealing with the asymmetry between “temporal” and “spatial” scaling we set, for convenience,

\[
L_\nu = \begin{cases} 
L^2 & \text{for } \nu = 0 \\
L & \text{for } \nu = 1, 2, 3
\end{cases}
\]

\[
\varepsilon_{n,\nu} = \begin{cases} 
\frac{1}{\varepsilon_{n,\nu}} = \frac{1}{L^{2n}} & \text{for } \nu = 0 \\
\frac{1}{\varepsilon_{n,\nu}} = \frac{1}{L_\nu} & \text{for } \nu = 1, 2, 3
\end{cases}
\]  

(II.6)

**Lemma II.2** Assume that \(|\text{Re } k_\nu| \leq \pi, |\text{Im } k_\nu| \leq 2\) for each \(0 \leq \nu \leq 3\).

(a) \(|u_n(k + \ell)| \leq \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi}\) for all \(\ell \in \hat{B}_n\). We use \(|\ell_\nu|\) to denote the magnitude of the smallest representative of \(\ell_\nu\) in its equivalence class, as an element of \(\hat{B}_n\). There is a constant \(\Gamma_1\), depending only on \(q\), such that \(\|Q_n\|_{m=1} \leq \Gamma_1\).

(b) \(|u_n(k + \ell)| \leq |k| \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi}\) if \(0 \neq \ell \in \hat{B}_n\).

(c) \(|u_n(k) - 1| \leq 4^3|k|^2\).

(d) If \(\ell \in \hat{B}_n\) and \(\ell_\nu \neq 0\) for some \(0 \leq \nu \leq 3\), then \(u_n(k + \ell) = \sin \left(\frac{1}{2}k_\nu\right)v_{n,\nu}(k + \ell)\) with \(|v_{n,\nu}(k + \ell)| \leq \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi}\).

(e) For all \(\ell \in \hat{B}_n\),

\[
|\text{Im } u_n(k + \ell)| \leq 16 \ |\text{Im } k| \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi}
\]

\[
|\text{Im } u_n(k + \ell)^q| \leq 16 q \ |\text{Im } k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi} \right]^q
\]

(f) If \(k\) is real

\[
\left(\frac{2}{\pi}\right)^4 \leq u_n(k) \leq \left(\frac{2}{\pi}\right)^4
\]

**Proof**: Set \(s(x) = \frac{\sin x}{x}\). By the definitions of \(u_n(p)\) in Remark II.1.b, \(\sigma(k)\) in (II.3) and \(\varepsilon_{n,\nu}\) in (II.6),

\[
u_n(p) = \frac{\sin \frac{1}{2}p_0}{\frac{1}{\varepsilon_n}} \sin \frac{1}{2}p_0 \prod_{\nu=1}^{3} \frac{\sin \frac{1}{2}p_\nu}{\frac{1}{\varepsilon_n} \sin \frac{1}{2}p_\nu} = \prod_{\nu=0}^{3} \frac{s(p_\nu/2)}{s(\varepsilon_{n,\nu}p_\nu/2)}
\]

(II.7)

(a) We may assume without loss of generality that \(\ell_\nu\) is bounded, as a real number, by \(\frac{\pi}{\varepsilon_{n,\nu}} - \pi\). (Recall that \(\frac{1}{\varepsilon_{n,\nu}}\) is an odd natural number.) So \(|\text{Re } k_\nu + \ell_\nu|\) is always bounded by \(\frac{\pi}{\varepsilon_{n,\nu}}\). Consequently, the hypotheses of Lemma B.1.c, with \(x + iy = k_\nu + \ell_\nu\) and \(\varepsilon = \varepsilon_{n,\nu}\), are satisfied and

\[
\left| \frac{1}{\varepsilon_{n,\nu}} \sin \frac{1}{2}(k_\nu + \ell_\nu) \right| \leq \begin{cases} 
4 & \text{if } |\ell_\nu| = 0 \\
8 & \text{if } |\ell_\nu| \geq 2\pi \end{cases} \leq \frac{24}{|\ell_\nu| + \pi}
\]
since $|\text{Re} \ k_\nu| \leq \pi$ and $\ell_\nu \in 2\pi \mathbb{Z}$. As $q > 1$, $|u_n(k + \ell)|^q$ is summable in $\ell$ and the bound on $\|Q_n\|_{m=1}$ follows from Lemma A.11.c.

(b) If $\ell_\nu \neq 0$,

$$\frac{1}{2} |k_\nu + \ell_\nu| \geq \frac{1}{2} |\text{Re} \ k_\nu + \ell_\nu| \geq \frac{1}{6} (\pi + |\ell_\nu|)$$

so that, by Lemma B.1.a, the denominator

$$\frac{1}{\varepsilon_{n,\nu}} \left| \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu) \right| \geq \frac{1}{\varepsilon_{n,\nu}} \frac{\sqrt{2}}{\pi} \frac{1}{6} \varepsilon_{n,\nu} (\pi + |\ell_\nu|) = \frac{1}{3\sqrt{2}\pi} (\pi + |\ell_\nu|)$$

On the other hand, the numerator, by Lemma B.1.b,

$$|\sin \frac{1}{2} (k_\nu + \ell_\nu)| = |\sin \left(\frac{1}{2} k_\nu\right)| \leq |k_\nu|$$

As $\ell \neq 0$, there is at least one $\nu$ with $\ell_\nu \neq 0$. Pick one and call it $\tilde{\nu}$. Bound the factor

$$\left| \frac{\sin \frac{1}{2} (k_{\tilde{\nu}} + \ell_{\tilde{\nu}})}{\varepsilon_{n,\tilde{\nu}} \sin \frac{1}{2} \varepsilon_{n,\tilde{\nu}} (k_{\tilde{\nu}} + \ell_{\tilde{\nu}})} \right| \leq \frac{3\sqrt{2}\pi |k_{\tilde{\nu}}|}{|\ell_{\tilde{\nu}}| + \pi} \leq \frac{24|k|}{|\ell_{\tilde{\nu}}| + \pi}$$

Bound the remaining factors, with $\nu \neq \tilde{\nu}$, by $\frac{24}{|\ell_\nu| + \pi}$ as in part (a). All together

$$\prod_{\nu=0}^{3} \left| \frac{\sin \frac{1}{2} (k_\nu + \ell_\nu)}{\varepsilon_{n,\nu} \sin \frac{1}{2} \varepsilon_{n,\nu} (k_\nu + \ell_\nu)} \right| \leq |k| \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi}$$

(c) For both $z = \frac{1}{2} \varepsilon_{n,\nu} k_\nu$ and $z = \frac{1}{2} k_\nu$, $|z|^2 \leq \frac{\pi^2}{4} + 1 < 4$ so that, by Lemma B.1.e,

$$|\sin \frac{z}{z} - 1| \leq \frac{1}{2} |z|^2$$

Using $\frac{a}{b} - 1 = \frac{(a-1)-(b-1)}{b}$ we have, by Lemma B.1.a,

$$\left| \frac{\sin \left(\frac{1}{2} k_\nu\right)}{\varepsilon_{n,\nu} \sin \left(\frac{1}{2} \varepsilon_{n,\nu} k_\nu\right)} - 1 \right| \leq \frac{1}{2} \frac{|\frac{1}{2} k_\nu|^2 + \frac{1}{2} \varepsilon_{n,\nu} k_\nu|^2}{|\sin \left(\frac{1}{2} \varepsilon_{n,\nu} k_\nu\right)|/\left|\frac{1}{2} \varepsilon_{n,\nu} k_\nu\right|} \leq \frac{\pi}{\sqrt{2}} \frac{1}{8} (1 + \varepsilon_{n,\nu}) |k_\nu|^2 \leq |k_\nu|^2$$

Finally, using

$$\prod_{\nu=0}^{3} A_\nu - 1 = (A_0 - 1) \prod_{\nu=1}^{3} A_\nu + (A_1 - 1) \prod_{\nu=2}^{3} A_\nu + \cdots + (A_3 - 1)$$

we have, by Lemma B.1.c,

$$|u_n(k) - 1| \leq 4^3 |k_0|^2 + 4^3-1 |k_1|^2 + \cdots + |k_3|^2 \leq 4^3 |k|^2$$
(d) The proof is the same as that for part (b), except that the factor \( \sin \left( \frac{1}{2} k_\tilde{\nu} \right) \) in the numerator

\[
\sin \frac{1}{2}(k_\tilde{\nu} + \ell_\tilde{\nu}) = (-1)^{\frac{k_\tilde{\nu}}{2}} \sin \left( \frac{1}{2} k_\tilde{\nu} \right)
\]

is pulled out of \( u_n \), leaving \( v_{n, \tilde{\nu}} \), rather than being bounded by \( |k_\tilde{\nu}| \).

(e) By Lemma B.1.c

\[
\left| \text{Im} \sin \frac{1}{2}(k_\nu + \ell_\nu) \right| \leq 8 \left| \text{Im} k_\nu \right| \frac{24}{|\ell_\nu| + \pi}
\]

In general, for any complex numbers \( z_j = r_j e^{i\theta_j}, 1 \leq j \leq J \),

\[
\left| \text{Im} \prod_{j=1}^{J} z_j \right| = \left| \sin \left( \sum_{j=1}^{J} \theta_j \right) \right| \prod_{j=1}^{J} r_j
\]

Repeatedly using

\[
|\sin(\theta + \theta')| = |\sin(\theta) \cos(\theta') + \cos(\theta) \sin(\theta')| \leq |\sin(\theta)| + |\sin(\theta')|
\]

we have

\[
\left| \text{Im} \prod_{j=1}^{J} z_j \right| \leq \sum_{j=1}^{J} |\sin(\theta_j)| \prod_{j=1}^{J} r_j = \sum_{j=1}^{J} |\text{Im} z_j| \prod_{j \neq j'} |z_{j'}|
\]

So

\[
|\text{Im} u_n (k + \ell)| \leq \left( \sum_{\nu=0}^{3} 8 |\text{Im} k_\nu| \right) \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi} \leq 16 |\text{Im} k| \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi}
\]

and

\[
|\text{Im} u_n (k + \ell)^q| \leq 16^{q} |\text{Im} k| \left[ \prod_{\nu=0}^{3} \frac{24}{|\ell_\nu| + \pi} \right]^{q}
\]

(f) Just apply Lemma B.1.d separately to all of the numerators and denominators in the right hand side of (II.7).

\[\Box\]

**Lemma II.3** Assume that \( |\text{Re} t_\nu| \leq \frac{\pi}{L_\nu} \) and \( |\text{Im} t_\nu| \leq \frac{\pi}{L_\nu} \) for each \( 0 \leq \nu \leq 3 \).

(a) \( |u_+ (t + \ell)| \leq \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu| + \pi} \) for all \( \ell \in \hat{B}^+ \). We use \( |\ell_\nu| \) to denote the magnitude of the smallest representative of \( \ell_\nu \) in its equivalence class, as an element of \( \hat{B}^+ \).

(b) \( |u_+ (t + \ell)| \leq \left[ \prod_{\nu \in I_+} L_\nu |t_\nu| \right] \left[ \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu| + \pi} \right] \) for all \( \ell \in \hat{B}^+ \) with \( \ell \neq 0 \). Here \( I_+ \) is any subset of \( \{ \nu \mid 0 \leq \nu \leq 3, \ell_\nu \neq 0 \} \).
(c) |u_+(t) - 1| \leq 4^3 \sum_{\nu=0}^{3} L^2_{\nu} |x_{\nu}|^2.

(d) If \( \ell \in \hat{B}^+ \) and \( \ell_{\nu} \neq 0 \) for some \( 0 \leq \nu \leq 3 \), then \( u_+(t + \ell) = \sin \left( \frac{L_{\nu}}{2} t_{\nu} \right) v_+, \hat{\nu} (t + \ell) \) with
\[
|v_+, \hat{\nu} (t + \ell)| \leq \frac{\pi}{3} \sum_{\nu=0}^{3} \frac{24}{L_{\nu} |\ell_{\nu}| + \pi}.
\]

(e) For all \( \ell \in \hat{B}^+ \),
\[
|\text{Im} u_+(t + \ell)| \leq 16 |\text{Im} \, t| \frac{\pi}{3} \sum_{\nu=0}^{3} \frac{24}{L_{\nu} |\ell_{\nu}| + \pi}.
\]
\[
|\text{Im} u_+(t + \ell)|^q \leq 16 q |\text{Im} \, t| \left[ \frac{\pi}{3} \sum_{\nu=0}^{3} \frac{24}{L_{\nu} |\ell_{\nu}| + \pi} \right]^q.
\]

**Proof:**

(a) We may assume without loss of generality that \( \ell_{\nu} \) is bounded, as a real number, by \( \pi - \frac{\pi}{\ell_{\nu}} \). (Recall that \( L \) is an odd natural number.) So we will always have \( |\text{Re} \, t_{\nu} + \ell_{\nu}| \leq \pi \). So, by Lemma B.1.c with \( \varepsilon = \frac{1}{\ell_{\nu}} \) and \( x + iy = L_{\nu} (t_{\nu} + \ell_{\nu}) \),
\[
\left| \sin \frac{L_{\nu}}{2} (t_{\nu} + \ell_{\nu}) \right| \leq \begin{cases} 4 \frac{8}{L_{\nu} |\text{Re} \, t_{\nu} + \ell_{\nu}|} & \text{if } \ell_{\nu} = 0 \\ \frac{24}{L_{\nu} |\ell_{\nu}| + \pi} & \text{if } |L_{\nu} \ell_{\nu}| \geq 2\pi \end{cases}
\]

since \( |\text{Re} \, t_{\nu}| \leq \frac{\pi}{L_{\nu}}, |\text{Im} \, t_{\nu}| \leq \frac{2}{L_{\nu}} \) and \( \ell_{\nu} \in \frac{2\pi}{L_{\nu}} \mathbb{Z} \).

(b) If \( \ell_{\nu} \neq 0 \),
\[
\frac{1}{2} |t_{\nu} + \ell_{\nu}| \geq \frac{1}{2} |\text{Re} \, t_{\nu} + \ell_{\nu}| \geq \frac{1}{6} \left( \frac{\pi}{L_{\nu}} + |\ell_{\nu}| \right)
\]

and \( \frac{1}{2} |\text{Re} \, t_{\nu} + \ell_{\nu}| \leq \frac{\pi}{2} \) so that, by Lemma B.1.a, the denominator
\[
L_{\nu} |\sin \frac{1}{2} (t_{\nu} + \ell_{\nu})| \geq \frac{\sqrt{2}}{6\pi} (\pi + L_{\nu} |\ell_{\nu}|)
\]

On the other hand, the numerator, by Lemma B.1.b,
\[
\left| \sin \frac{L_{\nu}}{2} (t_{\nu} + \ell_{\nu}) \right| = \left| \sin \left( \frac{L_{\nu}}{2} t_{\nu} \right) \right| \leq 2 \frac{L_{\nu}}{2} |t_{\nu}| = L_{\nu} |t_{\nu}|
\]

As \( \ell \neq 0 \), there is at least one \( \nu \) with \( \ell_{\nu} \neq 0 \), so that \( \mathcal{I}_+ \) is not empty. Bound each factor with \( \nu \in \mathcal{I}_+ \) by
\[
\frac{\sin \frac{L_{\nu}}{2} (t_{\nu} + \ell_{\nu})}{L_{\nu} |\sin \frac{1}{2} (t_{\nu} + \ell_{\nu})|} \leq \frac{6\pi}{\sqrt{2}} \frac{L_{\nu} |t_{\nu}|}{L_{\nu} |\ell_{\nu}| + \pi}
\]

Bound the remaining factors by \( \frac{24}{L_{\nu} |\ell_{\nu}| + \pi} \).

(c) has the same proof as that of Lemma II.2.c, with \( k_{\nu} \) replaced by \( L_{\nu} t_{\nu} \) and \( \frac{1}{\varepsilon_{n, \nu}} \) replaced by \( L_{\nu} \).
(d) The proof is similar to that for part (b), except that the factor \( \sin \left( \frac{L \ell}{2} \nu \right) \) in the numerator
\[
\sin \frac{L \ell}{2} (\ell - \nu) = (-1)^{\frac{L \ell}{2}} \sin \left( \frac{L \ell}{2} \nu \right)
\]
is pulled out of \( u_+ \), leaving \( u_{+, \nu} \), rather than being bounded by \( L \nu |L| \). Also, each
\[
\left| \frac{\sin \left( \frac{L \ell}{2} (\ell - \nu) \right)}{L \nu \sin \frac{L}{2} (\ell - \nu)} \right| \text{ with } \nu \neq \tilde{\nu} \text{ is bounded by } \frac{24}{L \nu |L| + \pi}.
\]

(e) By Lemma B.1.c
\[
\left| \frac{\Im \left( \frac{L \ell}{2} (\ell - \nu) \right)}{L \nu \sin \frac{L}{2} (\ell - \nu)} \right| \leq 8 |\Im L \nu | \frac{24}{L \nu |L| + \pi}
\]
The proof now continues as in Lemma II.2.e, just by substituting \( k_\nu \rightarrow L \nu t_\nu, \ell_\nu \rightarrow L \nu \ell_\nu \) and \( \varepsilon_{n, \nu} = \frac{1}{L \nu} \).

**Proposition II.4** There are constants \( \Gamma_2 \), depending only on \( q \), and \( \Gamma_3 \), depending only on \( a \), such that the following hold for all \( L > \Gamma_2 \).

(a) On the domain \( \{k \in \mathbb{C} \times \mathbb{C}^3 \mid |\Im k_\nu| < 2 \text{ for each } 0 \leq \nu \leq 3 \} \), \( \hat{\Omega}_n(k) \) is analytic, and invariant under \( k_\nu \rightarrow -k_\nu \) for each \( 0 \leq \nu \leq 3 \), and obeys
\[
\frac{5}{6} a \leq |\hat{\Omega}_n(k)| \leq \frac{5}{4} a \quad \text{Re} \hat{\Omega}_n(k) \geq \frac{a}{2}
\]
If \( k \) is real \( \frac{5}{6} a \leq \hat{\Omega}_n(k) \leq a \).

(b) If \( |\Re k_\nu| \leq \pi \) and \( |\Im k_\nu| \leq 2 \) for each \( 0 \leq \nu \leq 3 \), then
\[
|\hat{\Omega}_n(k) - a_n| \leq \frac{a}{3003} |k|^2 \quad \text{where} \quad a_n = a \frac{1 - L^{-2}}{1 - L^{-2m}}
\]

(c) \( \|\Omega_n\|_{m=1} \leq \Gamma_3 \)

**Proof:** (a) Recall, from Remark II.1.e, that \( \hat{\Omega}_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2j}} u_j(k + \ell_j)^{2q} \right]^{-1} \).

By Lemma II.2.a,
\[
\sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2j}} |u_j(k + \ell_j)|^{2q} \leq \sum_{j=1}^{\infty} \frac{1}{L^{2j}} \sum_{\ell \in 2\pi \mathbb{Z} \times 2\pi \mathbb{Z}} \prod_{\nu=0}^{3} \left( \frac{24}{|L \nu| + \pi} \right)^{2q} = \frac{c_q}{L^{2-1}}
\]
where \( c_q = \left[ \sum_{j \in \mathbb{Z}} \left( \frac{24}{|2j| + 1} \right)^{2q} \right]^{4} \). Just pick \( L \) large enough that \( \frac{c_q}{L^{2-1}} < \frac{1}{5} \) and use that, if \( |z| \leq \frac{1}{5} \)
\[
\text{Re} \frac{1}{1 + z} = \frac{\text{Re}(1 + \overline{z})}{|1 + z|^2} \geq \frac{4/5}{(6/5)^2} = \frac{20}{36}
\]
(b) Using $O(|k|^2)$ to denote any function that is bounded by a constant, depending only on $q$,

$$
1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2q}} u_j(k + \ell_j)^{2q} = \frac{1}{L^{2q}} + \sum_{j=1}^{n-1} \frac{1}{L^{2q}} [u_j(k)^{2q} - 1] + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^{2q}} u_j(k + \ell_j)^{2q}
$$

\[
\le \frac{1 - L^{-2}}{1 - L^{-2}} + \sum_{j=1}^{n-1} \frac{1}{L^{2q}} O(|k|^2) \quad \text{by Lemma II.2.b,c}
\]

\[
\le \frac{1 - L^{-2}}{1 - L^{-2}} + \frac{1}{L^{2}} O(|k|^2)
\]

So

$$
\hat{Q}_n(k) = a \left[ \frac{1 - L^{-2}}{L^{2}} + \frac{1}{L^{2}} O(|k|^2) \right]^{-1} = a_n \left[ 1 + \frac{1}{L^{2}} \frac{1 - L^{-2}}{L^{2}} O(|k|^2) \right]^{-1}
$$

and it suffices to choose $\Gamma_2$ large enough that

$$
\frac{9}{10} \le \frac{1 - L^{-2}}{L^{2}} \le \frac{11}{10} \quad \left| \frac{1}{L} \frac{1 - L^{-2}}{L^{2}} O(|k|^2) \right| \le \frac{10}{22} \frac{1}{9002} |k|^2 \quad \left| \frac{1}{L} \frac{1 - L^{-2}}{L^{2}} O(|k|^2) \right| \le \frac{1}{2}
$$

for all allowed $k$’s and $L$’s.

(c) follows immediately from part (a) and Lemma A.11.b.

For a function $\alpha \in \mathcal{H}_j^{(n)}$ and $\nu = 0, 1, 2, 3$, we define the forward derivative by

$$
(\partial_\nu \alpha)(x) = \frac{1}{\varepsilon_j, \nu} [\alpha(x + \varepsilon_j, \nu e_\nu) - \alpha(x)]
$$

where $e_\nu$ is a unit vector in the $\nu$th direction. The Fourier transforms

$$
\left( \left( \partial_\nu \phi \right) \left( p \right) \right) = 2 i e^{i \varepsilon_{n, \nu} p_\nu / 2} \frac{\sin(\varepsilon_{n, \nu} p_\nu / 2)}{\varepsilon_{n, \nu}} \hat{\phi}(p) \quad \text{for all } \phi \in \mathcal{H}_n \quad \text{and } p \in \mathcal{X}_n
$$

$$
\left( \left( \partial_\nu \psi \right) \left( k \right) \right) = 2 i e^{i k_\nu / 2} \sin(k_\nu / 2) \hat{\psi}(k) \quad \text{for all } \psi \in \mathcal{H}_0^{(n)} \quad \text{and } k \in \mathcal{X}_0^{(n)}
$$

$$
\left( \left( \partial_\nu \theta \right) \left( t \right) \right) = 2 i e^{i L_\nu t_\nu / 2} \frac{\sin(L_\nu t_\nu / 2)}{L_\nu} \hat{\theta}(t) \quad \text{for all } \theta \in \mathcal{H}_0^{(n+1)} \quad \text{and } t \in \mathcal{X}_0^{(n+1)}
$$

We now define operators like, for example, $Q_{n, \nu}^{(-)}$ chosen so that $\partial_\nu Q_n = Q_{n, \nu}^{(-)} \partial_\nu$. Set

$$
u_{n, \nu}^{(+)}(p) = \prod_{0 \leq \nu' \leq 3 \atop \nu' \neq \nu} \frac{\sin \frac{1}{2} p_{\nu'}}{\sin \frac{1}{2} \varepsilon_{n, \nu'} p_{\nu'}} \quad \nu_{n, \nu}^{(+)}(k) = \prod_{0 \leq \nu' \leq 3 \atop \nu' \neq \nu} \frac{\sin \frac{1}{2} L_{\nu'} k_{\nu'}}{L_{\nu'} \sin \frac{1}{2} k_{\nu'}}
$$

$$
u_{n, \nu}^{(-)}(p) = \frac{1}{3} \frac{\sin \frac{1}{2} p_{\nu}}{\sin \frac{1}{2} \varepsilon_{n, \nu} p_{\nu}} \prod_{\nu' = 0}^{3} \frac{1}{3} \frac{\sin \frac{1}{2} p_{\nu'}}{\sin \frac{1}{2} \varepsilon_{n, \nu'} p_{\nu'}} \quad \nu_{n, \nu}^{(-)}(k) = \frac{1}{3} \frac{\sin \frac{1}{2} L_{\nu} k_{\nu}}{L_{\nu} \sin \frac{1}{2} k_{\nu}} \prod_{\nu' = 0}^{3} \frac{1}{3} \frac{\sin \frac{1}{2} L_{\nu'} k_{\nu'}}{L_{\nu'} \sin \frac{1}{2} k_{\nu'}}
$$

and

$$
\zeta_{n, \nu}^{(+)}(k, \ell_n) = e^{i \varepsilon_{n, \nu} (k + \ell_n) / 2} e^{-i k_\nu / 2} \cos \frac{1}{2} \ell_n, \nu \quad \zeta_{n, \nu}^{(+)}(t, \ell) = e^{i(t + \ell) / 2} e^{-i L_\nu t_\nu / 2} \cos \frac{1}{2} L_\nu \ell
$$

$$
\zeta_{n, \nu}^{(-)}(k, \ell_n) = e^{i k_\nu / 2} e^{-i \varepsilon_{n, \nu} (k + \ell_n) / 2} \cos \frac{1}{2} \ell_n, \nu \quad \zeta_{n, \nu}^{(-)}(t, \ell) = e^{i L_\nu t_\nu / 2} e^{-i (t + \ell) / 2} \cos \frac{1}{2} L_\nu \ell
$$

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Define the operators \( Q_{n,\nu}^{(+)} : \mathcal{H}_0^{(n)} \to \mathcal{H}_n \) and \( Q_{n,\nu}^{(-)} : \mathcal{H}_n \to \mathcal{H}_0^{(n)} \) by

\[
\begin{align*}
(Q_{n,\nu}^{(+)} \psi)(k + \ell_n) &= \zeta_{n,\nu}^{(+)}(k; \ell_n) u_{n,\nu}^{(+)}(k + \ell_n) u_n(k + \ell_n)^q \psi(k) \\
(Q_{n,\nu}^{(-)} \phi)(k) &= \sum_{\ell_n \in \mathcal{B}_n} \zeta_{n,\nu}^{(-)}(k; \ell_n) u_{n,\nu}^{(-)}(k + \ell_n) u_n(k + \ell_n)^q \phi(k + \ell_n)
\end{align*}
\]

(II.10)

and the operators \( Q_{+\nu}^{(+)} : \mathcal{H}_{-1}^{(n+1)} \to \mathcal{H}_0^{(n)} \) and \( Q_{+\nu}^{(-)} : \mathcal{H}_0^{(n)} \to \mathcal{H}_{-1}^{(n+1)} \) by

\[
\begin{align*}
(Q_{+\nu}^{(+)} \theta)(\ell + \ell) &= \zeta_{+\nu}^{(+)}(\ell, \ell) u_{+\nu}^{(+)}(\ell + \ell) u_+^{(+)}(\ell + \ell)^q \theta(\ell) \\
(Q_{+\nu}^{(-)} \psi)(\ell) &= \sum_{\ell \in \mathcal{B}^+} \zeta_{+\nu}^{(-)}(\ell, \ell) u_{+\nu}^{(-)}(\ell + \ell) u_+^{(-)}(\ell + \ell)^q \psi(\ell + \ell)
\end{align*}
\]

(II.11)

**Remark II.5** Let \( 0 \leq \nu \leq 3 \). We have

\[
\partial_\nu Q_n^* = Q_{n,\nu}^{(+)} \partial_\nu \quad \partial_\nu Q_n = Q_{n,\nu}^{(-)} \partial_\nu \quad \partial_\nu Q^* = Q_{+\nu}^{(+)} \partial_\nu \quad \partial_\nu Q = Q_{+\nu}^{(-)} \partial_\nu
\]

If \( S : \mathcal{H}_n \to \mathcal{H}_n \) and \( T : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)} \) are linear operators that are translation invariant with respect to \( \mathcal{X}_n \) and \( \mathcal{X}_0^{(n)} \), respectively, then

\[
Q_{n,\nu}^{(-)} SQ_{n,\nu}^{(+)} = Q_n S Q_n^* \quad Q_{+\nu}^{(-)} T Q_{+\nu}^{(+)} = QTQ^*
\]

**Proof:** For the “\( Q_n^* \)” and “\( Q_n \)” cases, it suffices to observe that

\[
(2ie^{i\varepsilon_n,\nu(k+\ell)/2} \sin(\varepsilon_n,\nu(k+\ell)/2)) u_n(k + \ell) = \zeta_{n,\nu}^{(+)}(k, \ell) u_{n,\nu}^{(+)}(k + \ell)(2ie^{ik\nu/2} \sin(k\nu/2))
\]

and

\[
(2ie^{ik\nu/2} \sin(k\nu/2)) u_n(k + \ell) = \zeta_{n,\nu}^{(-)}(k, \ell) u_{n,\nu}^{(-)}(k + \ell)(2ie^{i\varepsilon_n,\nu(k+\ell)/2} \sin(\varepsilon_n,\nu(k+\ell)/2))
\]

and

\[
\zeta_{n,\nu}^{(-)}(k, \ell) u_{n,\nu}^{(-)}(k + \ell) \zeta_{n,\nu}^{(+)}(k, \ell) u_{n,\nu}^{(+)}(k + \ell) = u_n(k + \ell)^2
\]

for all \( k, \ell, \nu \). We remark that “\( Q_{n,\nu}^{(-)} SQ_{n,\nu}^{(+)} = Q_n S Q_n^* \)” should not be surprising since \( \partial_\nu Q_n S Q_n^* = Q_{n,\nu}^{(-)} SQ_{n,\nu}^{(+)} \partial_\nu \) and \( Q_n S Q_n^* \) is translation invariant on the unit scale and so commutes with \( \partial_\nu \). The proof for the “\( Q^* \)” and “\( Q \)” cases are virtually identical.
Lemma II.6  Let $0 \leq \nu \leq 3$, $\ell \in \hat{B}^+$ and $\ell_n \in \hat{B}_n$.

(a) $\zeta^{(+)}_{n,\nu}(k, \ell_n) u^{(+)}_{n,\nu}(k + \ell_n)$ and $\zeta^{(-)}_{n,\nu}(k, \ell_n) u^{(-)}_{n,\nu}(k + \ell_n)$ are entire in $k$ and $\zeta^{(+)}_{+,\nu}(\ell, \ell) u^{(+)}_{+,\nu}(\ell + \ell)$ and $\zeta^{(-)}_{+,\nu}(\ell, \ell) u^{(-)}_{+,\nu}(\ell + \ell)$ are entire in $\ell$

(b) Assume that $|\Re k_{\nu'}| \leq \pi$, $|\Im k_{\nu'}| \leq 2$, $|\Re k_{\nu'}| \leq \frac{\pi}{L_{\nu'}}$ and $|\Im k_{\nu'}| \leq \frac{2}{L_{\nu'}}$ for each $0 \leq \nu' \leq 3$. Then

\[
|\zeta_{n,\nu}^{(+)}(k, \ell_n) u_{n,\nu}^{(+)}(k + \ell_n)| \leq e^2 \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{|\ell_{n,\nu'}| + \pi} \\
|\zeta_{+,\nu}^{(+)}(\ell, \ell) u_{+,\nu}^{(+)}(\ell + \ell)| \leq e^2 \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{L_{\nu}|\ell_{\nu'}| + \pi} \\
|\zeta_{n,\nu}^{(-)}(k, \ell_n) u_{n,\nu}^{(-)}(k + \ell_n)| \leq e^2 \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{|\ell_{n,\nu'}| + \pi} \prod_{\nu' = 0}^{3} \frac{24}{|\ell_{\nu'}| + \pi} \\
|\zeta_{+,\nu}^{(-)}(\ell, \ell) u_{+,\nu}^{(-)}(\ell + \ell)| \leq e^2 \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{L_{\nu}|\ell_{\nu'}| + \pi} \prod_{\nu' = 0}^{3} \frac{24}{L_{\nu'}|\ell_{\nu'}| + \pi}
\]

(c) There is a constant $\Gamma_4$, depending only on $q$, such that $\|Q_{n,\nu}^{(\pm)}\|_{m=1} \leq \Gamma_4$.

**Proof:** (a) The proof is virtually identical to that of Remark II.1.d.

(b) The proof is virtually identical to that of Lemmas II.2.a and II.3.a.

(c) By (II.10), the Fourier transform of $Q_{n,\nu}^{(\pm)}$ is $\zeta_{n,\nu}^{(\pm)}(k, \ell_n) u_{n,\nu}^{(\pm)}(k + \ell_n) u_n(k + \ell_n)^q - 1$, which by part (b) and Lemma II.2.a, is bounded in magnitude by

\[
e^2 \prod_{0 \leq \nu' \leq 3, \nu' \neq \nu} \frac{24}{|\ell_{n,\nu'}| + \pi} \prod_{\nu' = 0}^{3} \frac{24}{|\ell_{\nu'}| + \pi} (q - 1)^{-1}
\]

As $q > 2$, the claim now follows by Lemma A.11.c.

Remark II.7  The principle obstruction to allowing $q = 1$ arises when a differential operator $\partial_\nu$ is intertwined with the block spin averaging operator $Q_n$, as happens in Remark II.5. See, for example, the proof of Lemma II.6.c. We use the condition $q > 1$ in Lemma II.2.a and then starting at Lemma IV.3 in §IV and in §V, VI. (See Lemma V.5.) We use the condition $q > 2$ in Proposition VI.1 and Lemma II.6.c.
III. Differential Operators

In [parabolic-all.tex, Definition I.5.a] we associated to an operator $h_0$ on $L^2(\mathbb{Z}^3/L_{sp}\mathbb{Z}^3)$ the operators

$$D_n = L^{2n} \mathbb{L}^{-n}_s (1 - e^{-h_0} - e^{-h_0} \partial_0)^n$$  \hspace{1cm} (III.1)

Here $\partial_0$ is the forward time derivative of (II.8). In this chapter we assume that $h_0$ is the periodization of a translation invariant operator $h_0$ on $L^2(\mathbb{Z}^3)$ whose Fourier transform $\hat{h}_0(\mathbf{p})$

- is entire in $\mathbf{p}$ and invariant under $\mathbf{p}_\nu \to -\mathbf{p}_\nu$ for each $1 \leq \nu \leq 3$
- is nonnegative when $\mathbf{p}$ is real
- obeys $\hat{h}_0(0) = \frac{\partial \hat{h}_0}{\partial \mathbf{p}_\mu} (0) = 0$ for $1 \leq \nu \leq 3$ and has strictly positive Jacobian matrix

$$H = \left[ \frac{\partial^2 \hat{h}_0}{\partial \mathbf{p}_\mu \partial \mathbf{p}_\nu} (0) \right]_{1 \leq \mu, \nu \leq 3}$$

Remark III.1

(a) The operator $D_n$ is the periodization of a translation invariant operator $D_n$, acting on $L^2(\varepsilon_\nu \mathbb{Z} \times \varepsilon_\nu^2 \mathbb{Z}^3)$, whose Fourier transform is

$$\hat{D}_n(\mathbf{p}) = \frac{1}{2} \varepsilon_\nu^2 p_0^2 e^{-\hat{h}_0(\varepsilon_\nu \mathbf{p})} \left[ \frac{\sin \frac{1}{2} \varepsilon_\nu^2 p_0}{\frac{1}{2} \varepsilon_\nu^2 p_0} \right]^2 + p^2 \frac{1 - e^{-\hat{h}_0(\varepsilon_\nu \mathbf{p})}}{\varepsilon_\nu^2 p_0^2} - i p_0 e^{-\hat{h}_0(\varepsilon_\nu \mathbf{p})} \frac{\sin \varepsilon_\nu^2 p_0}{\varepsilon_\nu^2 p_0}$$

with $\mathbf{p} = (p_0, \mathbf{p}) \in \Phi \times \Phi^3$.

(b) $\hat{D}_n(\mathbf{p})$ is entire in $\mathbf{p}$ and invariant under $\mathbf{p}_\nu \to -\mathbf{p}_\nu$ for each $1 \leq \nu \leq 3$.

(c) $\hat{D}_n(\mathbf{p})$ has nonnegative real part when $\mathbf{p}$ is real.

Proof:  (a) follows from (I.3) and the observation, by (II.9), that the Fourier transform of $\partial_0$, on $\mathbb{Z}$, is

$$2 i e^{ik_0/2} \sin(k_0/2) = -2 \sin^2(k_0/2) + i \sin(k_0)$$

(b) and (c) are obvious.
Lemma III.2 There are constants $\gamma_1$, $\Gamma_5$ and a function $\bar{m}(c) > 0$ that depend only on $\hat{h}_0$ and in particular are independent of $n$ and $L$, such that the following hold.

(a) For all $p \in \mathbb{R} \times \mathbb{R}^3$,
$$|\hat{D}_n(p)| \geq \gamma_1(|p_0| + \sum_{\nu=1}^3 |p_\nu|^2)$$

We use $|p_0|$, $|p_\nu|$ and $|p|$ to refer to the magnitudes of the smallest representatives of $p_0 \in \mathfrak{c}$, $p_\nu \in \mathfrak{c}$ and $p \in \mathfrak{c}^3$ in $\mathfrak{c}/2\pi \mathbb{Z}$, $\mathfrak{c}/2\pi \mathbb{Z}$ and $\mathfrak{c}^3/2\pi \mathbb{Z}^3$, respectively.

(b) For all $p \in \mathfrak{c} \times \mathfrak{c}^3$ with $\varepsilon_n^2 |p_0| \leq 1$ and $\varepsilon_n |\text{Im} \, p| \leq 1$,
$$\hat{D}_n(p) = -ip_0 + \frac{1}{2} \varepsilon_n^2 p_0^2 + \frac{1}{2} \sum_{\nu, \nu' = 1}^3 H_{\nu, \nu'} p_\nu p_{\nu'} + O\left(\varepsilon_n |p|^3 + \varepsilon_n^4 |p_0|^3\right)$$

The higher order part $O(\cdot)$ is uniform in $n$ and $L$.

(c) We have, for all $p \in \mathfrak{c} \times \mathfrak{c}^3$ with $\varepsilon_n^2 |\text{Im} \, p_0| \leq 1$ and $\varepsilon_n |\text{Im} \, p| \leq 1$,
$$|\hat{D}_n(p)| \leq \Gamma_5 \left(|p_0| + \sum_{\nu=1}^3 |p_\nu|^2\right)$$

and
$$\left|\frac{\partial^{\ell_i}}{\partial p_\nu} \hat{D}_n(p)\right| \leq \Gamma_5 \left\{ \begin{array}{ll}
\frac{1}{|1+|p_0|+|p|^2|^{1/2-1}} & \text{if } \nu = 0, \ell_i = 1, 2 \\
\frac{1}{|1+|p_0|+|p|^2|^{1/2-1}} & \text{if } 1 \leq \nu \leq 3, 1 \leq \ell_i \leq 4
\end{array}\right.$$  

(d) For all $c > 0$ and $p \in \mathfrak{c} \times \mathfrak{c}^3$, with $|p_0| + |p| \geq c$ and $|\text{Im} \, p| \leq \bar{m}(c)$,
$$|\hat{D}_n(p)| \geq \gamma_1 \left(|p_0| + \sum_{\nu=1}^3 |p_\nu|^2\right)$$

(e) For all $c > 0$ and all $p$ in the set
$$\{ \, p \in \mathfrak{c} \times \mathfrak{c}^3 \mid |\text{Im} \, p| \leq \bar{m}(c), \, |p| \geq c \} \cup \{ \, p \in \mathfrak{c} \times \mathfrak{c}^3 \mid |\text{Im} \, p| \leq \bar{m}(c), \, |\varepsilon_n p_0| \geq c \}$$

we have
$$\text{Re} \, \hat{D}_n(p) \geq \gamma_1 \left(\varepsilon_n^2 |p_0|^2 + \sum_{\nu=1}^3 |p_\nu|^2\right)$$

**Proof:** (a) By the hypotheses on $\hat{h}$,
$$\frac{1 - e^{-\bar{h}_0(\varepsilon_n p)}}{\varepsilon_n^2 p^2} = \frac{\bar{h}_0(\varepsilon_n p) + O(\bar{h}_0(\varepsilon_n p)^2)}{\varepsilon_n^2 p^2} = \frac{\frac{1}{2} \varepsilon_n^2 p : H p + O(|\varepsilon_n p|^3)}{\varepsilon_n^2 p^2}$$
for $\varepsilon_n p$ is a real neighbourhood of 0. This is strictly positive and bounded away from 0 on some real neighbourhood of 0, uniformly in $\varepsilon_n$. Since $\hat{h}$ is continuous and strictly positive on $\mathbb{R}^3 \setminus \frac{2\pi}{\varepsilon_n} \mathbb{Z}^3$, there is a constant $\gamma'_1 > 0$, independent of $\varepsilon_n$, such that

$$\frac{1-e^{-\hat{h}_0(\varepsilon_n p)}}{\varepsilon_n |p|^2} \geq \gamma'_1 \quad \text{and} \quad e^{-\hat{h}_0(\varepsilon_n p)} \geq \gamma'_1$$

for all $p \in \mathbb{R}^3$. The claim now follows from Lemma B.1.a.

(b) Expanding $\frac{\sin z}{z} = 1 + O(|z|^2)$, for $|z| \leq 1$, and

$$e^{-h(\varepsilon_n p)} = 1 - \frac{1}{2} \sum_{\nu,\nu'=1}^3 H_{\nu,\nu'} p_{\nu} p_{\nu'} + O\left( (\varepsilon_n |p|)^3 \right) \quad \text{for} \quad \varepsilon_n |p| \leq 1$$

gives, using Remark III.1.a,

$$\hat{D}_n(p) = \left[ -ip_0 + \frac{1}{2} \varepsilon_n^2 p_0 \right] \left[ 1 + O\left( \left( \varepsilon_n |p| \right)^2 + (\varepsilon_n^2 p_0)^2 \right) \right] + \frac{1}{2} \sum_{\nu,\nu'=1}^3 H_{\nu,\nu'} p_{\nu} p_{\nu'} + O\left( \varepsilon_n |p|^3 \right)$$

$$= -ip_0 + \frac{1}{2} \varepsilon_n^2 p_0^2 + \frac{1}{2} \sum_{\nu,\nu'=1}^3 H_{\nu,\nu'} p_{\nu} p_{\nu'} + O\left( \varepsilon_n |p|^3 + \varepsilon_n^2 |p_0| |p|^2 + \varepsilon_n^4 |p_0|^3 \right)$$

The claim now follows from

$$\varepsilon_n^2 |p_0| |p|^2 \leq \left( \varepsilon_n^4 |p_0|^3 \right)^{1/3} \left( \varepsilon_n^3 |p|^3 \right)^{2/3}$$

(c) Since $\hat{D}_n(p)$ is periodic with respect to $\frac{2\pi}{\varepsilon_n} \mathbb{Z} \times \frac{2\pi}{\varepsilon_n} \mathbb{Z}^3$, we may assume that $\varepsilon_n^2 \text{Re} p_0$ and each $\varepsilon_n \text{Re} p_{\nu}$, $1 \leq \nu \leq 3$ is bounded in magnitude by $\pi$. By Lemma B.1.b, $|\sin \frac{z}{2}| \leq 2$ for all $z \in \mathbb{C}$ with $|\text{Im} z| \leq 1$. Since $\varepsilon_n p$ runs over a compact set (independently of $n$ and $L$), $e^{-h(\varepsilon_n p)}$ is bounded. So

$$\left| \frac{1}{2} \varepsilon_n^2 p_0^2 e^{-\hat{h}_0(\varepsilon_n p)} \frac{\sin \frac{1}{2} \varepsilon_n^2 p_0}{\frac{1}{2} \varepsilon_n^2 p_0} \right| \leq \text{const} |p_0|$$

and

$$\left| \frac{1-e^{-\hat{h}_0(\varepsilon_n p)}}{\varepsilon_n^2} \right| \leq \text{const} \frac{\varepsilon_n |p|^2}{\varepsilon_n^2} \leq \text{const} |p|^2$$

This gives the bound on $|\hat{D}_n(p)|$.

The bounds

$$\frac{\partial}{\partial p_0} \varepsilon_n^2 \left[ \frac{\sin(\varepsilon_n^2 p_0)}{\frac{1}{2} \varepsilon_n^2} \right] = 2 \sin(\varepsilon_n^2 p_0) = O(1) \quad \text{and} \quad \frac{\partial}{\partial p_0} \left[ \frac{\sin(\varepsilon_n^2 p_0)}{\varepsilon_n^2} \right] = \cos(\varepsilon_n^2 p_0) = O(1)$$

$$\frac{\partial^2}{\partial p_0^2} \varepsilon_n^2 \left[ \frac{\sin(\varepsilon_n^2 p_0)}{\frac{1}{2} \varepsilon_n^2} \right] = 2 \varepsilon_n^2 \cos(\varepsilon_n^2 p_0) = O(\varepsilon_n^2) \quad \text{and} \quad \frac{\partial^2}{\partial p_0^2} \left[ \frac{\sin(\varepsilon_n^2 p_0)}{\varepsilon_n^2} \right] = -\varepsilon_n^2 \sin(\varepsilon_n^2 p_0) = O(\varepsilon_n^2)$$

$$\frac{\partial}{\partial p_{\nu}} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(|p|) \quad \frac{\partial^2}{\partial p_{\nu}^2} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(1+\varepsilon_n^2 |p|^2) = O(1)$$

$$\frac{\partial^3}{\partial p_{\nu}^3} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(\varepsilon_n^2 |p| + \varepsilon_n^4 |p|^3) \quad \frac{\partial^4}{\partial p_{\nu}^4} \left[ \frac{1}{\varepsilon_n^2} e^{-\hat{h}_0(\varepsilon_n p)} \right] = O(\varepsilon_n^2 + \varepsilon_n^4 |p|^2 + \varepsilon_n^6 |p|^4)$$
together with
\[ \varepsilon_n^2 \leq \frac{\text{const}}{1+|p_0|+|p|^2} \quad \varepsilon_n^2 |p_0| \leq \text{const} \]
yield the bounds on the derivatives.

(d) Write \( p = p + iQ \) with \( p = (p_0, p) \), \( Q = (Q_0, Q) \in \mathbb{R} \times \mathbb{R}^3 \). We may choose \( \bar{m}(c) \) sufficiently small that, if \( |p + iQ| \geq c \) and \( |Q| \leq \bar{m}(c) \), then
\[
|p_0 + iQ_0| + \sum_{\nu=1}^{3} |p_\nu + iQ_\nu|^2 \leq \left( |p_0| + \sum_{\nu=1}^{3} |p_\nu|^2 \right) + \left( |Q_0| + \sum_{\nu=1}^{3} |Q_\nu|^2 \right) \leq 2 \left( |p_0| + \sum_{\nu=1}^{3} |p_\nu|^2 \right)
\]
If \( \gamma > 0 \) is chosen small enough and \( \Gamma_5 \) is chosen large enough, then, for all such \( p, Q \), we have, by parts (a) and (c),
\[
\left| \hat{D}_n(p) \right| \geq 4\gamma_1 \left( |p_0| + \sum_{\nu=1}^{3} |p_\nu|^2 \right) \\
\geq 2\gamma_1 \left( |p_0 + iQ_0| + \sum_{\nu=1}^{3} |p_\nu + iQ_\nu|^2 \right)
\]
\[
\left| \hat{D}_n(p + iQ) - \hat{D}_n(p) \right| \leq 2\Gamma_5 \left( 1 + |p_0 + iQ_0| + \sum_{\nu=1}^{3} |p_\nu + iQ_\nu|^2 \right) |Q|
\]
Recalling that \( |p_0 + iQ_0| + \sum_{\nu=1}^{3} |p_\nu + iQ_\nu|^2 \geq \min \left\{ \frac{\varepsilon^2}{2}, \frac{\varepsilon^2}{4} \right\} \), it now suffices to choose \( \bar{m}(c) \) small enough that
\[
2\gamma_1 \bar{m}(c) \leq \frac{1}{2} \gamma_1 \min \left\{ \frac{\varepsilon^2}{2}, \frac{\varepsilon^2}{4} \right\} \quad \text{and} \quad 2\Gamma_5 \bar{m}(c) \leq \frac{1}{2} \gamma_1
\]
(e) Again write \( p = p + iQ \) with \( p = (p_0, p) \), \( Q = (Q_0, Q) \in \mathbb{R} \times \mathbb{R}^3 \). Since \( \hat{D}_n(p + iQ) \) is periodic with respect to \( p \in \frac{2\pi}{\varepsilon_n} \mathbb{Z} \times \frac{2\pi}{\varepsilon_n} \mathbb{Z}^3 \), we may assume that \( |\varepsilon_n^2 p_0| \leq \pi \) and \( |\varepsilon_n p_\nu| \leq \pi \), for each \( 1 \leq \nu \leq 3 \). If the constant \( \gamma_1 \) was chosen small enough, then, as in part (a),
\[
\text{Re} \hat{D}_n(p) = \frac{1}{2} \varepsilon_n^2 p_0^2 e^{-\hat{h}_0(\varepsilon_n p)} \left[ \sin \frac{1}{2} \varepsilon_n^2 p_0 \right]^2 + p^2 \frac{1 - e^{-\hat{h}_0(\varepsilon_n p)}}{\varepsilon_n^2 p^2} \geq 2\gamma_1 \left( \varepsilon_n^2 p_0^2 + |p|^2 \right)
\]
Hence it suffices to prove that it is possible to choose \( \bar{m} = \bar{m}(c) \) so that
\[
\left| \text{Re} \hat{D}_n(p + iQ) - \text{Re} \hat{D}_n(p) \right| \leq \gamma_1 |p|^2 \quad \text{when} \ |p| \geq c \ \text{and} \ |Q| \leq \bar{m} \quad (\text{III.2})
\]
and that
\[
\left| \text{Re} \hat{D}_n(p + iQ) - \text{Re} \hat{D}_n(p) \right| \leq \gamma_1 c^2 \quad \text{when} \ |p| \leq c, \ |\varepsilon_n p_0| \geq c \ \text{and} \ |Q| \leq \bar{m} \quad (\text{III.3})
\]
This is a consequence of the following bounds on the derivatives of the real parts of the three terms making up \( \tilde{D}_n(p + iQ) \) in Remark III.1.a. For the first term,

\[
\frac{d}{dt} \varepsilon_n^2(P_0 + i t Q_0)^2 e^{-\hat{h}_0(\varepsilon_n P)} \left[ \sin \frac{1}{2} \varepsilon_n^2(P_0 + i t Q_0) \right]^2 \leq \text{const} \left[ \varepsilon_n^2 |P_0 + i Q_0| |Q_0| + \varepsilon_n^2 |P_0 + i Q_0|^2 \varepsilon_n^2 |Q_0| \right] \\
\leq \text{const} \tilde{m}
\]

\[
\frac{d}{dt} \text{Re} \varepsilon_n^2(P_0 + i t Q_0)^2 \left[ e^{-\hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q)} - e^{-\hat{h}_0(\varepsilon_n P)} \right] \left[ \sin \frac{1}{2} \varepsilon_n^2(P_0 + i t Q_0) \right]^2 \bigg|_{t=0} = 0
\]

\[
\frac{d^2}{dt^2} \varepsilon_n^2(P_0 + i t Q_0)^2 \left[ e^{-\hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q)} - e^{-\hat{h}_0(\varepsilon_n P)} \right] \left[ \sin \frac{1}{2} \varepsilon_n^2(P_0 + i t Q_0) \right]^2 \leq \text{const} \left[ \varepsilon_n^2 |Q_0|^2 + \varepsilon_n^2 |P_0 + i Q_0| |Q_0| (\varepsilon_n |Q| + \varepsilon_n^2 |Q_0|) + \varepsilon_n^2 |P_0 + i Q_0|^2 (\varepsilon_n |Q| + \varepsilon_n^2 |Q_0|)^2 \right] \\
\leq \text{const} \tilde{m}^2
\]

For the second term,

\[
\frac{d}{dt} \text{Re} \frac{1 - e^{-\hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q)}}{\varepsilon_n^2} \bigg|_{t=0} = 0
\]

\[
\frac{d^2}{dt^2} \frac{1 - e^{-\hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q)}}{\varepsilon_n^2} \leq \text{const} \frac{1}{\varepsilon_n^2} \left( \varepsilon_n |Q| \right)^2 \leq \text{const} \tilde{m}^2
\]

For the third term,

\[
\frac{d}{dt} \text{Re} i(P_0 + i t Q_0) e^{-\hat{h}_0(\varepsilon_n P)} \sin \varepsilon_n^2(P_0 + i t Q_0) \bigg|_{t=0} \\
\leq \text{const} |Q_0| + \text{const} |P_0 + i Q_0| \varepsilon_n^2 |Q_0| \\
\leq \text{const} \tilde{m}
\]

\[
\frac{d}{dt} \text{Re} i(P_0 + i t Q_0) \left[ e^{-\hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q)} - e^{-\hat{h}_0(\varepsilon_n P)} \right] \left[ \sin \frac{1}{2} \varepsilon_n^2(P_0 + i t Q_0) \right]^2 \bigg|_{t=0} = \text{Re} i P_0 \left[ \frac{d}{dt} \hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q) \bigg|_{t=0} \right] e^{-\hat{h}_0(\varepsilon_n P)} \sin \varepsilon_n^2(P_0) \bigg| \varepsilon_n^2 P_0 \\
\leq \text{const} \varepsilon_n^2 |P_0| |P| \tilde{m}
\]

\[
\leq \text{const} |P| \tilde{m}
\]

\[
\frac{d^2}{dt^2} i(P_0 + i t Q_0) \left[ e^{-\hat{h}_0(\varepsilon_n P + i t \varepsilon_n Q)} - e^{-\hat{h}_0(\varepsilon_n P)} \right] \left[ \sin \frac{1}{2} \varepsilon_n^2(P_0 + i t Q_0) \right]^2 \leq \text{const} \left[ |Q_0| (\varepsilon_n |Q| + \varepsilon_n^2 |Q_0|) + |P_0 + i Q_0| (\varepsilon_n |Q| + \varepsilon_n^2 |Q_0|)^2 \right] \\
\leq \text{const} \tilde{m}^2
\]

Now choose \( \tilde{m} = \tilde{m}(c) \) small enough that (III.2) and (III.3) are satisfied.
IV. The Covariance

The covariance for the fluctuation integral in \[ \text{parabolic-all.tex} \] is

\[ C^{(n)} = \left( \frac{a}{L^2} Q^* Q + \Delta^{(n)} \right)^{-1} \]

where

\[ \Delta^{(n)} = \begin{cases} (I + \Omega_n Q_n D_n^{-1} Q_n^*)^{-1} \Omega_n & \text{if } n \geq 1 \\ D_0 & \text{if } n = 0 \end{cases} : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)} \]

See \[ \text{parabolic-all.tex}, \text{(I.15) and (I.14)} \]. In Lemma IV.2 we study the properties of \( \Delta^{(n)} \) and in Corollary IV.5 we study the properties of \( C^{(n)} \) and its square root.

Remark IV.1 Let \( n \geq 1 \).

(a) The operator \( \Delta^{(n)} \) is the periodization of a translation invariant operator \( \Delta \), acting on \( L^2(\mathbb{Z} \times \mathbb{Z}^3) \), whose Fourier transform is

\[ \hat{\Delta}^{(n)}(k) = \hat{\Omega}_n(k) \left( 1 + \hat{\Omega}_n(k) \sum_{\ell \in B_n} u_n(k + \ell) 2^q \hat{D}_n^{-1}(k + \ell) \right)^{-1} \]

\[ = \frac{\hat{\Omega}_n(k) \hat{D}_n(k)}{\hat{D}_n(k) + \hat{\Omega}_n(k) \sum_{\ell} u_n(k + \ell) 2^q \hat{D}_n^{-1}(k + \ell) \hat{D}_n(k)} \]

with \( k \in \mathbb{C} \times \mathbb{C}^3 \), where \( u_n(p) \) and \( \hat{B}_n \) were defined in parts (b) and (e) of Remark II.1, respectively.

(b) \( \hat{\Delta}^{(n)}(k) \) is invariant under \( k_\nu \to -k_\nu \) for each \( 1 \leq \nu \leq 3 \).

(c) \( \hat{\Delta}^{(n)}(k) \) has nonnegative real part when \( k \) is real.

Lemma IV.2 There are constants\(^{(1)}\) \( m_1 > 0, \gamma_2, \Gamma_6 \) and \( \Gamma_7 \), such that, for \( L > \Gamma_2 \), the following hold.

(a) \( \hat{\Delta}^{(n)}(k) \) is analytic on \(|\text{Im}k| < 3m_1|\).

(b) For all \( k \in \mathbb{C} \times \mathbb{C}^3 \) with \(|\text{Im}k| \leq 3m_1|\).

\[ \hat{\Delta}^{(n)}(k) = -ik_0 + \left( \frac{1}{a_n} + \frac{e_n^2}{2} \right) k_0^2 + \frac{1}{2} \sum_{\nu, \nu' = 1}^3 H_{\nu, \nu'} k_\nu k_{\nu'} + O(|k|^3) \]

\[ \hat{\Delta}^{(n)}(k) \hat{D}_n^{-1}(k) = 1 + \frac{ik_0}{a_n} + O(|k|^2) \]

The higher order part \( O(\cdot) \) is uniform in \( n \) and \( L \).

\(^{(1)}\) Recall Convention I.2.
(c) \(|\hat{\Delta}^{(n)}(k)| \leq 2a\) and \(|\frac{\partial}{\partial k} \hat{\Delta}^{(n)}(k)|, |\frac{\partial^2}{\partial k \partial k'} \hat{\Delta}^{(n)}(k)| \leq \Gamma_7\) for all 0 \leq \nu, \nu' \leq 3 and \(k \in \mathbb{C} \times \mathbb{C}^3\) with |Im\(k)| < 3m_1.

(d) There is a function \(\rho(c) > 0\), which is defined for all \(c > 0\) and which depends only on \(m_1, q, \hat{h}_0\) and \(a\) and, in particular, is independent of \(n\) and \(L\), such that

\[\text{Re} \hat{\Delta}^{(n)}(k) \geq \rho(c)\]

for all \(k \in \mathbb{C} \times \mathbb{C}^3\) with |\(k| \geq c\) and |Im\(k)| \leq 3m_1.

(e) For all \(k \in \mathbb{C} \times \mathbb{C}^3\) with |Im\(k)| \leq 3m_1 and |Re\(k_\nu)| \leq \pi\) for all \(0 \leq \nu \leq 3\),

\[|\hat{\Delta}^{(n)}(k)| \geq \gamma_2 |D_n(k)|\]

(f) \(\hat{D}_n^{-1}(p) \hat{\Delta}^{(n)}(p)\) is analytic on |Im\(p)| \leq 3m_1. Furthermore, for all \(p \in \mathbb{C} \times \mathbb{C}^3\) with |Im\(p)| \leq 3m_1,

\[|\hat{D}_n^{-1}(p) \hat{\Delta}^{(n)}(p)| \leq \frac{\Gamma_6}{1 + |p_0| + \sum_{\nu=1}^{3} |p_\nu|^2}\]

Here, as usual, |\(p_0| and |p_\nu| refer to the magnitudes of the smallest representatives of \(p_0 \in \mathbb{C}\) and \(p_\nu \in \mathbb{C}\) in \(\mathbb{C}/\frac{2\pi}{\nu} \mathbb{Z}\) and \(\mathbb{C}/\frac{2\pi}{\nu} \mathbb{Z}\) respectively.

(g) \(|\frac{\partial}{\partial k_\nu} \hat{\Delta}^{(n)}(k)| \leq \Gamma_7 |k_\nu|\) for all \(1 \leq \nu \leq 3\) and \(k \in \mathbb{C} \times \mathbb{C}^3\) with |Im\(k)| < 3m_1.

**Proof:** We first prove part (b). Using that

- \(u_n(k)^{2q} = 1 + O(|k|^2)\) by Lemma II.2.b
- \(\hat{\Delta}_n(k) = a_n + O(|k|^2)\) by Proposition II.4.b
- \(|\hat{D}_n(k)| \leq \text{const} \left( |k_0| + |k|^2 \right)\) by Lemma III.2.c

and that, for \(\ell \neq 0,\)

- \(|u_n(k + \ell)|^{2q} \leq \text{const} \left|k_0\right|^2 \prod_{k_\nu = 0}^{3} \frac{1}{(1 + |k_\nu| + \pi)^2q}\) by Lemma II.2.a
- \(|\hat{D}_n^{-1}(k + \ell)| \leq \text{const}\) by Lemma III.2.d

we obtain, by Remark IV.1.a and Lemma III.2.b,

\[
\hat{\Delta}^{(n)}(k) = \frac{\hat{\Delta}_n(k)D_n(k)}{\hat{\Delta}_n(k)u_n(k)^{2q} + \hat{D}_n(k) + O(|k|^3)}
\]

\[= \hat{D}_n(k) \left( a_n + O(|k|^2) \right) \]

\[= \hat{D}_n(k) \left\{ 1 - \frac{1}{a_n} \hat{D}_n(k) + O(|k|^2) \right\} \]

\[= -ik_0 + \frac{1}{2} \varepsilon_n^2 k_0^2 + \frac{1}{2} \sum_{\nu, \nu' = 1}^{3} H_{\nu, \nu'} k_\nu k_{\nu'} - \frac{1}{a_n} \hat{D}_n(k)^2 + O(|k|^3) \]

\[= -ik_0 + \left( \frac{1}{a_n} + \frac{\varepsilon_n^2}{2} \right) k_0^2 + \frac{1}{2} \sum_{\nu, \nu' = 1}^{3} H_{\nu, \nu'} k_\nu k_{\nu'} + O(|k|^3) \]
This also shows that, in a neighbourhood of the origin, $\Delta^{(n)}(k)$ is analytic and bounded in magnitude by $2a$. For the second expansion,

$$\hat{\Delta}^{(n)}(k)\hat{D}_n^{-1}(k) = \frac{a_n + O(|k|^2)}{a_n + \hat{D}_n(k) + O(|k|^2)}$$

$$= 1 - \frac{1}{a_n} \hat{D}_n(k) + O(|k|^2)$$

$$= 1 + \frac{i k}{a_n} + O(|k|^2)$$

(a), (c) We first prove the analyticity and the bound $|\hat{\Delta}^{(n)}(k)| \leq 2a$ of part (c). We have already done so for a neighbourhood of the origin. So it suffices to consider $|k| > c_0$, for some suitably small $c_0$. Since $\hat{\Delta}^{(n)}$ is periodic with respect to $2\pi \mathbb{Z} \times 2\pi \mathbb{Z}^3$, it suffices to consider $k$ in the set

$$M(m_1) = \{ k \in \mathbb{C} \times \mathbb{C}^3 \mid |\Re k| \leq \pi \text{ for all } 0 \leq \nu \leq 3, |\Im k| \leq 3m_1, |k| > c_0 \}$$

Recall, from Remarks II.1.d and III.1.b, that $u_n(p)$ and $\hat{D}_n(p)$ are entire. If $m_1$ is small enough, then, by Lemma III.2.d, the functions $k \mapsto D_n(k + \ell), \ell \in \hat{B}_n, n \geq 1$, have no zeroes in $M(m_1)$. Hence each term in the infinite sum $1 + \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell)$ is analytic and it suffices to prove that, for $k \in M(m_1)$, the sum converges uniformly and

$$|1 + \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell)| \geq \frac{6}{10}.$$ 

By Proposition II.4.a, Remark II.1.d and Lemma III.2.d, there is an $l > 0$ such that

$$|\hat{\Omega}_n(k)| \sum_{\ell \in \hat{B}_n, |\ell| \geq l} |u_n(k + \ell)|^2 \hat{D}_n^{-1}(k + \ell) \leq \frac{6}{50} \sum_{\ell \in \hat{B}_n, |\ell| \geq l} \frac{1}{\gamma_1 \pi} \prod_{\nu=0}^3 \left( \frac{24}{|\nu_i + \pi|} \right)^{2q} \leq \frac{\text{const}}{l} \leq \frac{2}{10} \quad (IV.1)$$

Hence we have uniform convergence and

$$\left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right| \geq \left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n, |\ell| < l} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right| - \frac{2}{10}$$

For real $k$ and every $\ell \in \hat{B}_n$, the real part of $\hat{\Omega}_n(k)u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell)$ is nonnegative. Consequently, if $m_1$ is small enough and $|\Im k| \leq 3m_1$

$$\Re \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n, |\ell| < l} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \geq -\frac{2}{10} \quad (IV.2)$$

since, for all $k \in M(m_1)$ and $\ell \in \hat{B}_n$ with $|\ell| < l$,

$\circ \hat{D}_n(k + \ell)$ is bounded away from zero (by Lemma III.2.d) and has bounded first derivative (by Lemma III.2.c) and
\( u_n(k + \ell) \) and \( \hat{\Omega}_n(k) \) are bounded with bounded first derivatives (by analyticity, Remark II.1.d and Proposition II.4.a).

So, by (IV.2)
\[
\left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in B_n, |\ell| < l} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right| \geq \frac{8}{10}
\]

All of this is uniform in \( n \) and \( L \).

Shrinking \( m_1 \) by a factor of 2, the bounds \( |\partial_{k \nu} \hat{\Delta}^{(n)}(k)|, |\partial_{k \nu} \partial_{k \nu'} \hat{\Delta}^{(n)}(k)| \leq \Gamma_7 \), with \( \Gamma_7 \) being the maximum of \( 4a \) divided by the original \( 3m_1 \) (for first order derivatives) and \( 16a \) divided by the square of the original \( 3m_1 \) (for second order derivatives), follow by the Cauchy integral formula.

(d) Denote the real and imaginary parts of the numerator and denominator by
\[
R_n = \text{Re} \hat{\Omega}_n(k) \quad R_d = \text{Re} \left[ 1 + \hat{\Omega}_n(k) \sum_{\ell \in B_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right]
\]
\[
I_n = \text{Im} \hat{\Omega}_n(k) \quad I_d = \text{Im} \left[ 1 + \hat{\Omega}_n(k) \sum_{\ell \in B_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right]
\]

By Proposition II.4.a and (IV.1), (IV.2),
\[
R_n \geq \frac{a}{2} \quad R_d \geq \frac{6}{10}
\]

By Proposition II.4.a, Remark II.1.d and Lemma III.2.d,
\[
I_d \leq \left| \hat{\Omega}_n(k) \sum_{\ell \in B_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right| \leq a \tilde{I}_d(c)
\]

where \( \tilde{I}_d(c) = \frac{a}{2} s \sum_{\ell \in B_n} \frac{1}{\gamma_{\min} |c|^{2c/4}} \prod_{\nu=0}^{2q} \frac{24}{24 - \nu \pi} \). When \( k \) is real, \( \hat{\Omega}_n(k) \) is real. By analyticity and Proposition II.4.a, the first order derivatives of \( \hat{\Omega}_n(k) \) are bounded. So if \( m_1 \) is small enough, \( |I_n| \leq \frac{2}{10 I_d(c)} \). So
\[
\text{Re} \hat{\Delta}^{(n)}(k) = \frac{R_n R_d + I_n I_d}{R_d^2 + I_d^2} \geq \frac{a (3/10) - a (2/10)}{(1 + a I_d(c))^2} = \frac{a}{10 (1 + a I_d(c))^2}
\]

(e) By Remark II.1.d, Proposition II.4.a and Lemma III.2.d,
\[
\left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in B_n} u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right|
\]
\[
\leq \left| \hat{\Omega}_n(k) u_n(k)^{2q} \hat{D}_n^{-1}(k) \right| + 1 + \sum_{0 \neq \ell \in B_n} \left| \hat{\Omega}_n(k) u_n(k + \ell)^{2q} \hat{D}_n^{-1}(k + \ell) \right|
\]
\[
\leq \frac{6}{3} a \left( \frac{24}{\pi} \right)^{8q} \left| \hat{D}_n^{-1}(k) \right| + 1 + \frac{6}{3} a \sum_{0 \neq \ell \in B_n} \frac{1}{\gamma_{1\pi}} \prod_{\nu=0}^{3} \left( \frac{24}{24 - \nu \pi} \right)^{2q}
\]
and so, by Lemma III.2.c,

\[ |\hat{D}_n(k)| \left| 1 + \hat{\Omega}_n(k) \sum_{\ell \in \hat{B}_n} u_n(k + \ell)^2q \hat{D}_n^{-1}(k + \ell) \right| \leq \frac{6}{5} a \left( \frac{24}{\pi} \right)^{8q} + \frac{\Gamma_5}{6} \left( \pi + 1 + 3(\pi + 1)^2 \right) \left[ 1 + \frac{6}{5} a \sum_{0 \neq \ell \in \hat{B}_n} \frac{1}{\gamma_3} \prod_{\nu = 0}^{3} \left( \frac{24}{|\nu \nu + \pi|} \right)^{2q} \right] \]

(f) By part (c) of this lemma, Remark III.1.b and Lemma III.2.d, it suffices to consider \( p = k \) with \(|\text{Im}\, k| \leq 3m_1 \) and \(|k| < c_0 < \pi\) where \( c_0 \) is sufficiently small. Now

\[ \hat{D}_n^{-1}(k) \hat{\Delta}^{(n)}(k) = \frac{\hat{\Omega}_n(k)}{\hat{\Omega}_n(k) u_n(k)^2q + \hat{D}_n(k) \left[ 1 + \hat{\Omega}_n(k) \sum_{0 \neq \ell \in \hat{B}_n} u_n(k + \ell)^2q \hat{D}_n^{-1}(k + \ell) \right]} \]

By Proposition II.4.a, Remark II.1.d, Lemma II.2.b and parts (c) and (d) of Lemma III.2, we may choose \( c_0 > 0 \) so that

\[ |\hat{\Omega}_n(k)| \left| u_n(k)^2q - 1 \right| < \frac{a}{5} \]

for all \(|k| < c_0\) with \(|\text{Im}\, k| \leq 3m_1\). That does it.

(g) Fix any \( 1 \leq \nu \leq 3 \). Observe that

\[ \frac{\partial \hat{\Delta}^{(n)}(k)}{\partial k_\nu}(k) = k_\nu \int_0^1 \frac{\partial^2 \hat{\Delta}^{(n)}(k(t))}{\partial k_\nu^2} \, dt \quad \text{with} \quad k(t)_{\nu'} = \begin{cases} k_{\nu'} & \text{if } \nu' \neq \nu \\ tk_\nu & \text{if } \nu = \nu' \end{cases} \]

since \( \frac{\partial \hat{\Delta}^{(n)}(k(0))}{\partial k_\nu} = 0 \), by Remark IV.1.b. Now apply the bound on the second derivative from part (c).

\[ \square \]

We next consider the “resolvents”

\[ R^{(n)}_{\zeta} = \left( \zeta \mathbb{I} - \frac{a}{2\pi} Q^*Q - \Delta^{(n)} \right)^{-1} \]

in preparation for studying \( C_n \) and \( \sqrt{C_n} \). We wish to apply Lemma A.13, with \( A \rightarrow D_n = \frac{a}{2\pi} Q^*Q + \Delta^{(n)} \) (scaled). By Lemmas A.9.b and A.5.b, for each \( \ell, \ell' \in \hat{B} = \hat{B}^+ \),

\[ a_{\ell}(\ell, \ell') \rightarrow D_{n,\ell}(\ell, \ell') = \frac{a}{2\pi} u_+(\ell + \ell') u_+(\ell + \ell')^q + \delta_{\ell,\ell'} \hat{\Delta}^{(n)}(\ell + \ell') \quad \text{(IV.3)} \]
Here we are denoting

- momenta dual to the \( L \)-lattice by \( \mathfrak{v} \in (\mathbb{R}/\frac{2\pi}{L} \mathbb{Z}) \times (\mathbb{R}^3/\frac{2\pi}{L} \mathbb{Z}^3) \) and
- momenta dual to the unit lattice \( \mathbb{Z} \times \mathbb{Z}^3 \) by \( k \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}^3/2\pi \mathbb{Z}^3) \) and decompose \( k = \mathfrak{v} + \ell \) or \( k = \mathfrak{v} + \ell' \) with \( \mathfrak{v} \) in a fundamental cell for \( (\mathbb{R}/\frac{2\pi}{L} \mathbb{Z}) \times (\mathbb{R}^3/\frac{2\pi}{L} \mathbb{Z}^3) \) and \( \ell, \ell' \in (\frac{2\pi}{L} \mathbb{Z}/2\pi \mathbb{Z}) \times (\frac{2\pi}{L} \mathbb{Z}^3/2\pi \mathbb{Z}^3) = \mathcal{B}^+ \).

We also use \( d_{n,\mathfrak{v}} \) to denote the \( \mathcal{B}^+ \times \mathcal{B}^+ \) matrix \( [d_{n,\mathfrak{v}}(\ell, \ell')]_{\ell, \ell' \in \mathcal{B}^+} \). Observe, by Remark II.1.d and Lemma IV.2.a, that \( d_{n,\mathfrak{v}}(\ell, \ell') \) is analytic in the strip \( \{ \text{Im} \mathfrak{v} \} < 3m_1 \).

Let \( [v_\ell]_{\ell \in \mathcal{B}^+} \) and \( [w_\ell]_{\ell \in \mathcal{B}^+} \) be any vectors in \( L^2(\mathcal{B}^+) \). Then, if \( \mathfrak{v} = L^{-1}(k) \),

\[
\langle \hat{v}, d_{n,\mathfrak{v}}w \rangle = \frac{\alpha}{L^2} \left[ \sum_{\ell \in \mathcal{B}^+} u_+(L^{-1}(k)+\ell)^q v_\ell \right] \left[ \sum_{\ell \in \mathcal{B}^+} u_+(L^{-1}(k)+\ell)^q w_\ell \right] + \sum_{\ell \in \mathcal{B}^+} \Delta^{(n)}(L^{-1}(k)+\ell) \bar{v}_\ell w_\ell \\
= \sum_{\ell, \ell' \in \mathcal{B}^1} \bar{v}_{\ell}(\ell') d^{(s)}_{n,k}(\ell, \ell') w_{\ell}(\ell')
\]

where the “scaled” matrix

\[
d^{(s)}_{n,k}(\ell, \ell') = \frac{\alpha}{L^2} u_+(L^{-1}(k+\ell))^q u_+(L^{-1}(k+\ell'))^q + \delta_{\ell, \ell'} \Delta^{(n)}(L^{-1}(k+\ell))
\]

**Lemma IV.3** There are constants \( m_2, \lambda_0, \Gamma_8 > 0 \), such that, for all \( L > \Gamma_2 \) and \( k \in \Phi \times \Phi^3 \) with \( \{ \text{Im} k \} < 3m_2 \), the following hold.

(a) Write \( \mathfrak{v} = L^{-1}(k) \). For both the operator and (matrix) \( \ell^1-\ell^\infty \) norms

\[
\|d_{n,\mathfrak{v}}\| \leq \Gamma_8 \quad \|d^{(s)}_{n,k}\| \leq \Gamma_8
\]

(b) Let \( \lambda \in \Phi \) be within a distance \( \frac{\lambda_0}{L^2} \) of the negative real axis. Then the resolvent

\[
\left\| (\lambda \mathbb{I} - d^{(s)}_{n,k})^{-1} \right\| \leq \Gamma_8 L^2
\]

This is true for both the operator and (matrix) \( \ell^1-\ell^\infty \) norms.

**Proof:** Since \( d_{n,\mathfrak{v}+p}(\ell, \ell') = d_{n,\mathfrak{v}}(\ell+p, \ell'+p) \), for all \( p, \ell, \ell' \in \mathcal{B}^+ \), we may always assume that \( \{ \text{Re} \mathfrak{v}_0 \} \leq \frac{\pi}{L^2} \) and \( \{ \text{Re} \mathfrak{v}_\nu \} \leq \frac{\pi}{L} \) for \( 1 \leq \nu \leq 3 \).

(a) By Lemmas II.3.a and IV.2.c, the matrices

\[
\left[ \frac{\alpha}{L^2} u_+(\mathfrak{v}+\ell)^q u_+(\mathfrak{v}+\ell')^q \right]_{\ell, \ell' \in \mathcal{B}^+} \quad \text{and} \quad \left[ \frac{\alpha}{L^2} u_+(L^{-1}(k+\ell))^q u_+(L^{-1}(k+\ell'))^q \right]_{\ell, \ell' \in \mathcal{B}^1}
\]

have finite \( L^1-L^\infty \) norms and the matrix elements of

\[
\left[ \delta_{\ell, \ell'} \Delta^{(n)}(\mathfrak{v}+\ell) \right]_{\ell, \ell' \in \mathcal{B}^+} \quad \text{and} \quad \left[ \delta_{\ell, \ell'} \Delta^{(n)}(L^{-1}(k+\ell)) \right]_{\ell, \ell' \in \mathcal{B}^1}
\]

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are all bounded.

(b) Case $|k| \leq c_0$, with $c_0$ being chosen later in this case: We first consider those diagonal matrix elements of $d_{n,k}(\ell, \ell')$ having $k, \ell$ such that $|L^{-1}(k + \ell)| < c_0$, where $c_0 > 0$ is a small number to be chosen shortly. This, and all other constants chosen in the course of this argument are to be independent of $L$. Then, by Lemma IV.2.b, we have the following.

- If at least one of $\ell_{\nu}$, $1 \leq \nu \leq 3$ is nonzero, then $\Re \hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq \frac{\pi}{2L^2}$, provided $m_2$ and $\tilde{c}_0$ are chosen small enough. Here $c_1$ and the constraints on $m_2$ and $\tilde{c}_0$ depend only on the largest and smallest eigenvalues of $[H_{\nu,\nu'}]$, $a_n$ and the $O(|k|^3)$. To see this, denote by $k$ and $\ell$ the spatial parts of $k$ and $\ell$ and observe that
  $$\begin{align*}
  &\Re L^{-1}(k + \ell) \geq \frac{1}{L} \max \{ \pi, \frac{1}{2}|\ell| \}, \\
  &\Im L^{-1}(k + \ell) = \frac{1}{L} \Im k \leq \frac{3m_2}{L} \text{ and} \\
  &\Im L^{-1}(k + \ell) = \frac{1}{L^2} \Im k \leq \frac{3m_2}{L^2}.
  \end{align*}$$
  In controlling the contribution from $O\left(\left|L^{-1}(k + \ell)\right|^3\right)$ when $\frac{\ell_0}{L^2}$ is larger than $\frac{|\ell|}{L^2}$, we have to use that, in this case, the real part of $\left(\frac{1}{a_n} + \frac{\sigma^2}{2}\right)\left(L^{-1}(k + \ell)\right)^2_0$ is at least a strictly positive constant times $\frac{\ell_0}{L^2}$.

- If $\ell_{\nu} = 0$ for all $1 \leq \nu \leq 3$ but $\ell_0 \neq 0$, then $-\text{sgn} \ell_0 \Im \hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq \frac{\pi}{2L^2}$, provided $m_2$ and $\tilde{c}_0$ is chosen small enough. To see this observe that
  $$\begin{align*}
  &\Re L^{-1}(k + \ell) = \frac{1}{L} \Re k \leq \sqrt{\frac{3\pi}{L}}, \\
  &\Im L^{-1}(k + \ell) = \frac{1}{L} \Im k \leq \frac{3m_2}{L}, \\
  &\text{sgn} \ell_0 \Re L^{-1}(k + \ell)_0 \geq \frac{1}{L^2} \max \{ \pi, \frac{1}{2}|\ell_0| \}, \\
  &\Re L^{-1}(k + \ell)_0 \leq \frac{3}{2L^2} |\ell_0| \text{ and} \\
  &\Im L^{-1}(k + \ell)_0 \leq \frac{3m_2}{L^2}.
  \end{align*}$$

- If $\ell \neq 0$, then, by parts (a) and (b) of Lemma II.3, the spatial parts of $k$ and $\ell$ and observe that
  $$\begin{align*}
  &\frac{\alpha}{L^2} |u_+(L^{-1}(k + \ell))|^{2q} \leq \frac{\alpha}{L^2} |k| \left( \prod_{\nu=0}^{n} \frac{24}{\ell_\nu |k| + \pi} \right)^{2q}
  \end{align*}$$

- If $\ell = 0$ then $\Re \left\{ \frac{\alpha}{L^2} u_+(L^{-1}(k))^{2q} + \hat{\Delta}^{(n)}(L^{-1}(k)) \right\} \geq \frac{\alpha}{2L^2}$, provided $m_2$ and $\tilde{c}_0$ are chosen small enough. To see this observe that
  $$\begin{align*}
  &\left| u_+(L^{-1}(k)) - 1 \right| \leq 3|k|^2, \text{ by Lemma II.3.c and} \\
  &\left| u_+(L^{-1}(k)) \right| \leq \left( \frac{24}{\pi} \right)^3 \text{ by Lemma II.3.a.}
  \end{align*}$$

  $$\begin{align*}
  &\Re \Delta^{(n)}(L^{-1}(k)) \geq -c_2 \left( \frac{m_2}{L^2} + \tilde{c}_0 \left( \frac{|k|}{L} \right)^3 \right) \text{ where } c_2 \text{ depends only on } a_n, \text{ the largest eigenvalue of } [H_{\nu,\nu'}], \text{ and the } O(|k|^3).
  \end{align*}$$
  Note that we have now fixed $\tilde{c}_0$. Now we consider the remaining matrix elements.

- For the remaining diagonal matrix elements we have $|L^{-1}(k + \ell)| \geq \tilde{c}_0$ and then, by Lemma IV.2.d, $\Re \hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq \rho(\tilde{c}_0)$
Finally, the off–diagonal matrix elements of $d_{n,k}^{(s)}$ obey, by parts (a) and (b) of Lemma II.3,

$$\frac{a}{L^2} |u_+((L^{-1}(k + \ell)) \delta | \leq \frac{a}{L^2} |k| \left[ \prod_{\nu=0}^{3} \left( \frac{24}{|c_{\nu} + \pi|} \right)^{q} \right]$$

Hence the off–diagonal part of $d_{n,k}^{(s)}$ has Hilbert–Schmidt, matrix and, as $q > 1$, $L^1$–$L^\infty$ norms all bounded by a universal constant times $\frac{a}{L^2} |k|$. Thus $\lambda \mathbb{I} - d_{n,k}^{(s)}$ has diagonal matrix elements of magnitude at least

$$\min \left\{ \frac{c_1}{L^2}, \frac{\pi}{2L^2}, \frac{a}{2L^2}, \rho(\bar{c}_0) \right\} - \frac{a}{L^2} |k| \left[ \prod_{\nu=0}^{3} \left( \frac{24}{|c_{\nu} + \pi|} \right)^{2q} \right] - \lambda_0 \frac{1}{L^2}$$

and off diagonal part with $L^1$–$L^\infty$ norm bounded by a universal constant times $\frac{a}{L^2} |k|$. It now suffices to choose $c_0$ and $\lambda_0$ small enough that every diagonal matrix element has magnitude at least $\frac{1}{4L^2} \min \{ c_1, \frac{\pi}{2}, \frac{a}{2}, \rho(\bar{c}_0) \}$ and the off diagonal part has $L^1$–$L^\infty$ norm bounded by $\frac{1}{4L^2} \min \{ c_1, \frac{\pi}{2}, \frac{a}{2}, \rho(\bar{c}_0) \}$ and then do a Neumann expansion.

(b) Case $|k| \geq c_0$, with the $c_0$ just chosen: We may assume that $|\Re k_\nu| \leq \pi$ for each $0 \leq \nu \leq 3$.

- If $|L^{-1}(k + \ell)| < \bar{c}_0$ and $\ell_\nu \neq 0$ for at least one $1 \leq \nu \leq 3$ or if $|L^{-1}(k + \ell)| \geq \bar{c}_0$, then
  $$\Re \frac{\hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq \min \{ \rho(\bar{c}_0), \frac{\pi}{2L^2} \}}.$$
  The proof of this given in the case $|k| \leq c_0$ applies now too.

- If $|L^{-1}(k + \ell)| < \bar{c}_0$, $\ell_\nu = 0$ for all $\nu \geq 1$ then
  $$\Re \frac{\hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq 0}{\hat{\Delta}^{(n)}(L^{-1}(k + \ell)) - \hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \leq 4\pi \Gamma_7 \frac{1}{L^2} |\Im k|}$$

The first bound follows immediately from Remark IV.1.c. The second bound follows from Lemma IV.2.c (for the $\frac{1}{L^2} \Im k_0 = \Im (L^{-1}(k + \ell)_0$ contribution) and Lemma IV.2.g (for the $\frac{1}{L^2} \Im k_\nu$ contribution, with $1 \leq \nu \leq 3$, — note that on the line segment from $L^{-1}(\Re k + \ell)$ to $L^{-1}(k + \ell)$, $|\frac{\partial \hat{\Delta}^{(n)}}{\partial k_\nu}|$ is bounded by $\Gamma_7 |L^{-1}(k + \ell)_\nu| = \frac{\Gamma_7}{L^2} |k_\nu| \leq \Gamma_7 \frac{\pi + 3m_2}{L}$. Furthermore

$$\frac{-\text{sgn} \ell_0 \Im \hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq \frac{\pi}{L^2}}{\hat{\Delta}^{(n)}(L^{-1}(k + \ell)) \geq \gamma_1 \gamma_2 \min \left\{ \frac{c_0}{\sqrt{2}}, \frac{c_0}{2} \right\} \frac{1}{L^2}}$$

In the case $\ell_0 \neq 0$, the proof of the bound given in the case $|k| \leq c_0$ applies now too. (Just apply it to $\Re k$.) The bound for the case $\ell_0 = 0$ follows from Lemma IV.2.e and Lemma III.2.a.
We have shown above that
\[
\lambda \text{ and the constant } \Gamma
\]
where
\[
\lambda \text{ with } \lambda
\]
provided we choose \(0 < \lambda_1 \leq \min \{c_1, \rho(\tilde{c}_0)\} \frac{\pi}{2}, \gamma_1 \gamma_2 \frac{\alpha_0}{\sqrt{2}}, \gamma_1 \gamma_2 \frac{\alpha_1}{2}\).

Since \(\Re z \leq 0 \Rightarrow \Re \frac{1}{z} \leq 0\), and \(v(\ell) \in \mathbb{R}\) for all \(\ell\), we have that \(\Re \kappa \leq 0\) so that
\[
\frac{1}{1 - \kappa} - 1
\]
with the constant \(\lambda_2 > 0\) depending only on \(a\) and \(\lambda_1\) and the constant \(\Gamma'_3\) depending only on \(\Gamma_7\), \(a\) and \(q\). It now suffices to choose \(\lambda_0\) and \(m_2\) smaller than \(\frac{\lambda_2}{12 \Gamma'_3}\) and use a Neumann expansion to give
\[
\| (\lambda \mathbb{1} - d_{n,k})^{-1} \|_{\ell^1 - \ell^\infty} \leq \frac{2 L^2}{\lambda_2}
\]
Proposition IV.4 Let $m_2, \lambda_0, \Gamma_8$ be as in Lemma IV.3 and use $\mathbb{R}_-$ to denote the negative real axis in $\mathbb{C}$. Set

$$\mathcal{O}_C = \{ z \in \mathbb{C} \mid \text{dist}(z, \mathbb{R}_-) > \frac{\lambda_0}{2L^2}, \ |z| < \Gamma_8 + 1 \}$$

$$\mathcal{O} = \{ z \in \mathbb{C} \mid \text{dist}(z, \mathbb{R}_-) > \frac{\lambda_0}{3L^2}, \ |z| < \Gamma_8 + 2 \}$$

and let

- $C = \partial \mathcal{O}_C$, oriented counterclockwise
- $f: \mathcal{O} \to \mathbb{C}$ be analytic.

Then $f\left((\frac{\alpha}{L^2}Q^*Q + \Delta^{(n)}(s))\right)$, defined by (A.11) and Lemma A.14.a, exists and there is a constant\(^{(2)}\) $\Gamma_9$ such that

$$\|f\left((\frac{\alpha}{L^2}Q^*Q + \Delta^{(n)}(s))\right)\|_{m_2} \leq \Gamma_9 L^7 \sup_{\zeta \in C} |f(\zeta)|$$

Proof: Apply Lemma A.13 with $\hat{a}_k(\ell, \ell') = d^{(s)}_{n,k}(\ell, \ell')$ and

$$\mathcal{X}_{\text{fin}} = \mathcal{X}_{\ell}^{(n)} \quad \mathcal{Z}_{\text{fin}} = \varepsilon_1^2 \mathbb{Z} \times \varepsilon_1 \mathbb{Z}^3 \quad \mathcal{X}_{\text{crs}} = \mathcal{X}_{0}^{(n+1)} \quad \mathcal{Z}_{\text{crs}} = \mathbb{Z} \times \mathbb{Z}^3 \quad \mathcal{B} = \mathcal{B}_1$$

and $m = 3m_2, \ m' = 2m_2$ and $m'' = m_2$. Then $\text{vol}_c = 1, \ \hat{\mathcal{B}}_1 = L^5$ and the lemma gives

$$\|f\left((\frac{\alpha}{L^2}Q^*Q + \Delta^{(n)}(s))\right)\|_{m''} \leq \frac{C_{m''}}{2\pi \text{vol}_c} |C| \sup_{\zeta \in C} |f(\zeta)| \sup_{|\text{Im}|=m'} \sum_{\ell, \ell' \in \hat{\mathcal{B}}_1} |(\zeta - \hat{d}^{(s)}_{n,k})^{-1}(\ell, \ell')|$$

$$\leq \frac{C_{m''}}{2\pi}(\Gamma_8 + 1)(3\pi + 2) \sup_{\zeta \in C} |f(\zeta)| L^5 \sup_{|\text{Im}|=m'} \sum_{\ell \in \hat{\mathcal{B}}_1} \sum_{\ell' \in \hat{\mathcal{B}}_1} |(\zeta - \hat{d}^{(s)}_{n,k})^{-1}(\ell, \ell')|$$

$$\leq \frac{C_{m''}}{2\pi}(\Gamma_8 + 1)(3\pi + 2) \sup_{\zeta \in C} |f(\zeta)| L^5 \Gamma_8 L^2$$

by Lemma IV.3.b.

Applying Proposition IV.4 with $f(z) = \frac{1}{z}, f(z) = \frac{1}{\sqrt{z}}$ and $f(z) = \sqrt{z}$, where $\sqrt{z}$ is the principal value of the square root gives

Corollary IV.5 The operators $C^{(n)}$, $\sqrt{C^{(n)}}$ and $\left(\sqrt{C^{(n)}}\right)^{-1}$ all exist. There is a constant $\Gamma_{10}$ such that

$$\|L_2^{-1}C^{(n)}L_2\|_{m_2}, \ \|\sqrt{L_2^{-1}C^{(n)}L_2}\|_{m_2}, \ \|\left(\sqrt{L_2^{-1}C^{(n)}L_2}\right)^{-1}\|_{m_2} \leq \Gamma_{10} L^9$$

\(^{(2)}\) Recall Convention I.2.
V. The Green’s Functions

In this chapter, we discuss the inverses of the operators

\[ D_n + Q_n^* \Omega_n Q_n \]

These inverses, and variations thereof, are constituents of the leading part of the power series expansion of the background fields of [parabolic-all.tex], see [parabolic-all.tex, Proposition I.16 and Proposition G.3]. In Proposition V.1, below, we show that for sufficiently small \( \mu \), the operators \( D_n + Q_n^* \Omega_n Q_n - \mu \) are invertible, and we estimate the decay of the kernels \( S_n(\mu)(x, y) \) of their inverses

\[ S_n(\mu) = [D_n + Q_n^* \Omega_n Q_n - \mu]^{-1} \]

By Remark II.5

\[ \partial_\nu (S_n(\mu)^*)^{-1} = (S_{n,\nu}(\mu))^{-1} \partial_\nu \quad \partial_\nu S_n(\mu)^{-1} = (S_{n,\nu}(\mu))^{-1} \partial_\nu \quad \text{(V.1)} \]

where

\[ S_{n,\nu}^{(\pm)}(\mu) = [D_n^* + Q_{n,\nu}^{(\pm)} \Omega_n Q_{n,\nu}^{(\pm)} - \mu]^{-1} \quad S_{n,\nu}(\mu) = [D_n + Q_{n,\nu}^{(\pm)} \Omega_n Q_{n,\nu}^{(\pm)} - \mu]^{-1} \quad \text{(V.2)} \]

and \( Q_{n,\nu}^{(\pm)} \) were defined in (II.10). We shall write

\[ S_n = S_n(0) \quad S_{n,\nu}^{(\pm)} = S_{n,\nu}^{(\pm)}(0) \quad \text{(V.3)} \]

The main result extends the statement of [parabolic-all.tex, Theorem I.15]. It is

**Proposition V.1** There are constants \( \mu_{up}, m_3 > 0 \) and \( \Gamma_{11} \), depending only on \( q, h_0 \) and \( a \), and in particular independent of \( n \) and \( L > \Gamma_2 \), such that, for \( |\mu| \leq \mu_{up} \),

the operators \( D_n + Q_n^* \Omega_n Q_n - \mu \) and \( D_n^* + Q_{n,\nu}^{(\pm)} \Omega_n Q_{n,\nu}^{(\pm)} - \mu \), \( D_n + Q_{n,\nu}^{(\pm)} \Omega_n Q_{n,\nu}^{(\pm)} - \mu \)

are invertible, and their inverses \( S_n(\mu) \) and \( S_{n,\nu}^{(\pm)}(\mu) \), \( S_{n,\nu}(\mu) \), respectively, fulfill

\[ \| S_n(\mu) \|_{m_3}, \| S_{n,\nu}^{(\pm)}(\mu) \|_{m_3} \leq \Gamma_{11} \]
\[ \| S_n(\mu) - S_n \|_{m_3}, \| S_{n,\nu}^{(\pm)}(\mu) - S_{n,\nu}^{(\pm)} \|_{m_3} \leq |\mu| \Gamma_{11} \]

This Proposition is proven following the proof of Lemma V.5.
Example V.2 As a model computation, we evaluate the inverse transform

\[ s(x) = \int_{\mathbb{R} \times \mathbb{R}^3} \frac{e^{ip \cdot x}}{-ip_0 + m^2 + p_1^2 + p_2^2 + p_3^2} \frac{dp_0}{(2\pi)^4} \]

of \( \hat{s}(p) = \frac{1}{-ip_0 + m^2 + p^2} \). It is designed to mimic the behaviour of \( S_n \) in the limit \( n \to \infty \). Write \( x = (t, x) \in \mathbb{R} \times \mathbb{R}^3 \). We first compute the \( p_0 \) integral. Observe that the integrand has exactly one pole, which is at \( p_0 = -i(m^2 + \textbf{p}^2) \), and that the \( e^{ip_0 t} \) in the integrand forces us to close the contour in the upper half plane when \( t > 0 \) and in the lower half plane when \( t < 0 \). Thus

\[ \int_{-\infty}^{\infty} \frac{e^{ip_0 t + ip \cdot x}}{-ip_0 + m^2 + p_1^2 + p_2^2 + p_3^2} \frac{dp_0}{(2\pi)^4} = \begin{cases} 0 & \text{if } t > 0 \\ e^{(m + \textbf{p}^2)t + i \textbf{p} \cdot \textbf{x}} \frac{1}{(2\pi)^3} & \text{if } t < 0 \end{cases} \]

Hence \( s(x) = 0 \) for \( t > 0 \) and, for \( t < 0 \),

\[ s(x) = \int_{\mathbb{R}^3} e^{-(m^2 + \textbf{p}^2)|t|} e^{i \textbf{p} \cdot \textbf{x}} \frac{d^3p}{(2\pi)^3} = e^{-m^2|t|} \prod_{j=1}^{3} \int_{-\infty}^{\infty} e^{-|t|p_j^2} e^{i p_j x_j} \frac{dp_j}{2\pi} \]

\[ = \frac{1}{2^{3/2} \pi |t|^{3/2}} e^{-m^2|t|} e^{-\frac{x^2}{4|t|}} \]

Here are some observations about \( s(x) \).

- Since \( m^2|t| + \frac{x^2}{4|t|} \geq m|x| \) (the minimum is at \( |t| = \frac{|x|}{2m} \)), \( s(x) \) decays exponentially for large \( |x| \) in all directions.
- For \( x \neq 0 \), \( \lim_{t \to 0} e^{-\frac{x^2}{4|t|}} = 0 \), so \( s(x) \) is continuous everywhere except at \( x = 0 \).
- \( s(x) \) has an integrable singularity at \( x = 0 \). There are a number of ways to see this. For example, the inequality \( e^{-\frac{x^2}{4|t|}} \leq \text{const} \left( \frac{|t|}{|x|^2} \right)^{\kappa/2} \) implies that \( |s(x)| \) is bounded near \( x = 0 \) by a constant times \( \frac{1}{|t|^{(3-\kappa)/2}} \frac{1}{|x|^{\kappa}} \). This is integrable if \( 1 < \kappa < 3 \).
- If we send \( x \to 0 \) along a curve with \( x^2 = -4\gamma |t| \ln |t| \), \( s(x) \approx \text{const} \frac{1}{|t|^{3/2-\gamma}} \).

- We see, using \( x_j^4 e^{-\frac{x_j^2}{4|t|}} \leq \text{const} \ |t|^2 \), that \( (t^2 + \sum_{j=1}^{3} x_j^4)|s(x)| \) is bounded and exponentially decaying. Note that \( \frac{1}{t^2 + \sum_{j} x_j^4} \) has an integrable singularity at the origin, since

\[ \int_{-\infty}^{\infty} \frac{1}{t^2 + \sum_{j} x_j^4} dt = \frac{\text{const}}{\sqrt{\sum_{j} x_j^4}}. \]

As preparation for and in addition to the position space estimates of Proposition V.1, we also derive bounds on the Fourier transforms of these and related operators. To convert bounds in momentum space into bounds in position space, we shall use Lemma A.11, with

\[ \mathcal{X}_{\text{fin}} = \mathcal{X}_n \quad \mathcal{Z}_{\text{fin}} = \varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3 \quad \mathcal{X}_{\text{crs}} = \mathcal{X}_0^{(n)} \quad \mathcal{Z}_{\text{crs}} = \mathbb{Z} \times \mathbb{Z}^3 \quad \mathcal{B} = \mathcal{B}_n \quad \text{(V.4)} \]
We shall routinely use \(|p_0|, |p_\nu|\) and \(|p|\) to refer to the magnitudes of the smallest representatives of \(p_0 \in \mathbb{C}, p_\nu \in \mathbb{C}\) and \(p \in \mathbb{C}^3\) in \(\mathbb{C}/2\pi\mathbb{Z}, \mathbb{C}/2\pi\mathbb{Z}\) and \(\mathbb{C}^3/2\pi\mathbb{Z}^3\), respectively.

The operators \(S_n(\mu)\) act on functions on the lattice \(\mathcal{X}_n\), but they are only translation invariant with respect to the sublattice \(\mathcal{X}_n^{(n)}\). An exponentially decaying operator which is fully translation invariant, and has the same local singularity as \(S_n\), is the operator

\[
S'_n = [D_n + a_n \exp\{-\Delta_n\}]^{-1}
\]

where

\[
\Delta_n = \partial_n^* \partial_0 + (\partial_1^* \partial_1 + \partial_2^* \partial_2 + \partial_3^* \partial_3)
\]

and \(a_n = a \frac{1-L^2}{1-L^{-2n}}\) as in Proposition II.4.b, and the forward derivatives \(\partial_n\) are defined in (II.8). Obviously \(S'_n\) has Fourier transform

\[
\hat{S}'_n(p) = [\hat{D}_n(p) + a_n \exp\{-\Delta_n(p)\}]^{-1}
\]

where \(\Delta_n(p) = \left[\frac{\sin^2 \varepsilon_n p_0}{2\varepsilon_n}\right]^2 + \sum_{\nu=1}^{3} \left[\frac{\sin^2 \varepsilon_n p_\nu}{2\varepsilon_n}\right]^2\)

Before we discuss the properties of \(S'_n\) and of the difference \(\delta S = S_n - S'_n\) we note

**Remark V.3** The Fourier transform \(\Delta_n(p)\) of the four dimensional Laplacian \(\Delta_n\) is entire. For \(p \in \mathbb{R} \times \mathbb{R}^3\),

\[
\Delta_n(p) \geq \frac{2}{\pi^2} \left[|p_0|^2 + |p|^2\right]
\]

For \(p \in \mathbb{C} \times \mathbb{C}^3\) with \(\varepsilon_n^2 |\text{Im} p_0| \leq 1\) and \(\varepsilon_n |\text{Im} p| \leq 1\)

\[
|\Delta_n(p)| \leq 4 |p_0|^2 + |p|^2
\]

\[
|\frac{\partial}{\partial p_\nu} \Delta_n(p)| \leq 4 |p_\nu|
\]

\[
|\frac{\partial^\ell}{\partial p_\nu^\ell} \Delta_n(p)| \leq 4 \varepsilon^\ell - 2\varepsilon_n,\nu\quad \text{if } \ell \geq 2
\]

with the \(\varepsilon_n,\nu\) of (II.6). For \(p \in \mathbb{C} \times \mathbb{C}^3\) with \(|\text{Im} p| \leq 1\),

\[
\text{Re} \Delta_n(p) \geq -5 \pi^2 + \frac{1}{\pi^2} \left[|p_0|^2 + |p|^2\right]
\]

**Proof:** For the first two claims, just apply parts (a) and (b) of Lemma B.1. For the derivatives, use

\[
\frac{d}{d\theta} \left[\frac{\sin(\eta \theta)}{\eta}\right]^2 = \frac{\sin(2\eta \theta)}{\eta} \quad \Rightarrow \quad \frac{d^\ell}{d\theta^\ell} \left[\frac{\sin(\eta \theta)}{\eta}\right]^2 = \pm (2\eta)^{\ell-1} \frac{1}{\eta} \left\{ \begin{array}{ll}
\sin(2\eta \theta) & \text{for } \ell \text{ odd} \\
\cos(2\eta \theta) & \text{for } \ell \text{ even}
\end{array} \right.
\]

For the final claim, write \(p = P + iQ\) with \(P, Q \in \mathbb{R} \times \mathbb{R}^3\). Then

\[
\text{Re} \Delta_n(p + iQ) \geq \text{Re} \Delta_n(p) - \left|\text{Re} \Delta_n(p + iQ) - \Delta_n(p)\right|
\]

\[
\geq \frac{2}{\pi^2} \left[|p_0|^2 + |p|^2\right] - 4|Q||P + iQ|
\]

\[
\geq \frac{2}{\pi^2} |P + iQ|^2 - 2(2 + \frac{2}{\pi^2}) |Q||P + iQ| - \frac{2}{\pi^2} |Q|^2
\]

\[
\geq \frac{1}{\pi^2} |P + iQ|^2 - \left\{ \pi^2 (2 + \frac{2}{\pi^2})^2 + \frac{2}{\pi^2} \right\} |Q|^2
\]
By the resolvent identity

$$\delta S = S_n - S'_n = -S'_n \left[ Q_n^* \Omega_n Q_n - a_n \exp \{-\Delta_n\} \right] S_n$$

$S_n$ and $\delta S$ are translation invariant with respect to the sublattice $\Lambda_0^{(n)}$ of $\mathcal{X}_n$. By “Floquet theory” (see Lemma A.1), their Fourier transforms $\hat{S}_n(p, p')$, $\delta \hat{S}(p, p')$, $p, p' \in \hat{\mathcal{X}}_n$ vanish unless $\pi_n^{(n,0)}(p) = \pi_n^{(n,0)}(p')$, i.e. unless there are $k \in \hat{\mathcal{X}}_0^{(n)}$ and $\ell, \ell' \in \hat{\mathcal{B}}_n$ such that $p = k + \ell$, $p' = k + \ell'$. The blocks $\hat{S}^{-1}_{n,k}(\ell, \ell') = \hat{S}^{-1}_{n}(k + \ell, k + \ell')$ and $\hat{\delta}S_{k}(\ell, \ell') = \hat{\delta}S(k + \ell, k + \ell')$ are given by

$$\hat{S}^{-1}_{n,k}(\ell, \ell') = \hat{D}_n(k + \ell) \delta_{\ell, \ell'} + u_n(k + \ell) \hat{\Omega}_n(k) u_n(k + \ell')$$

$$\hat{\delta}S_{k}(\ell, \ell') = -\sum_{\ell'' \in \hat{\mathcal{B}}_n} \hat{S}^\prime_{n}(k + \ell) \left[ u_n(k + \ell) \hat{\Omega}_n(k) u_n(k + \ell'') - a_n e^{-\Delta_n(k + \ell)} \delta_{\ell, \ell''} \right] \hat{S}_{n,k}(\ell'', \ell')$$

where $u_n$ and $\hat{\Omega}_n$ are given in parts (b) and (e) of Remark II.1.

**Lemma V.4** There are constants$^{(1)}$ $m_4 > 0$ and $\Gamma_{12}$, such that the following hold for all $L > \Gamma_2$.

(a) $\hat{S}^\prime_n(p)$ is analytic in $|\text{Im} p| < 3m_4$ and obeys

$$|\hat{S}^\prime_n(p)| \leq \frac{\Gamma_{12}}{1 + |p_0| + |p|^2}$$

and

$$\left| \frac{\partial_{\ell'}}{\partial_{p'}} \hat{S}^\prime_n(p) \right|, \left| \frac{\partial_{\ell}}{\partial_{p}} \hat{S}^\prime_n(p) \right| \leq \frac{\Gamma_{12}}{(1 + |p_0| + |p|^2)^3}$$

for $1 \leq \nu \leq 3$ there.

(b) For all $\ell, \ell' \in \hat{\mathcal{B}}_n$, $\hat{S}_{n,k}(\ell, \ell')$ is analytic in $|\text{Im} k| < 3m_4$ and obeys

$$|\hat{S}_{n,k}(\ell, \ell')| \leq \frac{\Gamma_{12}}{1 + |\ell_0| + \sum_{\nu=1}^3 |\ell_{\nu}|^2} \left\{ \delta_{\ell, \ell'} + \frac{1}{1 + |\ell_0| + \sum_{\nu=1}^3 |\ell_{\nu}|^2} \prod_{\nu=0}^3 \frac{1}{(|\ell_0'| + 1)^3} \prod_{\nu=0}^3 \frac{1}{(|\ell_{\nu}'| + 1)^3} \right\} \delta_{\ell, \ell'}$$

there.

(c) For all $\ell, \ell' \in \hat{\mathcal{B}}_n$, $\hat{\delta}S_{k}(\ell, \ell')$ is analytic in $|\text{Im} k| < 3m_4$ and obeys

$$|\hat{\delta}S_{k}(\ell, \ell')| \leq \Gamma_{12} \exp \left\{ -\frac{1}{40} \sum_{\nu=0}^3 |\ell_{\nu}|^2 \right\} \delta_{\ell, \ell'}$$

$$+ \frac{\Gamma_{12}}{1 + |\ell_0| + \sum_{\nu=1}^3 |\ell_{\nu}|^2} \left\{ \prod_{\nu=0}^3 \frac{1}{(|\ell_0'| + 1)^3} \prod_{\nu=0}^3 \frac{1}{(|\ell_{\nu}'| + 1)^3} \right\} \frac{1}{1 + |\ell_0'| + \sum_{\nu=1}^3 |\ell_{\nu}'|^2}$$

there.

(d) For all $u, u' \in \mathcal{X}_n$.

$$|S_n(u, u') - S'_n(u, u')| \leq \Gamma_{12} e^{-2m_4|u - u'|}$$

$$|S'_n(u, u')| \leq \Gamma_{12} \min \left\{ \frac{e^{-2m_4|u - u'|}}{|u_0 - u_0'|^2 + |u - u'|^2}, L^{5n} \right\}$$

$^{(1)}$ Recall Convention I.2.
Proof: (a) Obviously $\hat{S}'_n(p)^{-1}$ is entire. For real $p$

$$|\hat{S}'_n(p)^{-1}| \geq a_n \exp\{-\Delta_n(p)\} + \text{const} \{ |p_0| + |p|^2 \} \geq \text{const} \{ 1 + |p_0| + |p|^2 \} \quad (V.7)$$

by Remark V.3, Lemma III.2.a, and the fact that $\Delta_n(p), \text{Re} \hat{D}_n(p) \geq 0$ for real $p$. The bound on $|\frac{\partial}{\partial p_0} \hat{D}_n(p)|$ of Lemma III.2.c and Remark V.3 shows that (V.7) is valid for all $|\text{Im} p| < 3m_4$, if $m_4$ is chosen sufficiently small.

We now bound the derivatives. For any $0 \leq \nu \leq 3$ and $\ell \in \mathbb{N}$, $\frac{\partial^\ell}{\partial p_\nu^\ell} \hat{S}'_n(p)$ is a finite linear combination of terms of the form

$$\hat{S}'_n(p)^{1+j} \prod_{i=1}^j \frac{\partial^\ell_i}{\partial p_\nu^\ell_i} [\hat{D}_n(p) + a_n \exp\{-\Delta_n(p)\}]$$

with each $\ell_i \geq 1$ and $\sum_{i=1}^j \ell_i = \ell$. By Remark V.3, all derivatives of $\exp\{-\Delta_n(p)\}$ of order $\ell_i \in \mathbb{N}$ are bounded by $\frac{\text{const}}{1+|p_0|+|p|^2}^{\ell_i}$. Hence, by Lemma III.2.c,

$$|\frac{\partial^\ell_i}{\partial p_\nu^\ell_i} [\hat{D}_n(p) + a_n \exp\{-\Delta_n(p)\}]| \leq \text{const} \begin{cases} \frac{1}{[1+|p_0|+|p|^2]^{\ell_i-i}} & \text{if } \nu = 0, \ell_i = 1, 2 \\ \frac{1}{[1+|p_0|+|p|^2]^{\ell_i/2-i}} & \text{if } \nu \geq 1, 1 \leq \ell_i \leq 4 \end{cases}$$

As $|\hat{S}'_n(p)| \leq \frac{\text{const}}{1+|p_0|+|p|^2}$,

$$\left|\hat{S}'_n(p)^{1+j} \prod_{i=1}^j \frac{\partial^\ell_i}{\partial p_\nu^\ell_i} [\hat{D}_n(p) + a_n \exp\{-\Delta_n(p)\}] \right| \leq \frac{\text{const}}{1+|p_0|+|p|^2} \prod_{i=1}^j \begin{cases} \frac{1}{[1+|p_0|+|p|^2]^{\ell_i-i}} & \text{if } \nu = 0, \ell_i = 1, 2 \\ \frac{1}{[1+|p_0|+|p|^2]^{\ell_i/2-i}} & \text{if } \nu \geq 1, 1 \leq \ell_i \leq 4 \end{cases}$$

and the claim follows.

(b) For any $c'_0 > 0$ we have, for $|k| \geq c'_0$ and $|\text{Im} k_0| < 3m_4$, analyticity and the bound

$$|\hat{S}_{n,k}(\ell, \ell')| \leq \frac{\Gamma_{10}}{1+|k_0|+\sum_{\nu=1}^3 |k_\nu|^2} \delta_{\ell, \ell'} + \frac{\Gamma_1}{1+|k_0|+\sum_{\nu=1}^3 |k_\nu|^2} \prod_{\nu=0}^3 \frac{1}{(|k_\nu|+1)^q} \prod_{\nu=0}^3 \frac{1}{(|k_\nu|+1)^q} \frac{1}{1+|k_0|+\sum_{\nu=1}^3 |k_\nu|^2}$$

(with $\Gamma_{10}'$ depending on $c'_0$) which follows from the representation

$$S_n = D_n^{-1} - Q_n \Delta^{(n)} Q_n D_n^{-1}$$

(see [parabolic-all.tex, Remark B.9.b]) and Lemmas III.2.d, II.2.a and IV.2.c.

For $|k| < c'_0$, with $c'_0$ to be shortly chosen sufficiently small, and $|\text{Im} k| < 3m_4$ we use the representation

$$\hat{S}^{-1}_{n,k}(\ell, \ell') = \hat{D}_n(k+\ell) \delta_{\ell, \ell'} + u_n(k+\ell)^q \hat{D}_n(k) u_n(k+\ell)^q = D_{\ell, \ell'} + B_{\ell, \ell'} \quad (V.8)$$
with
\[ D_{\ell,\ell'} = \mathbf{D}_n(k + \ell) \delta_{\ell,\ell'} + \begin{cases} a_n & \text{if } \ell, \ell' = 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ B_{\ell,\ell'} = \begin{cases} \hat{Q}_n(k)u_n(k)^2q - a_n & \text{if } \ell = \ell' = 0 \\ u_n(k + \ell)q \hat{Q}_n(k)u_n(k + \ell')q & \text{otherwise} \end{cases} \]

By parts (c) and (d) of Lemma III.2, assuming that \(|k| < c_0'\) with \(c_0'\) small enough, \(D\) is invertible and the inverse is a diagonal matrix with every diagonal matrix element obeying
\[ |D_{\ell,\ell}^{-1}| \leq \frac{\Gamma''_{10}}{1 + |\ell_0 + \Sigma_{\nu=1}^r|\ell_\nu|^2} \]
for some \(\Gamma''_{10}\) which is independent of \(c_0'\). By parts (b) and (c) of Lemma II.2 and parts (a) and (b) of Proposition II.4,
\[ |B_{\ell,\ell'}| \leq \text{const}_{a, q} |k| \frac{3}{\prod_{\nu=0}^{24} (|\ell_\nu| + \pi)^q} \frac{3}{\prod_{\nu=0}^{24} (|\ell_\nu| + \pi)^q} \]
So if \(|k| < c_0'\) with \(c_0'\) small enough, \(D + B\) is invertible with the inverse given by the Neumann expansion \(D^{-1} + \sum_{p=1}^{\infty} (-1)^p D^{-1} (BD^{-1})^p\). Since \(D\) and \(B\) are both analytic on \(|\text{Im } k| < 2\) and
\[ \sum_{\ell \in 2\pi \mathbb{Z}^4} \text{const}_{a, q} |k| \left( \prod_{\nu=0}^{24} \frac{1}{|\ell_\nu| + \pi} \right)^q \frac{\Gamma''_{10}}{1 + |\ell_0 + \Sigma_{\nu=1}^r|\ell_\nu|^2} \left( \prod_{\nu=0}^{24} \frac{1}{|\ell_\nu| + \pi} \right)^q < \frac{1}{2} \]
if \(c_0'\) is small enough, we again get the desired analyticity and bound on \(|\hat{S}_{n,k}(\ell, \ell')|\).

(c) Just apply Remark V.3 and parts (a) and (b) of this lemma, Lemma II.2.a and Proposition II.4.a and the fact that
\[ \sum_{\ell \in 2\pi \mathbb{Z}^4} \left( \prod_{\nu=0}^{24} \frac{1}{|\ell_\nu| + \pi} \right)^q \frac{\Gamma''_{10}}{1 + |\ell_0 + \Sigma_{\nu=1}^r|\ell_\nu|^2} \left( \prod_{\nu=0}^{24} \frac{1}{|\ell_\nu| + \pi} \right)^q \]
is bounded uniformly in \(n\) and \(L\) to (V.6).

(d) The bound on \(|S_n(u, u') - S_n'(u, u')|\) follows from part (c) by Lemma A.11.b with the replacements (V.4). The bound on \(|S_n'(u, u')|\) follows from part (a), noting in particular that \(\frac{\Gamma_{12}}{(1 + |p_0| + |p|^2)^3} \in L^1(\hat{Z}_{\text{fin}})\), and
\[ |u_\nu - u_\nu'|^2 S_n'(u, u') = \int_{\hat{Z}_{\text{fin}}} \frac{\partial^j S_n'}{\partial p_{\nu'}^j} (p) e^{-ip \cdot (u - u') \cdot d^4p \over (2\pi)^4} \]
and
\[ |S_n'(u, u')| \leq \int_{\hat{Z}_{\text{fin}}} \hat{S}_n'(p) e^{-ip \cdot (u - u') \cdot d^4p \over (2\pi)^4} \leq \Gamma_{12} \int_{\hat{Z}_{\text{fin}}} d^4p \over (2\pi)^4} = \Gamma_{12} L^5n \]

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We now prove the analog of Lemma V.4 for the operators $S_{n,\nu}^{(\pm)}$ and $\bar{S}_{n,\nu}^{(-)}$ of (V.3). As in (V.5)–(V.6), we decompose

\[ S_{n,\nu}^{(\pm)} = S_n^* + \delta S_{\nu}^{(\pm)} \quad S_{n,\nu}^{(-)} = S_n' + \delta S_{\nu}' \]

with

\[ \delta S_{\nu}^{(+)} = S_{n,\nu}^{(+)} - S_n^* = -S_n^*[Q_{\nu,\nu}^{(+)Q_{\nu,n,\nu}^{(-)} - a_n \exp\{-\Delta_n\}] S_{n,\nu}^{(+)} \]
\[ \delta S_{\nu}' = S_{n,\nu}^{(-)} - S_n' = -S_n'[Q_{\nu,\nu}^{(+)Q_{\nu,n,\nu}^{(-)} - a_n \exp\{-\Delta_n\}] S_{n,\nu}^{(-)} \]

They have Fourier representations

\[ \mathcal{S}_{\nu,k}^{(\pm)}(\ell, \ell') = -\sum_{\ell'' \in \mathcal{B}_n} \mathcal{S}_{n}(k+\ell)[U^{(+)\nu}_n(k, \ell)\hat{\Omega}_n(k)U^{(-)\nu}_n(k, \ell'') - a_n e^{-\Delta_n(k+\ell)} \hat{\delta}_{\ell,\ell''}] \mathcal{S}_{n,\nu,k}^{(\pm)}(\ell'', \ell') \]

\[ \mathcal{S}_{\nu,k}^{(-)}(\ell, \ell') = -\sum_{\ell'' \in \mathcal{B}_n} \hat{\mathcal{S}}_{n}(k+\ell)[U^{(+)\nu}_n(k, \ell)\hat{\Omega}_n(k)U^{(-)\nu}_n(k, \ell'') - a_n e^{-\Delta_n(k+\ell)} \hat{\delta}_{\ell,\ell''}] \hat{\mathcal{S}}_{n,\nu,k}^{(-)}(\ell'', \ell') \]  

(V.9)

where, by (II.10),

\[ U^{(+)\nu}_n(k, \ell) = \zeta^{(+)\nu}_n(k, \ell)u^{(+)\nu}_n(k + \ell)u_n(k + \ell)^{q-1} \]
\[ U^{(-)\nu}_n(k, \ell'') = \zeta^{(-)\nu}_n(k, \ell'')u^{(-)\nu}_n(k + \ell'')u_n(k + \ell'')^{q-1} \]

**Lemma V.5** There are constants $m_5 > 0$ and $\Gamma_13$ such that the following hold for all $L > \Gamma_2$.

(a) For all $\ell, \ell' \in \mathcal{B}_n$, $\hat{\mathcal{S}}_{n,\nu,k}^{(\pm)}(\ell, \ell')$ is analytic in $|\text{Im} k| < 3m_5$ and obeys

\[ |\hat{\mathcal{S}}_{n,\nu,k}^{(\pm)}(\ell, \ell')| \leq \frac{\Gamma_{13}}{1+|\ell_0|+\Sigma_{\nu=1}^{3}|\ell_\nu|^2} \left\{ \delta_{\ell,\ell'} + \frac{1}{1+|\ell_0|+\Sigma_{\nu=1}^{3}|\ell_\nu|^2} \prod_{\nu=0}^{3} \frac{1}{((|\ell_\nu|+1)^2)} \prod_{\nu=0}^{3} \frac{1}{(|\ell_\nu|+1)^2} \right\} \]

there.

(b) For all $\ell, \ell' \in \mathcal{B}_n$, $\hat{\delta}_{\nu,k}^{(\pm)}(\ell, \ell')$ is analytic in $|\text{Im} k| < 3m_5$ and obeys

\[ |\hat{\delta}_{\nu,k}^{(\pm)}(\ell, \ell')| \leq \Gamma_{13} \exp \left\{ -\frac{1}{40} \Sigma_{\nu=0}^{3}|\ell_\nu|^2 \right\} \delta_{\ell,\ell'} \]

\[ + \frac{\Gamma_{13}}{1+|\ell_0|+\Sigma_{\nu=1}^{3}|\ell_\nu|^2} \prod_{\nu=0}^{3} \frac{1}{((|\ell_\nu|+1)^2)} \prod_{\nu=0}^{3} \frac{1}{(|\ell_\nu|+1)^2} \]

there.

(c) For all $u, u' \in \mathcal{X}_n$, $|S_{n,\nu}^{(\pm)}(u, u') - S_{n,\nu}'(u, u')| \leq \Gamma_{13} e^{-2m_5|u-u'|}$.  

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Proof: (a) For any $c''_0 > 0$ we have, for $|k| \geq c''_0$ and $|\text{Im} k_0| < 3m_5$, analyticity and the desired bound follows from the representations (apply [parabolic-all.tex, Remark B.9.b] with $R = Q_{n,n}(\cdot)$, $R_\ast = Q_{n,n}(\cdot)$ and use Remark II.5 to give $RD^{-1}R_\ast = Q_-D^{-1}Q_-$)

$$S_{n,\nu}^{(+)} = D_n^{-1} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1} \quad S_{n,\nu}^{(-)} = D_n^{-1} - D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}Q_{n,\nu}^{(-)}D_n^{-1}$$

and Lemmas III.2.d, II.2.a, II.6.b and IV.2.c.

For $|k| < c'_0$, with $c'_0$ to be shortly chosen sufficiently small, and $|\text{Im} k| < 3m_5$ we use the representation

$$(\hat S_{n,\nu,k}^{(-)})^{-1}(\ell, \ell') = \hat D_n(k + \ell) \delta_{\ell,\ell'} + U_{n,\nu}^{(+)}(k, \ell)\hat \Omega_n(k)U_{n,\nu}^{(-)}(k, \ell') = D_{\ell,\ell'} + B_{\ell,\ell'} \quad \text{(V.10)}$$

with

$$D_{\ell,\ell'} = \hat D_n(k + \ell) \delta_{\ell,\ell'} + \begin{cases} a_n & \text{if } \ell = \ell' = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$B_{\ell,\ell'} = \begin{cases} \hat \Omega_n(k)u_n(k)^{2q} - a_n & \text{if } \ell = \ell' = 0 \\ U_{n,\nu}^{(+)}(k, \ell)\hat \Omega_n(k)U_{n,\nu}^{(-)}(k, \ell') & \text{otherwise} \end{cases}$$

and the obvious analog for $(\hat S_{n,\nu,k}^{(+)})^{-1}$ — just replace $\hat D_n(k + \ell)$ with its complex conjugate. As in the proof of Lemma V.4.b, $D$ is invertible and the inverse is a diagonal matrix with every diagonal matrix element obeying

$$|D_{\ell,\ell'}^{-1}| \leq \frac{\Gamma'_{11}}{1 + |\ell_0| + \sum_{\nu=1}^{\nu_0} |\ell_\nu|^2}$$

for some $\Gamma'_{11}$ which is independent of $c'_0$. By parts (b) and (c) of Lemma II.2, parts (a) and (b) of Proposition II.4, and part (b) of Lemma II.6,

$$|B_{\ell,\ell'}| \leq \text{const}_{a,q} |k| \prod_{\nu=0}^{3} \left( \frac{24}{|\ell_{\nu}| + \pi} \right)^{q-1} \prod_{\nu=0}^{3} \left( \frac{24}{|\ell'_{\nu}| + \pi} \right)^{q}$$

So if $|k| < c'_0$ with $c'_0$ small enough, $D + B$ is invertible with the inverse given by the Neumann expansion $D^{-1} + \sum_{p=1}^{\infty} (-1)^p D^{-1}(BD^{-1})^p$. As $q > 1$, the desired analyticity and the desired bound on $|\hat S_{n,\nu,k}^{(\pm)}(\ell, \ell')|$ follow as in the proof of Lemma V.4.b

(b) is proven just as Lemma V.4.c.

(c) follows from part (b) by Lemma A.11.b. \qed
Proof of Proposition V.1: Set \(m_3 = \min\{m_4, m_5\}\). As \(\frac{1}{|\nu_0|^2 + |\nu_1|^2}\) is locally integrable in \(\mathbb{R}^4\), the pointwise bounds on \(|S_n(u, u') - S_n'(u, u')|\) and \(|S_n'(u, u')|\), given in Lemma V.4.d, and on \(|S_n'(u, u') - S_n'(u, u')|\), given in Lemma V.5.c, imply
\[
\| S_n \|_{m_3}, \| S_n'(\pm) \|_{m_3} \leq \tilde{\Gamma}_{11}
\]
with \(\tilde{\Gamma}_{11}\) a constant, depending only on \(m_3\), times \(\max\{\Gamma_{12}, \Gamma_{13}\}\). Setting \(\mu_{\text{up}} = \frac{1}{2\tilde{\Gamma}_{11}}\) and \(\Gamma_{11}\) to be the maximum of \(2\tilde{\Gamma}_{11}\) (for \(\| S_n(\mu) \|_{m_3}\) and \(\| S_n^{(\pm)}(\mu) \|_{m_3}\) and \(2\tilde{\Gamma}_{11}\) (for \(\| S_n(\mu) - S_n \|_{m_3}\) and \(\| S_n^{(\pm)}(\mu) - S_n^{(\pm)} \|_{m_3}\) a Neumann expansion gives the specified bounds.

We now formulate and prove two more technical lemmas that will be used elsewhere.

Lemma V.6 There are constants \(m_6 > 0\) and \(\Gamma_{14}\) such that, for all \(L > \Gamma_2\),
\[
\left| (S_nQ_n^*)(y, x) \right| \leq \Gamma_{14}e^{-m_6|x-y|} \quad \|S_n Q_n^*\|_{m_6} \leq \Gamma_{14}
\]

Proof: From the definitions of \(S_n\), in (V.3), and \(\Delta^{(n)}\), at the beginning of §IV, one sees directly that
\[
S_n^{(\ast)}Q_n^* = D_n^{(\ast)^{-1}} Q_n^* \Delta^{(n)(\ast)} \Omega_n^{-1} : L^2(\mathcal{X}_{\text{crs}}) \rightarrow L^2(\mathcal{X}_{\text{fin}})
\]

The Fourier transform of the kernel, \(b(y, x)\), of the operator \(S_nQ_n^*\) is
\[
\hat{b}_k(\ell) = D_n^{-1}(k + \ell)u_n(k + \ell)^q \hat{\Delta}^{(n)}(k)^{-1} \Omega_n(k)
\]

By Lemma IV.2.a,c,f, Remark III.1.b and Lemma III.2.d, Proposition II.4.a, Remark II.1.d and Lemma II.2.a, \(\hat{b}_k(\ell)\) is analytic in \(|\text{Im} k| < 3m_6\) and
\[
\left| \hat{b}_k(\ell) \right| \leq \frac{5\Gamma_6}{4\delta} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_\nu| + \pi} \right)^q
\]
The bound is uniform in \(n\) and \(L\) and is summable in \(\ell\), so the claims follow from Lemma A.11.c.

Lemma V.7 There are constants \(m_7 > 0\) and \(\Gamma_{15}\) such that, for all \(L > \Gamma_2\), the operators
\[
(S_n(\mu)Q_n^*\Omega_n)^* (S_n(\mu)Q_n^*\Omega_n), \quad (S_n(\mu)^*(Q_n^*\Omega_n))^* (S_n(\mu)^*Q_n^*\Omega_n)
\]
\[
(S_n^{(\pm)}(\mu)Q_n^{(\pm)}\Omega_n)^* (S_n^{(\pm)}(\mu)Q_n^{(\pm)}\Omega_n)
\]
all have bounded inverses. The \(\| \cdot \|_{m_7}\) norms of the inverses are all bounded by \(\Gamma_{15}\).
Proof: We first consider the case that $\mu = 0$. By (V.11), the operator

$$S_n Q_n^* \mathcal{Q}_n = D_n^{-1} Q_n^* \Delta^{(n)} : L^2(\mathcal{X}_{crs}) \to L^2(\mathcal{X}_{fin})$$

has Fourier transform

$$\tilde{b}_k(\ell) = \hat{D}_n^{-1}(k + \ell) u_n(k + \ell)^q \hat{\Delta}^{(n)}(k)$$

The operator $(S_n Q_n^* \mathcal{Q}_n)^*(S_n Q_n^* \mathcal{Q}_n)$ maps $L^2(\mathcal{X}_{crs})$ to $L^2(\mathcal{X}_{crs})$ and has Fourier transform

$$\sum_{\ell \in \mathcal{B}} \tilde{b}_{-\ell}(-\ell) \tilde{b}_k(\ell) = \sum_{\ell \in \mathcal{B}} \hat{D}_n^{-1}(-k - \ell) \hat{D}_n^{-1}(k + \ell) u_n(k + \ell)^{2q} \hat{\Delta}^{(n)}(-k) \hat{\Delta}^{(n)}(k)$$

For $k$ real,

$$\sum_{\ell \in \mathcal{B}} \tilde{b}_{-\ell}(-\ell) \tilde{b}_k(\ell) = \sum_{\ell \in \mathcal{B}} \left| \hat{D}_n^{-1}(k + \ell) u_n(k + \ell)^q \hat{\Delta}^{(n)}(k) \right|^2$$

$$\geq \left| \hat{D}_n^{-1}(k) u_n(k)^q \hat{\Delta}^{(n)}(k) \right|^2$$

$$\geq \gamma_2^2 \inf_{|\nu| \leq \pi} |u_n(k)|^{2q}$$

$$\geq \gamma_2^2 \left( \frac{2}{\pi} \right)^{8q}$$

by Lemmas IV.2.e and II.2.f. To show that half this the lower bound extends into a strip along the real axis that has width independent of $n$ and $L$, we observe that

- all first order derivatives of $u_n(k + \ell)$ are uniformly bounded by $2 \prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi}$ on such a strip by the Cauchy integral formula and Remark II.1.d and Lemma II.2.a and
- $u_n(k + \ell)$ itself is uniformly bounded by $\prod_{\nu=0}^{3} \frac{24}{|\nu| + \pi}$ on such a strip by Lemma II.2.a and
- all first order derivatives of $\hat{\Delta}^{(n)}(k)$ are uniformly bounded on such a strip by Lemma IV.2.c and
- for $\ell \neq 0$, all first order derivatives of $\hat{D}_n^{-1}(k + \ell)$ are uniformly bounded on such a strip by parts (c) and (d) of Lemma III.2 and
- for $\ell = 0$, all first order derivatives of $\hat{D}_n^{-1}(k) \hat{\Delta}^{(n)}(k)$ are uniformly bounded on such a strip by Lemma IV.2.f.

The operator $S_n^* Q_n^* \mathcal{Q}_n = D_n^{-1} Q_n^* \Delta^{(n)*}$ has Fourier transform $\tilde{b}_{-\ell}(-\ell)$. So

$$(S_n Q_n^* \mathcal{Q}_n)^*(S_n Q_n^* \mathcal{Q}_n) = (S_n^* Q_n^* \mathcal{Q}_n)^*(S_n^* Q_n^* \mathcal{Q}_n)$$

The operators $S_n^{(-)} Q_n^{(+)} \mathcal{Q}_n = D_n^{-1} Q_n^{(+)\Delta^{(n)}}$ and $S_n^{(+)} Q_n^{(+)} \mathcal{Q}_n = D_n^{-1} Q_n^{(+)\Delta^{(n)*}}$ map $L^2(\mathcal{X}_{crs})$ to $L^2(\mathcal{X}_{fin})$ and have Fourier transforms

$$\tilde{c}_k(\ell) = \hat{D}_n^{-1}(\pm k \pm \ell) s_{n,\nu}^{(+)}(k, \ell) u_n^{(+)}(k + \ell) u_n(k + \ell)^{q-1} \hat{\Delta}^{(n)}(\pm k)$$
So the operators \((S^{(\pm)} Q^{(\pm)} \Omega_n)^* (S^{(\pm)} Q^{(\pm)} \Omega_n)\) both map \(L^2(\mathcal{X}_{crs})\) to \(L^2(\mathcal{X}_{crs})\) and have Fourier transform

\[
\sum_{\ell \in \hat{B}} \hat{c}_{-k}(-\ell) \hat{c}_{k}(\ell)
\]

\[
= \sum_{\ell \in \hat{B}} \hat{D}_n^{-1}(-k - \ell) \hat{D}_n^{-1}(k + \ell) u^{(+)}_{n,\nu}(k + \ell) u_n(k + \ell) 2^{(q-1)} \hat{\Delta}^{(n)}(-k) \hat{\Delta}^{(n)}(k)
\]

This is bounded just as \(\sum_{\ell \in \hat{B}} \hat{b}_{-k}(-\ell) \hat{b}_{k}(\ell)\) was. The specified bounds, in the special case that \(\mu = 0\) follow.

By Proposition V.1, a Neumann expansion gives the desired bounds when \(\mu\) is nonzero.
VI. The Degree One Part of the Critical Field

In [parabolic-all.tex, Proposition G.13 and (G.29)] we derive an expansion for the critical fields of the form

\[
\psi_n(\theta^*, \theta, \mu, \nu) = \frac{a}{L^2} C^{(n)}(\mu)^{(s)} Q^* \theta^{(s)} + \psi_{(s)n}(\theta^*, \theta, \mu, \nu)
\]

with the \(C^{(n)}(\mu)\) of [parabolic-all.tex, Proposition I.17] and with \(\psi_{(s)n}^{(\geq 3)}\) being of degree at least 3 in \(\theta^{(s)}\). In this section we derive bounds on a scaled version of \(C^{(n)}(\mu)^{(s)} Q^* \theta^{(s)}\) and some related operators. To do so we use the representation

\[
\frac{a}{L^2} C^{(n)}(\mu)^{(s)} Q^* = \left( \frac{a}{L^2} Q^* Q + \Omega_n \right)^{-1} \left\{ \frac{a}{L^2} Q^* + \Omega_n Q \tilde{S}_{n+1}(\mu)^{(s)} \tilde{Q}_{n+1} \right\}^* \tilde{Q}_{n+1} \tilde{Q}_{n+1} \]

of (G.28). Here, as in [parabolic-all.tex, Lemma III.4 and (G.27)],

\[
\tilde{Q}_{n+1} = L_n Q_{n+1} L_n^{-1}, \quad \tilde{Q}_{n+1} = L_n Q_{n+1} L_n^{-1}, \quad \tilde{Q}_{n+1} = \left\{ D_n - \mu \right\} \tilde{Q}_{n+1} \tilde{Q}_{n+1} \tilde{Q}_{n+1} \tilde{Q}_{n+1} \]

These operators are all translation invariant with respect to \(X_{-1}^{(n+1)}\). As

\[
\tilde{S}_{n+1}(\mu) = \tilde{S}_{n+1} + \mu \tilde{S}_{n+1} \tilde{S}_{n+1}(\mu) \quad \text{with} \quad \tilde{S}_{n+1} = \tilde{S}_{n+1}(0)
\]

we have

\[
\frac{a}{L^2} C^{(n)}(\mu)^{(s)} Q^* = \frac{a}{L^2} C^{(n)}(\mu)^{(s)} Q^* + \mu A_{\psi, \phi} \tilde{S}_{n+1}(\mu)^{(s)} \tilde{Q}_{n+1} \tilde{Q}_{n+1} \tilde{Q}_{n+1} \tilde{Q}_{n+1} \]

where

\[
A_{\psi, \phi} = (aL^{-2} Q^* Q + \Omega_n)^{-1} \Omega_n Q_n : \mathcal{H}_n \to \mathcal{H}_0^{(n)} \]

The operator \(A_{\psi, \phi}\) is also used in the course of bounding \(\psi_{(s)n}^{(\geq 3)}\) in [parabolic-all.tex, Proposition G.13].

The main results of this section are

**Proposition VI.1** There are constants\(^{(1)}\) \(m_8 > 0\) and \(\Gamma_16, \Gamma_17\) such that the following holds, for each \(L > \Gamma_17\) and each \(\mu\) obeying \(|L^2 \mu| \leq \mu_{up}\).

\(^{(1)}\) Recall Convention I.2.
(a) \[ \left\| \mathbb{L}^{-1} A_{\psi, \phi} \mathbb{L}_* \right\|_{m=1} \leq \Gamma_{16} \quad \text{and} \quad \left\| \mathbb{L}_*^{-1} \frac{\alpha}{L^2} C^{(n)}(\mu)^{(s)} Q^* \mathbb{L}_* \right\|_{m=1} \leq \Gamma_{16} \]

(b) Let \( 0 \leq \nu \leq 3 \). There are operators \( A_{\psi, \phi, \nu} \) and \( A_{\psi_{(\nu)}} \nu(\mu) \) such that
\[
\partial_{\nu} A_{\psi, \phi} = A_{\psi, \phi, \nu} \partial_{\nu} \quad \text{and} \quad \partial_{\nu} \frac{\alpha}{L^2} C^{(n)}(\mu)^{(s)} Q^* = A_{\psi_{(\nu)}}(\mu) \partial_{\nu}
\]

and
\[
\left\| \mathbb{L}^{-1} A_{\psi, \phi, \nu} \mathbb{L}_* \right\|_{m=1} \leq \Gamma_{16} \quad \left\| \mathbb{L}_*^{-1} A_{\psi_{(\nu)}}(\mu) \mathbb{L}_* \right\|_{m=1} \leq \Gamma_{16}
\]

This proposition is proven at the end of this section, after Lemma VI.6. In this proof we write \( \frac{\alpha}{L^2} C^{(n)}(\mu)^{(s)} Q^* = A_{\psi_{(\nu)}} \) so that
\[
A_{\psi_{(\nu)}} = (aL^{-2}Q^*Q + \Omega_n)^{-1}(aL^{-2}Q^* + \Omega_n Q_n S_{n+1} D_{n+1} Q_{n+1} \tilde{\Delta}_{n+1}) : \mathcal{H}_0^{(n+1)} \rightarrow \mathcal{H}_0^{(n)}
\]

Remark VI.2

(a) \( A_{\psi_{(\nu)}} = \Omega_n^{-1} Q^* \tilde{\Delta}_{n+1} + A_{\psi, \phi} D_{n+1}^{-1(\psi)} \tilde{\Delta}_{n+1} \)

\[
= \frac{1}{L^2} \mathbb{L}_* \Delta^{(n+1)(\psi)} \mathbb{L}_*^{-1} = \left\{ \mathbb{I} + \tilde{\Delta}_{n+1} D_{n+1}^{-1(\psi)} \tilde{\Delta}_{n+1} \right\}^{(n+1)(\psi)}
\]

being a fully translation invariant operator on \( \mathcal{H}_0^{(n+1)} \).

(b) Let \( 0 \leq \nu \leq 3 \). We have \( \partial_{\nu} A_{\psi, \phi} = \partial_{\nu} A_{\psi_{(\nu)}}(\mu) \) and \( \partial_{\nu} A_{\psi_{(\nu)}}(\mu) = A_{\psi_{(\nu)}}(\mu) \partial_{\nu} \) where
\[
A_{\psi, \phi} = [\mathbb{I} - \Omega_n^{-1} Q_{n+1(\psi)} \tilde{\Delta}_{n+1} Q_{n+1(\psi)}] Q_{n+1(\nu)}
\]
\[
A_{\psi_{(\nu)}}(\mu) = \Omega_n^{-1} Q_{n+1(\psi)} \tilde{\Delta}_{n+1} + A_{\psi, \phi} \mathbb{L}_* D_{n+1}^{-1(\psi)} Q_{n+1(\nu)} \Delta^{(n+1)(\psi)} \mathbb{L}_*^{-1}
\]

Proof: (a) First observe that, by (V.11),
\[
Q_n S_{n+1} D_{n+1}^{-1(\psi)} \tilde{\Delta}_{n+1} = Q_n \mathbb{L}_* D_{n+1}^{-1(\psi)} Q_{n+1} \Delta^{(n+1)(\psi)} \mathbb{L}_*^{-1} = (Q_n D_{n+1}^{-1(\psi)} Q_{n+1} Q^* \tilde{\Delta}_{n+1})
\]

Using (VI.5), the operator
\[
A_{\psi, \phi} = (aL^{-2}Q^*Q + \Omega_n)^{-1}(aL^{-2}Q^* + \Omega_n Q_n D_{n+1}^{-1(\psi)} \tilde{\Delta}_{n+1} \Delta^{(n+1)(\psi)})
\]
\[
= \Omega_n^{-1}(Q^*Q \Omega_n + \frac{1}{aL^{-2}} \mathbb{I})^{-1} Q^* + A_{\psi, \phi} D_{n+1}^{-1(\psi)} \tilde{\Delta}_{n+1} \Delta^{(n+1)(\psi)}
\]
\[
= \Omega_n^{-1} Q^* \tilde{\Delta}_{n+1} + A_{\psi, \phi} D_{n+1}^{-1(\psi)} \tilde{\Delta}_{n+1} \Delta^{(n+1)(\psi)}
\]

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(b) By Remark II.5,
\[
\partial_\nu A_{\psi,\phi} = \partial_\nu (\mathbb{I} + aL^{-2} \Omega_n^{-1} Q^* Q)^{-1} Q_n = \partial_\nu Q_n - \partial_\nu aL^{-2} \Omega_n^{-1} Q^* (\mathbb{I} + aL^{-2} Q \Omega_n^{-1} Q^*)^{-1} Q Q_n = Q_n \partial_\nu - aL^{-2} \Omega_n^{-1} Q^* (\mathbb{I} + aL^{-2} Q \Omega_n^{-1} Q^*)^{-1} Q (-) \partial_\nu = [\mathbb{I} - \Omega_n^{-1} Q^* \partial_\nu \Omega_n^{-1} Q (-)] Q_n \partial_\nu
\]
Therefore by part (a), (III.1), (VI.2) and Remark II.5,
\[
\partial_\nu A_{\psi(\ast),\phi(\ast)} = \partial_\nu \left[ \Omega_n^{-1} Q^* \hat{\Omega}_{n+1} + A_{\psi,\phi} \partial_\nu D_{n+1}^{-1(\ast)} Q^* n+1 \Delta^{(n+1)(\ast) \Omega_n^{-1} \Omega_n^{-1}} \right] = A_{\psi(\ast),\phi(\ast)} \partial_\nu
\]
since \( \partial_\nu \mathbb{L}_+ = \frac{1}{L_\nu} \mathbb{L}_+ \partial_\nu \) by [parabolic-all.tex, Remark III.2.a,b]. To get \( \partial_\nu A_{\psi(\ast),\phi(\ast)} (\mu) = A_{\psi(\ast),\phi(\ast)} (\mu) \partial_\nu \) when \( \mu \neq 0 \), write, using (VI.2),
\[
A_{\psi(\ast),\phi(\ast)} (\mu) = A_{\psi(\ast),\phi(\ast)} + \mu A_{\psi,\phi} \hat{S}_{n+1}(\mu) \hat{Q}^* n+1 \hat{\Omega}_{n+1}
\]
and use (V.1), Remark II.5 and the fact that \( \Omega_{n+1} \) is fully translation invariant.

The operators of principal interest, \( A_{\psi,\theta} \) and \( A_{\psi,\theta} \), act from \( L^2(\mathcal{X}_{crs}) = \mathcal{H}_{-1}^{(n+1)} \) to \( L^2(\mathcal{X}_{fin}) = \mathcal{H}_0^{(n)} \) with
\[
\mathcal{X}_{fin} = \mathcal{X}_0^{(n)} \quad \mathcal{X}_{crs} = \mathcal{X}_{-1}^{(n+1)} \quad \mathcal{B} = \mathcal{B}^+
\]
We now give a bunch of Fourier transforms (in the sense of (A.6) and (A.7) – but we shall suppress the * from the notation). All of the operators above are periodic in the sense of Definition A.2. As before we denote
- momenta dual to the \( L^- \)-lattice \( L^Z \times L^Z \) by \( \ell \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}/2\pi \mathbb{Z}) \),
- momenta dual to the unit lattice \( Z \times Z \) by \( \ell \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}/2\pi \mathbb{Z}) \) and decompose \( k = \ell + \ell' \) with \( k \) in a fundamental cell for \( (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}/2\pi \mathbb{Z}) \) and \( \ell, \ell' \in (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}/2\pi \mathbb{Z}) = \hat{B}^+ \) and
- momenta dual to the \( \varepsilon_j \)-lattice \( \varepsilon_j Z \times \varepsilon_j Z \) by \( p_j \in (\mathbb{R}/2\pi Z) \times (\mathbb{R}/2\pi Z) \) and decompose \( p_j = k + \ell_j \) with \( k \) in a fundamental cell for \( (\mathbb{R}/2\pi Z) \times (\mathbb{R}/2\pi Z) \) and \( \ell_j \in (\varepsilon_j Z \times \varepsilon_j Z) = \hat{B}_j \). Here \( 1 \leq j \leq n \).

The Fourier transform of \( A_{\psi(\ast),\phi(\ast)} \) is
\[
(A_{\psi(\ast),\phi(\ast)})_{\ell} (\ell') = \sum_{\ell' \in \hat{B}^+} \left( \frac{a}{L^n} Q^* Q + \Omega_n \right)^{-1}_{\ell} (\ell, \ell') \left\{ \frac{a}{L^n} Q^* (\ell') + \Omega_n (\ell + \ell') \left( Q_n D_n^{-1(\ast)} Q^*_n \right) (\ell + \ell') \partial_\nu (\ell') \Delta^{(n+1)(\ast)} (\ell) \right\}
\]
(VI.6)
where, by Remark II.1, Remark VI.2.a and (VI.2)

\[(aL^{-2}Q^*Q + \Omega_n)\e_k(\ell, \ell') = \Omega_n(\e + \ell)\delta_{\ell,\ell'} + aL^{-2}u_+ (\e + \ell)^q u_+ (\e + \ell')^q \]

\[Q_\e^*(\ell) = u_+ (\e + \ell)^q \]

\[\Omega_n(k) = a \left[ 1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in B_j} \frac{1}{L^2} u_j (k + \ell_j)^2 q \right]^{-1} \]

\[(Q_nD_n^{-1(*)}Q_n^*)(k) = \sum_{\ell_n \in B_n} u_n(k + \ell_n)^2 q D_n^{-1(*)}(k + \ell_n) \]

\[\tilde{\Delta}^{(n+1)(*)}(\e) = \frac{\tilde{\Omega}_{n+1}(\e)}{1 + \tilde{\Omega}_{n+1}(\e) \sum_{\ell_n \in B_n} \frac{u_n(k + \ell_n)^2 q}{L^2} \Omega_n(\e + \ell_n)^{-1}} \]

\[\tilde{\Omega}_{n+1}(\e) = \frac{a}{L^2} \left[ 1 + \sum_{\ell \in B^+} \frac{a}{L^2} u_+ (\e + \ell)^2 q \Omega_n(\e + \ell)^{-1} \right]^{-1} \]

**Lemma VI.3** Let \(|\text{Im} \e_\nu| \leq \frac{2}{L^2} \) for each \(0 \leq \nu \leq 3\). There is a constant \(\Gamma_{17}\), depending only on \(q\), such that the following hold for all \(L > \Gamma_{17}\).

(a) We have \(\frac{6}{7} \leq |\tilde{\Omega}_{n+1}(\e)| \leq \frac{6}{5} \) and \(\text{Re} \tilde{\Omega}_{n+1}(\e) \geq \frac{a}{2L^2} \).

(b) We have

\[\left| (aL^{-2}Q^*Q + \Omega_n)^{-1}(\ell, \ell') - \Omega_n(\e + \ell)^{-1}\delta_{\ell,\ell'} \right| \leq \frac{2}{aL^2} \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|k_{\nu}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|k_{\nu}| + \pi} \right)^q \left\{ \begin{array}{ll} 1 & \text{for all } \ell \\ \prod_{0 \leq \nu \leq 3 \atop \ell_{\nu} \neq 0} L_{\nu}|k_{\nu}| & \text{if } \ell \neq 0 \end{array} \right. \]

(c) Let \(0 \leq \nu \leq 3\). Then

\[\left| [\Omega_n^{-1}Q_{+,\nu}^{-1} \tilde{\Omega}_{n+1} Q_{+,\nu}^+]_{\ell}(\ell, \ell') \right| \leq \frac{3a^4}{2L^2} \left( \frac{24\pi}{L_{\nu}|k_{\nu}| + \pi} \right)^q \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|k_{\nu}| + \pi} \right)^q -1 \prod_{\nu=0}^{3} \left( \frac{24}{L_{\nu}|k_{\nu}| + \pi} \right)^q \]

**Proof:** (a) is proven much as Proposition II.4.a was.
(b) The straight forward Neumann expansion gives
\[
\left| \left( \frac{\partial}{\partial \mathbf{r}} \mathbf{Q} \cdot \mathbf{Q} + \Omega_n \right) \right|^{\ell} (\ell, \ell') - \Omega_n (\ell + \ell)^{-1} \delta_{\ell, \ell'} \right| \\
\leq \frac{a}{L^2} \left| \Omega_n (\ell + \ell)^{-1} u_+ (\ell + \ell) q \right| \sum_{j=0}^{\infty} \left\{ \frac{\alpha L^2}{\Omega_n (\ell + \ell)^{-1}} \right\}^j \\
\leq \frac{a}{L^2} \left( \frac{5}{4a} \right)^2 \left| u_+ (\ell + \ell) q \right| \prod_{\nu=0}^{\infty} \left( \frac{L_{\nu} 24}{L_{\nu} (\nu + \pi)} \right)^q \sum_{j=0}^{\infty} \left\{ \frac{5L^2}{4L^2} \right\}^j \\
\leq \frac{2}{aL^2} \prod_{\nu=0}^{\infty} \left( \frac{L_{\nu} 24}{L_{\nu} (\nu + \pi)} \right)^q \prod_{\nu=0}^{\infty} \left( \frac{L_{\nu} 24}{L_{\nu} (\nu + \pi)} \right)^q \left\{ \right\} \frac{1}{L_{\nu} |\ell_{\nu}|} \text{ for all } \ell \neq 0
\]
by part (a), Lemma II.3.a,b and Proposition II.4.a, with L satisfying the conditions of part (a).

(c) The specified bound follows from (II.11), part (a), Proposition II.4.a and Lemmas II.6.b, II.3.a.

\[\]

**Lemma VI.4** For all
\[
t = \mathbb{L}^{-1} (k) \in \left( \mathbb{R} / \frac{2\pi}{\ell} \mathbb{Z} \right) \times \left( \mathbb{R}^3 / \frac{2\pi}{\ell} \mathbb{Z}^3 \right) \quad k \in \left( \mathbb{R} / 2\pi \mathbb{Z} \right) \times \left( \mathbb{R}^3 / 2\pi \mathbb{Z}^3 \right) \\
p = \mathbb{L}^{-1} (p) \in \left( \mathbb{R} / \frac{2\pi}{\epsilon_n} \mathbb{Z} \right) \times \left( \mathbb{R}^3 / \frac{2\pi}{\epsilon_n} \mathbb{Z}^3 \right) \quad p \in \left( \mathbb{R} / \frac{2\pi}{\epsilon_{n+1}} \mathbb{Z} \right) \times \left( \mathbb{R}^3 / \frac{2\pi}{\epsilon_{n+1}} \mathbb{Z}^3 \right) \\
\ell_{n+1} \in \left( 2\pi \mathbb{Z} / \frac{2\pi}{\epsilon_{n+1}} \mathbb{Z} \right) \times \left( 2\pi \mathbb{Z}^3 / \frac{2\pi}{\epsilon_{n+1}} \mathbb{Z}^3 \right)
\]
we have

(a) \( u_+ (\mathbb{L}^{-1} (k)) = u_1 (k) \) and \( u_n (\mathbb{L}^{-1} (k)) u_+ (\mathbb{L}^{-1} (k)) = u_{n+1} (k) \) for all \( n \in \mathbb{N} \).

(b) \( \tilde{\Omega}_{n+1} (\mathbb{L}^{-1} (k)) = \frac{1}{L^2} \Omega_{n+1} (k) \)

(c) \( D_n^{-1} (\mathbb{L}^{-1} (p)) = L^2 D_{n+1}^{-1} (p) \)

(d) \( \tilde{\Delta}^{(n+1)*} (\mathbb{L}^{-1} (k)) = \frac{1}{L^2} \Delta^{(n+1)*} (k) \)

(e) \( \tilde{S}_{n+1, \mathbb{L}^{-1} (k)} (\mathbb{L}^{-1} (\ell_{n+1}), \mathbb{L}^{-1} (\ell'_{n+1})) = L^2 \tilde{S}_{n+1, k} (\ell_{n+1}, \ell'_{n+1}) \)

**Proof:** (a) These all follow from (VI.2), Remark VI.2.a and

- \( Q_1 = \mathbb{L}^{-1}_s Q \mathbb{L}_s \) by (II.2)
- \( D_{n+1} = L^2 \mathbb{L}^{-1}_s D_n \mathbb{L}_s \) by (III.1)

and Lemmas A.14.b and A.15.b.

\[\]
Corollary VI.5  There are constants $m_9 > 0$ and $\Gamma_{18}$ such that, for all $L > \Gamma_{17}$,

$$\|L_*^{-1} \tilde{S}_{n+1} L_*\|_{m_9} \leq L^2 \Gamma_{18} \quad \|L_*^{-1} \tilde{S}_{n+1} \tilde{Q}_{n+1}^* L_*\|_{m_9} \leq L^2 \Gamma_{18}$$

Proof:  Set $m_9 = \min\{m_3, m_6\}$. Just combine Lemmas A.14.b and A.15.b and parts (a) and (e) of Lemma VI.4 to yield

$$\|L_*^{-1} \tilde{S}_{n+1} L_*\|_{m_9} = L^2 \|S_{n+1}\|_{m_9} \quad \|L_*^{-1} \tilde{S}_{n+1} \tilde{Q}_{n+1}^* L_*\|_{m_9} = L^2 \|S_{n+1} Q_{n+1}\|_{m_9}$$

and then apply Proposition V.1 and Lemma V.6.

Lemma VI.6  Assume that $L > \Gamma_{17}$, the constant of Lemma VI.3. There are constants $m_{10} > 0$ and $\Gamma_{19}$ such that the following hold for all $k \in \mathbb{Q}$ with $|\text{Im} k| \leq m_{10}$.

(a)  $\left| (L_*^{-1} A_{\psi(\cdot) \theta(\cdot)} L_*) k(\ell_1) \right| = \left| (A_{\psi(\cdot) \theta(\cdot)})_{L-1}(k)(L_*^{-1}(\ell_1)) \right| \leq \Gamma_{19} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu} + \pi|} \right)^q$

(b)  If $\ell_1 \neq 0$, then $\left| (L_*^{-1} A_{\psi(\cdot) \theta(\cdot)} L_*) k(\ell_1) \right| \leq \Gamma_{19} |k| \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu} + \pi|} \right)^q$

Proof:  By (VI.6), Lemma VI.4 and Remark II.1.e,

$$(A_{\psi(\cdot) \theta(\cdot)})_{L-1}(k)(L_*^{-1}(\ell_1)) = \sum_{\ell' \in \mathcal{B}_1} \left( \frac{\alpha}{L^2} Q^* Q + \Omega_n \right)_{L-1}(k)(L_*^{-1}(\ell_1), L_*^{-1}(\ell'_1)) \left\{ \frac{\alpha}{L^2} Q_{L-1}^*(k)(L_*^{-1}(\ell'_1)) \right. $$

$$+ \Omega_n(L_*^{-1}(k + \ell'_1))(Q_n D_n^{-1}(\cdot) Q_n^*)(L_*^{-1}(k + \ell'_1)) Q_{L-1}^*(k)(L_*^{-1}(\ell'_1)) \Delta^{(n+1)(\cdot)}(L_*^{-1}(k)) \}

$$= \sum_{\ell' \in \mathcal{B}_1} \left( \frac{\alpha}{L^2} Q^* Q + \Omega_n \right)_{L-1}(k)(L_*^{-1}(\ell_1), L_*^{-1}(\ell'_1)) B_1(k, \ell'_1)

$$= \Omega_n(L_*^{-1}(k + \ell_1)) B_1(k, \ell_1) + \sum_{\ell'_1 \in \mathcal{B}_1} C(k, \ell_1, \ell'_1) B_1(k, \ell'_1)

where

$$B_1(k, \ell_1') = u_1(k + \ell_1') q \left\{ \frac{\alpha}{L^2} + \Omega_n(L_*^{-1}(k + \ell'_1)) \right\} \sum_{\ell_n \in \mathcal{B}_n} B_2(k, \ell'_1, \ell_n)

$$B_2(k, \ell'_1, \ell_n) = u_n(L_*^{-1}(k + \ell'_1) + \ell_n) 2^q D_{n+1}^{-1}(k + \ell'_1 + \ell_n) \Delta^{(n+1)(\cdot)}(k)

$$C(k, \ell_1, \ell'_1) = \left( \frac{\alpha}{L^2} Q^* Q + \Omega_n \right)_{L-1}(k)(L_*^{-1}(\ell_1), L_*^{-1}(\ell'_1)) - \Omega_n(L_*^{-1}(k + \ell_1))^{-1} \delta_{\ell_1, \ell'_1}$$

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(a) Choose \( m_{10} = \min\{2, m_1, \bar{m}(\pi)\} \). Then

\[
\left| \left(\frac{a}{L} Q^* Q + \Omega_n \right)^{-1}_{L^{-1}(k)} (\mathbb{L}^{-1}(\ell_1), \mathbb{L}^{-1}(\ell_1')) \right| \leq \frac{6}{5} a \delta_{\ell_1, \ell_1'} + \frac{2}{a L^2} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q
\]

(by Lemma VI.3.b, Proposition II.4.a)

\[
|u_1(k + \ell_1')|^q \leq \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q
\]

(by Lemma II.2.a)

\[
|\Omega_n(\mathbb{L}^{-1}(k + \ell_1'))| \leq \frac{6}{5} a
\]

(by Proposition II.4.a)

\[
|u_n(\mathbb{L}^{-1}(k + \ell_1') + \ell_n)^{2q}| \leq \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{n,\nu}| + \pi} \right)^{2q}
\]

(by Lemma II.2.a)

\[
|\mathbf{D}^{-1(s)}_{n+1}(k + \ell_1' + \mathbb{L}(\ell_n))| \leq \frac{1}{\gamma_1 \pi} \text{ if } (\ell_1', \ell_n) \neq (0, 0)
\]

(by Lemma III.2.d)

\[
|\Delta^{(n+1)(s)}(k)| \leq 2a
\]

(by Lemma IV.2.c)

\[
|\mathbf{D}^{-1(s)}_{n+1}(k) \Delta^{(n+1)(s)}(k)| \leq \Gamma_6
\]

(by Lemma IV.2.f)

So

\[
\left| \left( A_{\psi_{(\cdot)^r} \theta_{(\cdot)^r}} \right)^{-1}_{L^{-1}(k)} (\mathbb{L}^{-1}(\ell_1)) \right|
\]

\[
\leq \sum_{\ell_1' \in \mathcal{B}_1} \left\{ \frac{6}{5} a \delta_{\ell_1, \ell_1'} + \frac{2}{a L^2} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \right\}
\]

\[
\prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \left\{ \frac{a}{L^2} + \frac{6}{5} a \sum_{\ell_n \in \mathcal{B}_n} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{n,\nu}| + \pi} \right)^q \max \left\{ \Gamma_6, \frac{2a}{\gamma_1 \pi} \right\} \right\}
\]

\[
\leq \text{const} \sum_{\ell_1' \in \mathcal{B}_1} \left\{ \frac{6}{5} a \delta_{\ell_1, \ell_1'} + \frac{2}{a L^2} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \right\}
\]

\[
\prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \leq \Gamma_{19} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q
\]

(b) Using the bounds of part (a) together with

\[
|u_1(k + \ell_1')|^q \leq |k| \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \text{ if } \ell_1' \neq 0
\]

(by Lemma II.2.b)

\[
|u_n(\mathbb{L}^{-1}(k + \ell_1') + \ell_n)^{2q}| \leq |\mathbb{L}^{-1}(k + \ell_1')| \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{n,\nu}| + \pi} \right)^{2q} \text{ if } \ell_n \neq 0
\]

(by Lemma II.2.b)

we have, if \( \ell_1' \neq 0 \),

\[
|B_1(k, \ell_1')| \leq |k| \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q \left\{ \frac{a}{L^2} + \frac{6}{5} a \sum_{\ell_n \in \mathcal{B}_n} \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{n,\nu}| + \pi} \right)^q \max \left\{ \Gamma_6, \frac{2a}{\gamma_1 \pi} \right\} \right\}
\]

\[
\leq \text{const} |k| \prod_{\nu=0}^3 \left( \frac{24}{|\ell_{1,\nu}| + \pi} \right)^q
\]
and
\[ |B_1(k,0)| \leq \prod \frac{3}{\nu_0} \left( \frac{24}{\pi} \right)^q \left\{ \frac{a}{L^2} + \frac{6}{5} a \sum_{\ell_n \in \mathcal{E}_n} \prod \frac{3}{\nu_0} \left( \frac{24}{|\ell_n|+\pi} \right)^{2q} \max \{ \Gamma_6, \frac{2a}{\gamma_1 \pi} \} \right\} \leq \text{const} \]

Using these bounds, the first bound of part (a) and Lemma VI.3.b, and assuming that \( \ell_1 \neq 0 \),
\[
(A_{\psi(\ast)} \theta \ast)_{L,k^{-1}}(\mathbb{L}^{-1}(\ell_1)) = \sum_{\ell'_1 \in \mathcal{B}_1} \left( \frac{a}{L} Q^*Q + \Omega_n \right)^{-1}_{L^{-1}(k)}(\mathbb{L}^{-1}(\ell_1), \mathbb{L}^{-1}(\ell'_1)) B_1(k,\ell'_1)
\]
\[
= \left( \frac{a}{L} Q^*Q + \Omega_n \right)^{-1}_{L^{-1}(k)}(\mathbb{L}^{-1}(\ell_1), 0) B_1(k,0) + O\left( |k| \prod_{\nu=0} \left( \frac{24}{|\ell_{1,\nu}|+\pi} \right)^q \right)
\]
\[
= \frac{2}{aL^2} \prod_{\nu=0} \left( \frac{24}{|\ell_{1,\nu}|+\pi} \right)^q \prod_{\nu=0} \left( \frac{24}{\pi} \right)^q \prod_{0 \leq \nu \leq 3} \prod_{\ell_{1,\nu} \neq 0} |k_\nu| B_1(k,0) + O\left( |k| \prod_{\nu=0} \left( \frac{24}{|\ell_{1,\nu}|+\pi} \right)^q \right)
\]
\[
= O\left( |k| \prod_{\nu=0} \left( \frac{24}{|\ell_{1,\nu}|+\pi} \right)^q \right)
\]

Proof of Proposition VI.1:

**Bound on \( \|\mathbb{L}_*^{-1} A_{\psi,\phi} \mathbb{L}_*\|_{m=1} \):** By Lemmas A.14.b and VI.3.b, if |Im\( k_{\nu'} \)| \leq 2 for each 0 \leq \nu' \leq 3 then
\[
\left| \left[ \mathbb{L}_*^{-1} \{ (aL^{-2}Q^*Q + \Omega_n)^{-1} - \Omega_n^{-1} \} \right] \mathbb{L}_* \right|_{m=1} \leq \text{const}_q
\]

So, by Lemma A.11.b,
\[
\|\mathbb{L}_*^{-1} \{ (aL^{-2}Q^*Q + \Omega_n)^{-1} - \Omega_n^{-1} \} \mathbb{L}_*\|_{m=1} \leq \text{const}_q
\]

By Proposition II.4.a and Lemmas II.2.a and A.11.b,c,
\[
\|\Omega_n\|_{m=1}, \|\Omega_n^{-1}\|_{m=1}, \|Q_n\|_{m=1} \leq \text{const}_q
\]


**Bound on \( \|\mathbb{L}_*^{-1} A_{\psi,\phi,\nu} \mathbb{L}_*\|_{m=1} \):** By Lemmas A.14.b and VI.3.c, if |Im\( k_{\nu'} \)| \leq 2 for each 0 \leq \nu' \leq 3 then
\[
\left| \left[ \mathbb{L}_*^{-1} (\Omega_n^{-1} Q_{+,-} \mathbb{L}_* \Omega_n^{-1} Q_{+,-}) \right] \mathbb{L}_* \right|_{k}(\ell, \ell') = \left| \left[ \Omega_n^{-1} Q_{+,-} \mathbb{L}_* \Omega_n^{-1} Q_{+,-} \right] \mathbb{L}_* \right|_{k}(\ell, \ell')
\]
\[
\leq \frac{3^4}{2L^4} \left( \frac{24}{\pi} \right)^3 \prod_{\nu=0} \left( \frac{24}{|\ell_{\nu}|+\pi} \right)^q \prod_{\nu=0} \left( \frac{24}{|\ell_{\nu}|+\pi} \right)^q
\]

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As \( q > 2 \), Lemma A.11.b yields

\[
\left\| \mathbb{L}^{-1}_n \mathcal{O}_n^{-1} Q^{(+)\nu} \mathcal{O}_{n+1} Q^{(-)\nu}_n \right\|_{m=1} \leq \text{const}_q
\]

By Lemmas II.6.b and A.11.c, \( \| Q^{(-)\nu}_n \|_{m=1} \leq \text{const}_q \) too, since \( q > 2 \). Now just apply Lemma A.15.c.

**Bound on** \( \| \mathbb{L}^{-1}_n A_{\psi(s)\theta(s)} \mathcal{L}_n \|_{m^*} \) : This follows from Lemma VI.6.a by Lemma A.11.c.

**Bound on** \( \| \mathbb{L}^{-1}_n \frac{a}{\ell^2} C(n)(\mu)_q \mathbb{L}_n \|_{m^*} \) : By (VI.3)

\[
\mathbb{L}_n^{-1} \frac{a}{\ell^2} C(n)(\mu)_q \mathbb{L}_n = \mathbb{L}_n^{-1} A_{\psi(s)\theta(s)} \mathbb{L}_n + L^2 \mu \mathbb{L}_n^{-1} A_{\psi(s)\theta(s)} \mathbb{L}_n S_{n+1}(L^2 \mu) Q_{n+1}^{(*)} Q_{n+1} Q_{n+1}
\]

Now just apply Proposition V.1, Lemma II.2 and Proposition II.4.c.

**Bound on** \( \| \mathbb{L}^{-1}_n A_{\psi(s)\theta(s),\nu} \mathbb{L}_n \|_{m^*} \) : It suffices to bound

- \( \mathbb{L}_n^{-1} A_{\psi(s),\nu} \mathbb{L}_n \) as above,
- bound \( \mathbb{L}_n^{-1} \Omega_n^{-1} Q^{(+)\nu} \mathcal{O}_{n+1} \mathcal{L}_n \) using

\[
\left[ \left[ \mathbb{L}_n^{-1} \left( \Omega_n^{-1} Q^{(+)\nu} \mathcal{O}_{n+1} \right) \mathcal{L}_n \right](k)(\ell) \right] = \left[ \left[ \Omega_n^{-1} Q^{(+)\nu} \mathcal{O}_{n+1} \right] \left( \mathcal{L}_n^{-1} \right)(k)(\ell) \right]
\]

\[
\leq \frac{3\pi^2 (24)}{27} \prod_{\nu=0}^{3} \left( \frac{24}{|\nu+1|} \right)^{q-1}
\]

(by Lemma A.15.b, (II.11), Proposition II.4.a and Lemmas II.6.b, II.3.a, VI.3.a) and Lemma A.11.c, and

- bound \( D_n^{-1}(s) Q_{n+1}(k+\ell+1) \Delta^{(n+1)(*)} \) using

\[
\left( \tilde{D}_n^{-1}(s) Q_{n+1}(k+\ell+1) \Delta^{(n+1)(*)} \right)(k)(\ell+1)
\]

\[
\leq \left( \tilde{D}_n^{-1}(s) Q_{n+1}(k+\ell+1) \Delta^{(n+1)(*)} \right)(k)(\ell+1) u_{n+1}(k+\ell+1)^{q-1}
\]

(by (II.10)) and

\[
\left| \Delta^{(n+1)(*)} \right| \leq e^2 (\frac{24}{\pi})^3 \]

(by Lemma II.6.b)

\[
\left| u_{n+1}(k+\ell+1) \right| \leq \prod_{\nu=0}^{3} \left( \frac{24}{|n+1|+\nu} \right)^{q-1}
\]

(by Lemma II.2.a)

\[
\left| \tilde{D}_n^{-1}(s) \right| \leq 1 \text{ if } \ell_{n+1} \neq 0
\]

(by Lemma III.2.d)

\[
\left| \tilde{D}_n^{-1}(s) \right| \leq 2a
\]

(by Lemma IV.2.c)

\[
\left| \tilde{D}_n^{-1}(s) \right| \leq \Gamma_6
\]

(by Lemma IV.2.f)

and Lemma A.11.c.

**Bound on** \( \| \mathbb{L}^{-1}_n A_{\psi(s)\theta(s),\nu}(\mu) \mathcal{L}_n \|_{m^*} \) : This follows from the previous bounds of this Proposition, Remark VI.2.b, Proposition V.1, Lemma II.6.c and Proposition II.4.c.
Appendix A: Bloch Theory for Periodic Operators

In the construction of [parabolic-all.tex] there are a good number of linear operators that act on functions defined on a finite lattice and that are translation invariant with respect to a sub-lattice. For example the lattice $X_{n+1}$ is a sublattice of $X_n$ and the block spin averaging operator $Q : \mathcal{H}_n \to \mathcal{H}_{n+1}$ of (II.1) is translation invariant with respect to $X_{n+1}$. Similarly, $X_n$ is a sublattice of $X_{n+1}$ and the block spin averaging operator $Q_n : \mathcal{H}_n \to \mathcal{H}_0$ of (II.2) is translation invariant with respect to $X_{n+1}$. As another example, the fluctuation integral covariance $C(n) : \mathcal{H}_n \to \mathcal{H}_n$ of §IV is translation invariant with respect to $X_{n+1}$. In this appendix, we use the Bloch theory approach to develop some general machinery for bounding such linear operators. In [parabolic-all.tex] the operators of interest tend to be periodizations of operators acting on $L^2$ of an infinite lattice. An important example is the “differential” operator $D_n$. See Remark III.1.a. We also develop general machinery for bounding such periodizations. We use the results of this appendix in the main body of this paper to bound many of the operators appearing in [parabolic-all.tex].

Periodic Operators in “Position Space” and “Momentum Space” Environments

We start by setting up a general environment consisting of a “fine” lattice and a “coarse” sub-lattice. We shall consider operators that act on functions defined on the former and that are translation invariant with respect to the latter. Let $\varepsilon_T, \varepsilon_X > 0$, $L_T, L_X \in \mathbb{N}$ and $\mathcal{L}_T \in L_T \mathbb{N}$, $\mathcal{L}_X \in L_X \mathbb{N}$ and define the (finite) $(d + 1)$-dimensional lattices

$$X_{\text{fin}} = \left(\varepsilon_T \mathbb{Z}/\varepsilon_T \mathcal{L}_T \mathbb{Z}\right) \times \left(\varepsilon_X \mathbb{Z}^d/\varepsilon_X \mathcal{L}_X \mathbb{Z}^d\right)$$

$$X_{\text{crs}} = \left(L_T \varepsilon_T \mathbb{Z}/\varepsilon_T \mathcal{L}_T \mathbb{Z}\right) \times \left(L_X \varepsilon_X \mathbb{Z}^d/\varepsilon_X \mathcal{L}_X \mathbb{Z}^d\right)$$

and the corresponding Hilbert spaces

$$\mathcal{H}_f = L^2(X_{\text{fin}}) \quad \langle \phi_1, \phi_2 \rangle_f = \text{vol}_f \sum_{u \in X_{\text{fin}}} \phi_1(u)^* \phi_2(u)$$

$$\mathcal{H}_c = L^2(X_{\text{crs}}) \quad \langle \psi_1, \psi_2 \rangle_c = \text{vol}_c \sum_{x \in X_{\text{crs}}} \psi_1(x)^* \psi_2(x)$$

where we use

$$\text{vol}_f = \varepsilon_T \varepsilon_X^d \quad \text{vol}_c = (\varepsilon_T L_T)(\varepsilon_X L_X)^d$$

to denote the volume of a single cell in $X_{\text{fin}}$, and $X_{\text{crs}}$, respectively. For the Bloch construction, it will also be useful to define the “single period” lattice

$$\mathcal{B} = \left(\varepsilon_T \mathbb{Z}/L_T \varepsilon_T \mathbb{Z}\right) \times \left(\varepsilon_X \mathbb{Z}^d/L_X \varepsilon_X \mathbb{Z}^d\right) \cong X_{\text{fin}}/X_{\text{crs}}$$

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The lattices dual to $\mathcal{X}_{\text{fin}}, \mathcal{X}_{\text{crs}}$ and $\hat{\mathcal{B}}$ are

\[
\hat{\mathcal{X}}_{\text{fin}} = \left(\frac{2\pi}{\varepsilon_T L_T} \mathbb{Z} / \frac{2\pi}{\varepsilon_T} \mathbb{Z}\right) \times \left(\frac{2\pi}{\varepsilon_X L_X} \mathbb{Z}^D / \frac{2\pi}{\varepsilon_X} \mathbb{Z}^D\right)
\]
\[
\hat{\mathcal{X}}_{\text{crs}} = \left(\frac{2\pi}{L_T \varepsilon_T} \mathbb{Z} / \frac{2\pi}{L_T \varepsilon_T} \mathbb{Z}\right) \times \left(\frac{2\pi}{\varepsilon_X L_X} \mathbb{Z}^D / \frac{2\pi}{\varepsilon_X} \mathbb{Z}^D\right)
\]
\[
\hat{\mathcal{B}} = \left(\frac{2\pi}{L_T \varepsilon_T} \mathbb{Z} / \frac{2\pi}{L_T \varepsilon_T} \mathbb{Z}\right) \times \left(\frac{2\pi}{L_X \varepsilon_X} \mathbb{Z}^D / \frac{2\pi}{L_X \varepsilon_X} \mathbb{Z}^D\right) \cong \hat{\mathcal{X}}_{\text{crs}} / \hat{\mathcal{X}}_{\text{fin}}
\]

We denote by

\[\hat{\pi} : \hat{\mathcal{X}}_{\text{fin}} \to \hat{\mathcal{X}}_{\text{crs}}\]

the canonical projection from $\hat{\mathcal{X}}_{\text{fin}}$ to $\hat{\mathcal{X}}_{\text{crs}}$. It has kernel $\hat{\mathcal{B}}$. Observe that

\[p \cdot x = \hat{\pi}(p) \cdot x \mod 2\pi \quad \text{for all } x \in \mathcal{X}_{\text{crs}}, \ p \in \hat{\mathcal{X}}_{\text{fin}}\]

The Fourier and inverse Fourier transforms are, for $\phi \in \mathcal{H}_f, \ \psi \in \mathcal{H}_c, \ \zeta \in L^2(\mathcal{B}), \ p \in \hat{\mathcal{X}}_{\text{fin}}, \ k \in \hat{\mathcal{X}}_{\text{crs}}, \ \ell \in \hat{\mathcal{B}}, \ u \in \mathcal{X}_{\text{fin}}, \ x \in \mathcal{X}_{\text{crs}}$ and $w \in \mathcal{B}$,

\[
\hat{\phi}(p) = \text{vol}_f \sum_{u \in \mathcal{X}_{\text{fin}}} \phi(u)e^{-ip \cdot u} \quad \phi(u) = \frac{\text{vol}_f}{(2\pi)^{1+D}} \sum_{p \in \hat{\mathcal{X}}_{\text{fin}}} \hat{\phi}(p)e^{iu \cdot p}
\]
\[
\hat{\psi}(k) = \text{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}}} \psi(x)e^{-ik \cdot x} \quad \psi(x) = \frac{\text{vol}_c}{(2\pi)^{1+D}} \sum_{k \in \hat{\mathcal{X}}_{\text{crs}}} \hat{\psi}(k)e^{ik \cdot x}
\]
\[
\hat{\zeta}(\ell) = \text{vol}_f \sum_{w \in \mathcal{B}} \zeta(w)e^{-i\ell \cdot w} \quad \zeta(w) = \frac{\text{vol}_b}{(2\pi)^{1+D}} \sum_{\ell \in \hat{\mathcal{B}}} \hat{\zeta}(\ell)e^{iw \cdot \ell}
\]

where

\[
\text{vol}_f = \frac{(2\pi)^{1+D}}{(\varepsilon_T L_T)(\varepsilon_X L_X)^D}, \quad \text{vol}_c = \frac{(2\pi)^{1+D}}{(\varepsilon_T L_T)(\varepsilon_X L_X)^D}, \quad \text{vol}_b = \frac{(2\pi)^{1+D}}{(\varepsilon_T L_T)(\varepsilon_X L_X)^D}
\]

denote the volume of a single cell in $\hat{\mathcal{X}}_{\text{fin}}, \hat{\mathcal{X}}_{\text{crs}}$ and $\hat{\mathcal{B}}$, respectively. Observe that

\[
\frac{\text{vol}_f \text{vol}_c}{(2\pi)^{1+D}} = \frac{L_T L_X^D}{(\varepsilon_T L_T)(\varepsilon_X L_X)^D} = \frac{1}{|\mathcal{X}_{\text{fin}}|} = \frac{1}{|\mathcal{X}_{\text{crs}}|} \quad \text{(A.1)}
\]

where $|\mathcal{X}_{\text{crs}}|$ denotes the number of points in $\mathcal{X}_{\text{crs}}$. By (A.1) and the fact that $\delta_{u,u'} = \frac{1}{|\mathcal{X}_{\text{fin}}|} \sum_{p \in \hat{\mathcal{X}}_{\text{fin}}} e^{ip \cdot u} e^{-ip \cdot u'}$,

\[
\langle \phi_1, \phi_2 \rangle_f = \text{vol}_f \sum_{u \in \mathcal{X}_{\text{fin}}} \phi_1(u)\phi_2(u) = \frac{\text{vol}_f}{(2\pi)^{1+D}} \sum_{p \in \hat{\mathcal{X}}_{\text{fin}}} \hat{\phi}_1(-p)\hat{\phi}_2(p)
\]
\[
\langle \psi_1, \psi_2 \rangle_c = \text{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}}} \psi_1(x)\psi_2(x) = \frac{\text{vol}_c}{(2\pi)^{1+D}} \sum_{k \in \hat{\mathcal{X}}_{\text{crs}}} \hat{\psi}_1(-k)\hat{\psi}_2(k)
\]
Let $A$ be any operator on $\mathcal{H}_f$ that is translation invariant with respect to $\mathcal{X}_{crs}$. We call such an operator a “periodic operator”. Denote by $A(u, u')$ its kernel, defined so that

$$(A\phi)(u) = \text{vol}_{\hat{f}} \sum_{u' \in \mathcal{X}_{\text{fin}}} A(u, u')\phi(u')$$

By “translation invariant with respect to $\mathcal{X}_{crs}$”, we mean that $A(u + x, u' + x) = A(u, u')$ for all $u, u' \in \mathcal{X}_{\text{fin}}$ and $x \in \mathcal{X}_{crs}$. Set

$$(A) \quad \text{for each fixed } k \quad \text{and, for } u, u' \in \mathcal{X}_{\text{fin}}, \quad \hat{A}(p, p') = \frac{\text{vol}_{\hat{f}}}{|\mathcal{X}_{\text{fin}}|} \sum_{u, u' \in \mathcal{X}_{\text{fin}}} e^{-i p \cdot u} A(u, u') e^{i p' \cdot u'}$$

(A.2) and, for $u, u' \in \mathcal{X}_{\text{fin}}$ and $k \in \frac{2\pi}{\varepsilon_T \varepsilon_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X \varepsilon_X} \mathbb{Z}^D$, the “universal cover” of $\hat{\mathcal{X}}_{crs}$,

$$(A) \quad A_k(u, u') = \text{vol}_{\hat{f}} \sum_{u'' \in \mathcal{X}_{\text{fin}}} e^{-i k \cdot u} A(u, u'') e^{i k \cdot u''}$$

(A.3) For each fixed $u, u' \in \mathcal{X}_{\text{fin}}$, $k \mapsto A_k(u, u')$ is not a function on the torus $\hat{\mathcal{X}}_{crs}$ since, for $p \in \frac{2\pi}{L_T \varepsilon_T} \mathbb{Z} \times \frac{2\pi}{L_X \varepsilon_X} \mathbb{Z}^D$,

$$A_{k+p}(u, u') = e^{-i p \cdot (u - u')} A_k(u, u')$$

This is why we defined $A_k$ for $k \in \frac{2\pi}{\varepsilon_T \varepsilon_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X \varepsilon_X} \mathbb{Z}^D$, rather than $k \in \hat{\mathcal{X}}_{crs}$. On the other hand, $k \mapsto e^{i k (u - u')} A_k(u, u')$ is a well–defined function on $\hat{\mathcal{X}}_{crs}$.

The following lemma is standard.

**Lemma A.1** Let $A$ be an operator on $\mathcal{H}_f$ that is translation invariant with respect to $\mathcal{X}_{crs}$.

(a) $A(u, u') = \frac{\text{vol}_{\hat{f}}}{(2\pi)^{1+D}} \sum_{p, p' \in \mathcal{X}_{\text{fin}}} e^{i p \cdot u} \hat{A}(p, p') e^{-i p' \cdot u'}$

(b) $A(u, u') = \frac{\text{vol}_{\hat{f}}}{(2\pi)^{1+D}} \sum_{[k] \in \mathcal{X}_{crs}} \sum_{\ell \in \hat{B}} e^{i \ell \cdot u} \hat{A}(k + \ell, k + \ell') e^{-i \ell' \cdot u'} e^{i k \cdot (u - u')}$

Here

$$\sum_{[k] \in \hat{\mathcal{X}}_{crs}} f(k) \text{ means that one sums } k \text{ over a subset of } \frac{2\pi}{\varepsilon_T \varepsilon_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X \varepsilon_X} \mathbb{Z}^D \text{ that contains exactly one (arbitrary) representative from each equivalence class of } \hat{\mathcal{X}}_{crs}. \text{ Note that if } k \in \frac{2\pi}{\varepsilon_T \varepsilon_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X \varepsilon_X} \mathbb{Z}^D \text{ and } \ell \in \hat{B}, \text{ then } k + \ell \in \hat{\mathcal{X}}_{\text{fin}}.$$

(1) The “normal prefactor” for $\hat{A}$ would be $\text{vol}_f^2$. We have chosen $\frac{\text{vol}_{\hat{f}}}{|\mathcal{X}_{\text{fin}}|} = \frac{\text{vol}_{\hat{f}}}{(2\pi)^{1+D}} \text{ vol}_f^2$ so as to replace approximate Dirac $(2\pi)^{1+D}\delta(p - p')$'s with simple Kronecker $\delta_{p,p'}$'s in the translation invariant case.
(c) \((\hat{A}\phi)(p) = \sum_{p' \in \hat{X}_{\text{fin}}} \hat{A}(p, p') \hat{\phi}(p')\) for all \(\phi \in H_f\).

(d) For each \(k \in \frac{2\pi}{\varepsilon_T \xi_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X \xi_X} \mathbb{Z}^D\), \(A_k(u, u')\) is periodic with respect to \(X_{\text{crs}}\) in both \(u\) and \(u'\) and
\[
A(u, u') = \frac{\text{vol}_c}{(2\pi)^{1+D}} \sum_{k \in \hat{X}_{\text{crs}}} e^{ik \cdot u} A_k(u, u') e^{-ik \cdot u'}
\]

(e) \(A_k(u, u') = \sum_{\ell, \ell' \in \hat{B}} e^{i\ell \cdot u} \hat{A}(k + \ell, k + \ell') e^{-i\ell' \cdot u'}\)

(f) Define the transpose of \(A\) by \(A^*(u, u') = A(u', u)\). Then
\[
A^*(u, u') = \frac{\text{vol}_c}{(2\pi)^{1+D}} \sum_{[k] \in \hat{X}_{\text{crs}}} \sum_{\ell, \ell' \in \hat{B}} e^{i\ell \cdot u} \hat{A}(-k - \ell', -k - \ell) e^{-i\ell' \cdot u'} e^{ik \cdot (u - u')}
\]

**Periodized Operators**

Define the (infinite) lattices
\[
\mathcal{Z}_{\text{fin}} = \varepsilon_T \mathbb{Z} \times \varepsilon_X \mathbb{Z}^D \quad \mathcal{Z}_{\text{crs}} = \xi_T \mathbb{Z} \times \xi_X \mathbb{Z}^D
\]

**Definition A.2 (Periodization)** Suppose that \(a(u, u')\) is a function on \(\mathcal{Z}_{\text{fin}} \times \mathcal{Z}_{\text{fin}}\) that
- is translation invariant with respect to \(\mathcal{Z}_{\text{crs}}\) in the sense that \(a(u + x, u' + x) = a(u, u')\) for all \(x \in \mathcal{Z}_{\text{crs}}\) and \(u, u' \in \mathcal{Z}_{\text{fin}}\) and
- has finite \(L^1\text{-}L^\infty\) norm (i.e. \(\sup_{u \in \mathcal{Z}_{\text{fin}}} \sum_{u' \in \mathcal{Z}_{\text{fin}}} |a(u, u')|\) and \(\sup_{u' \in \mathcal{Z}_{\text{fin}}} \sum_{u \in \mathcal{Z}_{\text{fin}}} |a(u, u')|\) are both finite)

and that the operator \(A\) (on \(H_f\)) acts by
\[
(A\phi)([u]) = \text{vol}_f \sum_{u' \in \mathcal{Z}_{\text{fin}}} a(u, u') \phi([u']) \quad (A.4)
\]

Here, for each \(u \in \mathcal{Z}_{\text{fin}}\), the notation \([u]\) means the equivalence class in \(X_{\text{fin}}\) that contains \(u\). Then we say that \(A\) is the periodization of \(a\). It is “\(a\) with periodic boundary conditions on a box of size \(\varepsilon_T \mathcal{L}_T \times \varepsilon_X \mathcal{L}_X \times \cdots \times \varepsilon_X \mathcal{L}_X\) ”.
Remark A.3

(a) The right hand side of (A.4) is independent of the representative $u$ chosen from $[u]$ (by translation invariance with respect to $\varepsilon T L T Z Z \times \varepsilon X L X Z Z \subset Z_{crs}$).

(b) The kernel of $A$ is given by

$$A([u], [u']) = \sum_{u'' \in Z_{fin}} a(u, u'') = \sum_{z \in \varepsilon T L T Z Z \times \varepsilon X L X Z Z} a(u, u' + z)$$

The sum converges because $a$ has finite $L^1 - L^\infty$ norm. This is the motivation for the name the “periodization of $a$”.

(c) If $A$ is the periodization of $a$ and $B$ is the periodization of $b$, then $C = AB$ is the periodization of

$$c(u, u') = \text{vol}_f \sum_{u'' \in Z_{fin}} a(u, u'')b(u'', u')$$

Let $\hat{Z}_{crs} = (\mathbb{R} / 2\pi T L) \times (\mathbb{R}^D / 2\pi X L X Z Z D)$ be the dual space of $Z_{crs}$. Its universal cover is $\mathbb{R} \times \mathbb{R}^D$. For each $k \in \mathbb{R} \times \mathbb{R}^D$, set, for $u, u' \in Z_{fin}$,

$$a_k(u, u') = \text{vol}_c \sum_{\substack{u'' \in Z_{fin} \\ u'' - u' \in Z_{crs}}} e^{-ik \cdot u''} a(u, u'') e^{ik \cdot u''} \tag{A.5}$$

and, for $\ell, \ell' \in \hat{B}$,

$$\hat{a}_k(\ell, \ell') = \frac{1}{|B|^2} \sum_{[u], [u'] \in \mathcal{B}} e^{-i\ell \cdot u} a_k(u, u') e^{i\ell' \cdot u'}$$

$$= \frac{\text{vol}_c}{|\mathcal{B}|} \sum_{\substack{u'' \in \mathbb{Z}_{fin} \\ u' \in \mathbb{Z}_{fin}}} e^{-i\ell \cdot u''} a(u, u') e^{i\ell' \cdot u'} e^{-ik \cdot (u - u')} \tag{A.6}$$

(Recall that $\frac{1}{|B|^2} = \frac{\text{vol}_c}{|\mathcal{B}|}$. We shall show in Lemma A.5.a, below, that $a_k(u, u')$ is periodic with respect to $Z_{crs}$ in both $u$ and $u'$. ) By the $L^1 - L^\infty$ hypothesis and the Lebesgue dominated convergence theorem, both $a_k(u, u')$ and $a_k(\ell, \ell')$ are continuous in $k$.

Remark A.4  As was the case for $A_k(u, u')$, for each fixed $u, u' \in Z_{fin}$, the map $k \mapsto a_k(u, u')$ is not a function on the torus $\hat{Z}_{crs}$ since, for $p \in \frac{2\pi}{\varepsilon T L T} Z \times \frac{2\pi}{\varepsilon X L X} Z^D$,

$$a_{k+p}(u, u') = e^{-ip \cdot (u - u')} a_k(u, u')$$
However
\[ k \in \hat{\mathbb{Z}}_{crs} \mapsto e^{ik \cdot (u-u')} a_k(u, u') = \text{vol}_{c} \sum_{x \in \mathbb{Z}_{crs}} a(u, u' + x)e^{ik \cdot x} \]
is a legitimate function on the torus \( \hat{\mathbb{Z}}_{crs} \) and is in fact the Fourier transform of the function
\[ x \in \mathbb{Z}_{crs} \mapsto a(u, u' + x) \]

Correspondingly, for \( p \in \frac{2\pi}{\varepsilon} L_T T \frac{2\pi}{\varepsilon} X \mathbb{Z}_{crs} \) and \( \ell, \ell' \in \hat{\mathcal{B}} \)
\[ \hat{a}_{k+p}(\ell, \ell') = \hat{a}_k(\ell + p, \ell' + p) \]

The following two lemmas are again standard.

**Lemma A.5** Let \( a(u, u') : \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}} \to \mathbb{C} \) obey the conditions of Definition A.2, and, in particular, be translation invariant with respect to \( \mathbb{Z}_{crs} \).

(a) For each \( k \in \mathbb{R} \times \mathbb{R}^D \), \( a_k(u, u') \) is periodic with respect to \( \mathbb{Z}_{crs} \) in both \( u \) and \( u' \) and
\[ a(u, u') = \int_{\mathbb{Z}_{crs}} a_k(u, u')e^{ik \cdot (u-u')} \frac{d^{1+D}k}{(2\pi)^{1+D}} = \sum_{\ell, \ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathbb{Z}}_{crs}} e^{i\ell \cdot u} \hat{a}_k(\ell, \ell')e^{-i\ell' \cdot u}e^{ik \cdot (u-u')} \frac{d^{1+D}k}{(2\pi)^{1+D}} \]

(b) If, in addition, \( a(u, u') = \alpha(u - u') \) is translation invariant with respect to \( \mathbb{Z}_{\text{fin}} \), then
\[ \hat{a}_k(\ell, \ell') = \delta_{\ell', \ell} \hat{\alpha}(k + \ell) \]

where \( \hat{\alpha}(p) = \text{vol}_{f} \sum_{u \in \mathbb{Z}_{\text{fin}}} \alpha(u)e^{-ip \cdot u} \).

(c) Let \( A \) be the periodization of \( a \). Then
\[ A_k([u], [u']) = a_k(u, u') \]
\[ \hat{A}(k + \ell, k + \ell') = \hat{a}_k(\ell, \ell') \]

for all \( k \in \frac{2\pi}{\varepsilon} L_T T \frac{2\pi}{\varepsilon} X \mathbb{Z}_{crs}, [u], [u'] \in \mathcal{X}_{\text{fin}} \) and \( \ell, \ell' \in \hat{\mathcal{B}} \).

**Lemma A.6**

(a) If \( a(u, u') = \frac{1}{\text{vol}_{f}} \delta_{u', u} \) is the kernel of the identity operator, then \( \hat{a}_k(\ell, \ell') = \delta_{\ell, \ell'} \).
(b) Let \( a(u, u') : \mathcal{Z}_{\text{fin}} \times \mathcal{Z}_{\text{fin}} \rightarrow \mathbb{C} \) both obey the conditions of Definition A.2, and set
\[
c(u, u') = \text{vol}_f \sum_{u'' \in \mathcal{Z}_{\text{fin}}} a(u, u'') b(u'', u')
\]
Then, for all \( k \in \frac{2\pi}{\varepsilon_T L_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X L_X} \mathbb{Z}^D \) and \( \ell, \ell' \in \hat{B} \),
\[
c_k(\ell, \ell') = \sum_{\ell'' \in \hat{B}} a_k(\ell, \ell'') b_k(\ell'', \ell')
\]

We now generalize the above discussion to include periodized operators from \( L^2(\mathcal{X}_{\text{crs}}) \) to \( L^2(\mathcal{X}_{\text{fin}}) \) and vice versa. If \( b(u, x) \) and \( c(x, u) \) are translation invariant with respect to \( \mathcal{X}_{\text{crs}} \) (with \( x \) running over \( \mathcal{X}_{\text{crs}} \) and with \( u \) running over \( \mathcal{Z}_{\text{fin}} \) as usual) and have finite \( L^1-L^\infty \) norm, we define, for \( k \in \mathbb{R} \times \mathbb{R}^D \) and \( \ell, \ell' \in \hat{B} \),
\[
\hat{b}_k(\ell) = \text{vol}_f \sum_{[u] \in \hat{B}} e^{-i(k+\ell) \cdot u} b(u, x) e^{i k \cdot x} \quad \hat{c}_k(\ell') = \text{vol}_f \sum_{[u] \in \hat{B}} e^{-i k \cdot x} c(x, u) e^{i (k+\ell') \cdot u}
\] (A.7)

For \( p \in \frac{2\pi}{\varepsilon_T L_T} \mathbb{Z} \times \frac{2\pi}{\varepsilon_X L_X} \mathbb{Z}^D \) and \( \ell, \ell' \in \hat{B} \)
\[
\hat{b}_{k+p}(\ell) = \hat{b}_k(\ell + p) \quad \hat{c}_{k+p}(\ell') = \hat{c}_k(\ell' + p)
\]
The inverse transforms are
\[
b(u, x) = \sum_{\ell \in \hat{B}} \int_{\hat{\mathcal{X}}_{\text{crs}}} e^{i \ell \cdot u} \hat{b}_k(\ell) e^{i k \cdot (u-x)} \frac{d^{1+D}k}{(2\pi)^{1+D}}
\]
\[
c(x, u) = \sum_{\ell' \in \hat{B}} \int_{\hat{\mathcal{X}}_{\text{crs}}} \hat{a}_k(\ell, \ell') e^{-i \ell' \cdot u} e^{i k \cdot (x-u)} \frac{d^{1+D}k}{(2\pi)^{1+D}}
\]
For \( \psi \in L^2(\mathcal{X}_{\text{crs}}) \) and \( \phi \in L^2(\mathcal{X}_{\text{fin}}) \)
\[
\hat{b}_k(\ell) \hat{\psi}(k) = \hat{b}_k(\ell) \hat{\psi}(k) \quad \hat{c}_k(\ell') = \sum_{\ell'' \in \hat{B}} \hat{\phi}(k) \hat{\phi}(\ell' + \ell')
\] (A.8)

If \( b^*(x, u) = b(u, x) \) and \( c^*(u, x) = c(x, u) \) are the transposes of \( b \) and \( c \), respectively, then
\[
\hat{b}_k^*(\ell') = \hat{b}_{-k}(-\ell') \quad \hat{c}_k^*(\ell) = \hat{c}_{-k}(-\ell)
\] (A.9)

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Averaging Operators

In this subsection, we analyze “averaging operators” as examples of periodic operators. Fix a function \( q : \mathcal{X}_{\text{fin}} \to \mathbb{R} \) and define the “averaging operator” \( Q : \mathcal{H}_f \to \mathcal{H}_c \) by

\[
(Q\phi)(x) = \operatorname{vol}_f \sum_{u \in \mathcal{X}_{\text{fin}}} q(x - u)\phi(u) \quad (A.10)
\]

Lemma A.7

(a) The adjoint \( Q^\ast \) is given by

\[
(Q^\ast \psi)(u) = \operatorname{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}}} \psi(x)q(x - u)
\]

(b) The composite operators \( QQ^\ast \) and \( Q^\ast Q \) are given by

\[
(QQ^\ast \psi)(x) = \operatorname{vol}_f \operatorname{vol}_c \sum_{u \in \mathcal{X}_{\text{fin}}} \sum_{x' \in \mathcal{X}_{\text{crs}}} q(x - u)q(x' - u)\psi(x')
\]

\[
(Q^\ast Q\phi)(u) = \operatorname{vol}_f \operatorname{vol}_c \sum_{u' \in \mathcal{X}_{\text{fin}}} \sum_{x \in \mathcal{X}_{\text{crs}}} q(x - u)q(x - u')\phi(u')
\]

Proof: trivial.

Example A.8 Assume that \( L_T \) and \( L_X \) are odd and choose \( q \) to be \( \frac{1}{\operatorname{vol}_c} \) times the characteristic function of the rectangle \( \varepsilon_T \left[ -\frac{L_T - 1}{2}, \frac{L_T - 1}{2} \right] \times \varepsilon_X \left[ -\frac{L_X - 1}{2}, \frac{L_X - 1}{2} \right] \) in \( \mathcal{X}_{\text{fin}} \). Observe that the number of points in this rectangle is exactly \( L_T L_X^D \). For \( x \in \mathcal{X}_{\text{crs}} \), denote by \( \boxed{x} \) the rectangle \( x + \varepsilon_T \left[ -\frac{L_T - 1}{2}, \frac{L_T - 1}{2} \right] \times \varepsilon_X \left[ -\frac{L_X - 1}{2}, \frac{L_X - 1}{2} \right] \) in \( \mathcal{X}_{\text{fin}} \). Also, for \( u \in \mathcal{X}_{\text{fin}} \), let \( \xi(u) \) be the point of \( \mathcal{X}_{\text{crs}} \) closest to \( u \). Then

\[
(Q\phi)(x) = \frac{1}{L_T L_X^D} \sum_{u \in \boxed{x}} \phi(u) \quad (Q^\ast \psi)(u) = \psi(\xi(u))
\]

The composite operators are

\[
(QQ^\ast \psi)(x) = \frac{1}{L_T L_X^D} \sum_{u \in \boxed{x}} (Q^\ast \psi)(u) = \frac{1}{L_T L_X^D} \sum_{u \in \boxed{x}} \psi(x) = \psi(x)
\]

\[
(Q^\ast Q\phi)(u) = (Q\phi)(\xi(u)) = \frac{1}{L_T L_X^D} \sum_{u' \in \xi(u)} \phi(u')
\]
Lemma A.9  Let \( Q : \mathcal{H}_f \to \mathcal{H}_c \) be the averaging operator of (A.10), but with \( q : \mathcal{Z}_{\text{fin}} \to \mathbb{R} \) and \( q(u) \) vanishing unless \( |u_0| < \frac{1}{2} \varepsilon_T \mathcal{L}_T \) and \( |u_\nu| < \frac{1}{2} \varepsilon_X \mathcal{L}_X \) for \( \nu = 1, 2, 3 \).

(a) For all \( \phi \in \mathcal{H}_f \) and \( \psi \in \mathcal{H}_c \),

\[
\hat{(Q\phi)}(k) = \sum_{p \in X_{\text{fin}} \atop \#(p) = k} \hat{q}(p) \hat{\phi}(p) \quad \quad \hat{(Q^*\psi)}(p) = \overline{q(p)} \hat{\psi}(\hat{\pi}(p))
\]

\[
(QQ^*\psi)(k) = \left( \sum_{p \in X_{\text{fin}} \atop \#(p) = k} |\hat{q}(p)|^2 \right) \hat{\psi}(k) \quad \quad (Q^*Q\phi)(p) = \overline{q(p)} \sum_{p' \in X_{\text{fin}} \atop \#(p') = \#(p)} \hat{q}(p') \hat{\phi}(p')
\]

(b) For \( A = Q^*Q \),

\[
\hat{a}_k(\ell, \ell') = \overline{q(k + \ell)} q(k + \ell')
\]

Proof:  (a) Using the definitions and (A.1),

\[
\hat{(Q\phi)}(k) = \text{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}}} (Q\phi)(x)e^{-ik \cdot x} = \text{vol}_f \text{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}} \atop u \in \mathcal{X}_{\text{fin}}} e^{-ik \cdot x} q(x - u) \phi(u)
\]

\[
= \frac{\text{vol}_f}{|\mathcal{X}_{\text{crs}}|} \sum_{x \in \mathcal{X}_{\text{crs}} \atop u \in \mathcal{X}_{\text{fin}} \atop p \in X_{\text{fin}}} e^{-ik \cdot x} e^{iu \cdot p} q(x - u) \hat{\phi}(p) = \frac{\text{vol}_f}{|\mathcal{X}_{\text{crs}}|} \sum_{x \in \mathcal{X}_{\text{crs}} \atop u \in \mathcal{X}_{\text{fin}} \atop p \in X_{\text{fin}}} e^{-ik \cdot x} e^{i(x - u) \cdot p} q(u) \hat{\phi}(p)
\]

\[
= \frac{1}{|\mathcal{X}_{\text{crs}}|} \sum_{x \in \mathcal{X}_{\text{crs}} \atop p \in X_{\text{fin}}} e^{-i(k - \hat{\pi}(p)) \cdot x} \hat{q}(p) \hat{\phi}(p) = \frac{1}{|\mathcal{X}_{\text{crs}}|} \sum_{x \in \mathcal{X}_{\text{crs}} \atop p \in X_{\text{fin}}} e^{-i(k - \hat{\pi}(p)) \cdot x} \hat{q}(p) \hat{\phi}(p)
\]

The computation for \( \hat{(Q^*\psi)}(p) \) is similar. For the composite operators

\[
(QQ^*\psi)(k) = \sum_{p \in X_{\text{fin}} \atop \#(p) = k} \hat{q}(p) (QQ^*\psi)(p) = \sum_{p \in X_{\text{fin}} \atop \#(p) = k} \hat{q}(p) \overline{q(p)} \hat{\psi}(\hat{\pi}(p)) = \sum_{p \in X_{\text{fin}} \atop \#(p) = k} |\hat{q}(p)|^2 \hat{\psi}(k)
\]

and similarly for \( \hat{(Q^*Q\phi)}(p) \).

(b) Since

\[
a(u, u') = \text{vol}_c \sum_{x \in \mathcal{Z}_{\text{crs}}} q(x - u)q(x - u')
\]
we have
\[ \hat{a}_k(\ell, \ell') = \frac{\text{vol}_f \text{vol}_c}{|B|} \sum_{[u] \in B} e^{i(k+\ell) \cdot (u - x)} q(x - u) q(x - u') e^{i(k+\ell') \cdot (u' - x)} \]
\[ = \text{vol}^2_f \sum_{[u] \in B} e^{-i(k+\ell) \cdot (u - x)} q(x - u) q(u') e^{-i(k+\ell') \cdot u'} \]
\[ = \text{vol}^2_f \sum_{u, u' \in \mathbb{Z}_{\text{fin}}} e^{i(k+\ell) \cdot u} q(u) q(u') e^{-i(k+\ell') \cdot u'} = q(k + \ell) q(k + \ell') \]

**Example A.8 (continued)** In the notation of Example A.8, the Fourier transform of \( q \) is
\[ \hat{q}(p) = \frac{\text{vol}_f}{\text{vol}_c} \sum_{u \in \mathbb{X}_{\text{fin}}, u \in [0]} e^{-ip \cdot u} = u_L T (\varepsilon_T p_0) \prod_{\ell=1}^D u_{L_x} (\varepsilon_{Xp_\ell}) \]
where
\[ u_L(\omega) = \frac{1}{L} \sum_{k=-L-1}^{L-1} e^{-i\omega k} = \begin{cases} \frac{1}{L} \sin \frac{L\omega}{2} \sin \frac{\omega}{2} & \text{if } \omega \notin 2\pi \mathbb{Z} \\ 1 & \text{otherwise} \end{cases} \]

![Graph of \( y = u_L(\omega) \)]

**Remark A.10** For the \( q \) of Example A.8, which is, up to a multiplicative constant, the characteristic function of a rectangle, the Fourier transform \( \hat{q}(p) \) decays relatively slowly for large \( p \). Choosing a smoother \( q \) increases the rate of decay of \( \hat{q}(p) \). A convenient way to “smooth off” \( Q \) is to select an even \( q \in \mathbb{N} \) and choose \( q \) to be the inverse Fourier transform of
\[ \hat{q}(p) = u_{LT} (\varepsilon_T p_0)^q \prod_{\ell=1}^D u_{Lx} (\varepsilon_{Xp_\ell})^q \]
For example, when \( q = 2 \), \( q \) is the convolution of (a constant times) the characteristic function of a rectangle with itself and so is a “tent” function. In the main text, we use \( q \geq 4 \).
Analyticity of the Fourier Transform and $L^1-L^\infty$ Norms

Define, for any $m \geq 0$ and $a: \mathbb{X} \times \mathbb{X}' \to \mathbb{C}$, with $\mathbb{X}$ and $\mathbb{X}'$ being any of our lattices,

$$\|a\|_m = \max \left\{ \sup_{y \in \mathbb{X}} \text{vol}_X \sum_{y' \in \mathbb{X}'} e^{m|y-y'|} |a(y, y')|, \sup_{y' \in \mathbb{X}'} \text{vol}_X \sum_{y \in \mathbb{X}} e^{m|y-y'|} |a(y, y')| \right\}$$

Here $\text{vol}_X$ and $\text{vol}_X'$ is the volume of a single cell in $X$ and $X'$, respectively.

**Lemma A.11** Let $a(u, u') : \mathcal{Z}_{\text{fin}} \times \mathcal{Z}_{\text{fin}} \to \mathbb{C}$, $b(u, x) : \mathcal{Z}_{\text{fin}} \times \mathcal{Z}_{\text{crs}} \to \mathbb{C}$ and $c(x, u) : \mathcal{Z}_{\text{crs}} \times \mathcal{Z}_{\text{fin}} \to \mathbb{C}$ be translation invariant with respect to $\mathcal{Z}_{\text{crs}}$ and have finite $L^1-L^\infty$ norms. Let $0 < m'' < m' < m$.

\(a\) If $\|a\|_m < \infty$, then, for each $\ell, \ell' \in \hat{\mathcal{B}}$, $\hat{a}_k(\ell, \ell')$ is analytic in $|\text{Im} k| < m$ and

$$\sup_{|\text{Im} k| < m} |\hat{a}_k(\ell, \ell')| \leq \|a\|_m$$

\(b\) If, for each $\ell, \ell' \in \hat{\mathcal{B}}$, $\hat{a}_k(\ell, \ell')$ is analytic in $|\text{Im} k| < m$, then,

$$\|A\|_{m''} \leq \|a\|_{m''} \leq \frac{C_{m'-m''}}{|\text{vol}_f|} \sum_{\ell, \ell' \in \hat{\mathcal{B}}} |\hat{a}_k(\ell, \ell')| \leq \frac{C_{m'-m''}}{|\text{vol}_f|} \sup_{|\text{Im} k| = m'} |\hat{a}_k(\ell, \ell')|$$

where $A$ is the periodization of $a$ and $C_{m'-m''} = |\text{vol}_f| \sum_{u \in \mathcal{Z}_{\text{fin}}} e^{-(m'-m'')|u|}$.

\(c\) If, for each $\ell \in \hat{\mathcal{B}}$, $\hat{b}_k(\ell)$ is analytic in $|\text{Im} k| < m$, then,

$$\sup_{u \in \mathcal{Z}_{\text{fin}}, x \in \mathcal{Z}_{\text{crs}}} |b(u, x)| e^{m'|u-x|} \leq \frac{1}{|\text{vol}_c|} \sup_{|\text{Im} k| = m'} |\hat{b}_k(\ell)| \leq \frac{1}{|\text{vol}_c|} \sup_{|\text{Im} k| = m'} |\hat{b}_k(\ell)|$$

If, for each $\ell' \in \hat{\mathcal{B}}$, $\hat{c}_k(\ell')$ is analytic in $|\text{Im} k| < m$, then,

$$\sup_{u \in \mathcal{Z}_{\text{fin}}, x \in \mathcal{Z}_{\text{crs}}} |c(x, u)| e^{m'|x-u|} \leq \frac{1}{|\text{vol}_c|} \sup_{|\text{Im} k| = m'} |\hat{c}_k(\ell')| \leq \frac{1}{|\text{vol}_c|} \sup_{|\text{Im} k| = m'} |\hat{c}_k(\ell')|$$

**Proof:** (a) If $|\text{Im} k| < m$, then

$$|\hat{a}_k(\ell, \ell')| \leq \frac{|\text{vol}_f|}{|\mathcal{B}|} \sum_{|u| \in \mathcal{B}} \sum_{u' \in \mathcal{Z}_{\text{fin}}} |a(u, u')| e^{m'|u-u'|} \leq \frac{1}{|\mathcal{B}|} \sum_{|u| \in \mathcal{B}} \|a\|_m \leq \|a\|_m$$
Analyticity in $k$ follows from the uniform convergence of the series on $|\text{Im } k| < m$.

(b) Fix any $u, u' \in \mathbb{Z}_{\text{fin}}$. Set $q = m' \frac{u-u'}{|u-u'|}$. Then

$$a(u, u') e^{m'|u-u'|} = \sum_{\ell, \ell' \in \hat{B}} \int_{\mathbb{Z}_{\text{crs}}} \hat{a}_k(\ell, \ell') e^{i(k-iq) \cdot (u-u')} e^{i\ell \cdot u - i\ell' \cdot u'} \frac{d^{1+D} k}{(2\pi)^{1+D}}$$

$$= \sum_{\ell, \ell' \in \hat{B}} \int_{\mathbb{Z}_{\text{crs}}} \hat{a}_{k+iq}(\ell, \ell') e^{i(k-u') \ell \cdot u - i\ell' \cdot u'} \frac{d^{1+D} k}{(2\pi)^{1+D}}$$

where we have applied Stokes' theorem, using analyticity in $k$ and the fact that

$$e^{i(k-u') \ell \cdot u - i\ell' \cdot u'} = e^{i(k-u') a_k(u, u')}$$

is periodic in the real part of $k$ with respect to $\frac{2\pi}{\varepsilon_T L_T} \times \frac{2\pi}{\varepsilon_X L_X} \mathbb{Z}^b$. Hence

$$|a(u, u')| e^{m'|u-u'|} \leq \int_{\mathbb{Z}_{\text{crs}}} \sum_{\ell, \ell' \in \hat{B}} |\hat{a}_{k+iq}(\ell, \ell')| \frac{d^{1+D} k}{(2\pi)^{1+D}}$$

$$\leq \frac{1}{\text{vol}_c} \sup_{k \in \mathbb{Z}_{\text{crs}}} \sum_{\ell, \ell' \in \hat{B}} |\hat{a}_{k+iq}(\ell, \ell')|$$

$$\leq \frac{|\hat{B}|}{\text{vol}_f} \sup_{k \in \mathbb{Z}_{\text{crs}}} \sum_{\ell, \ell' \in \hat{B}} |\hat{a}_{k+iq}(\ell, \ell')|$$

The second bound is obvious from

$$\text{vol}_f \sum_{y' \in \mathcal{X}_{\text{fin}}} |A([u], y')| e^{m'||u-y'||} = \text{vol}_f \sum_{y' \in \mathcal{X}_{\text{fin}}} \sum_{u' \in \mathbb{Z}_{\text{fin}}} a(u, u') |e^{m'||u-u'||}|$$

$$(\text{with the distance } ||u - y'|| \text{ measured in } \mathcal{X}_{\text{fin}} \text{ and the distance } |u - u'|| \text{ measure in } \mathbb{Z}_{\text{fin}}) \text{ and the similar bound with the roles of } u \text{ and } u' \text{ interchanged.}$$

(c) The proof is much the same as that of part (b).

\[\square\]

**Lemma A.12** Let $m > 0$. Let, for each $\ell, \ell' \in \hat{B}$, $\hat{b}_k(\ell, \ell')$ be analytic in $|\text{Im } k| < m$. Assume that

$$\hat{b}_{k+p}(\ell, \ell') = \hat{b}_k(\ell + p, \ell' + p)$$

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for all $p \in \frac{2\pi}{\varepsilon T_L T} \mathbb{Z} \times \frac{2\pi}{\varepsilon X_L X} \mathbb{Z}^D$ and $\ell, \ell' \in \hat{B}$. Set

$$a(u, u') = \sum_{\ell, \ell' \in \hat{B}} \int_{\hat{Z}} e^{i\ell \cdot u} \hat{b}_k(\ell, \ell') e^{-i\ell' \cdot u'} e^{i k \cdot (u - u')} \frac{d^{1+D}k}{(2\pi)^{1+D}}$$

Then $a(u, u') : \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}} \to \Phi$ obeys the conditions of Definition A.2 and

$$\hat{a}_k(\ell, \ell') = \hat{b}_k(\ell, \ell')$$

for all $k \in \mathbb{R} \times \mathbb{R}^D$ and $\ell, \ell' \in \hat{B}$.

**Proof:** The proof is straightforward. 

---

### Functions of Periodic Operators

Let $C$ be a simple, closed, positively oriented, piecewise smooth curve in the complex plane and denote by $\mathcal{O}_C$ its interior. Denote by $\sigma(A)$ the spectrum of the bounded operator $A$ and assume $\sigma(A) \subset \mathcal{O}_C$. Let $f(z)$ be analytic on the closure of $\mathcal{O}_C$. Then, by the Cauchy integral formula,

$$f(A) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - A} d\zeta$$

and, for any $m \geq 0$,

$$\|f(A)\|_m \leq \frac{1}{2\pi} |C| \sup_{\zeta \in \mathcal{C}} |f(\zeta)| \sup_{\zeta \in \mathcal{C}} \| (\zeta - A)^{-1} \|_m$$

(A.12)

**Lemma A.13** Let

- $a(u, u') : \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}} \to \Phi$ obey the conditions of Definition A.2,
- $C$ be a simple, closed, positively oriented, piecewise smooth curve in the complex plane with interior $\mathcal{O}_C$,
- $\mathcal{O}$ contain the closure of $\mathcal{O}_C$ and $f : \mathcal{O} \to \Phi$ be analytic, and
- $0 < m'' < m' < m$.

Suppose that

- for each $\ell, \ell' \in \hat{B}$, $\hat{a}_k(\ell, \ell')$ is analytic in $|\text{Im } k| < m$.
- for each $\zeta \in \Phi \setminus \mathcal{O}_C$ and each $k$ with $|\text{Im } k| < m$, the matrix $[\zeta \delta_{\ell, \ell'} - \hat{a}_k(\ell, \ell')]_{\ell, \ell' \in \hat{B}}$ is invertible.

Denote by $A$ the periodization of $a$. Then $f(A)$, defined by (A.11), exists and

$$\|f(A)\|_{m''} \leq \frac{C_{m', m''}}{2\pi \text{Vol}_T} |C| \sup_{\zeta \in \mathcal{C}} |f(\zeta)| \sup_{|\text{Im } k| = m'} \sum_{\ell, \ell' \in \hat{B}} \| (\zeta I - \hat{a}_k)^{-1}(\ell, \ell') \|_{m''}$$

$$\leq \frac{C_{m', m''} |\mathcal{B}|}{2\pi \text{Vol}_T} |C| \sup_{\zeta \in \mathcal{C}} |f(\zeta)| \sup_{|\text{Im } k| = m'} \sum_{\ell, \ell' \in \hat{B}} \| (\zeta I - \hat{a}_k)^{-1}(\ell, \ell') \|_{m''}$$

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Here \((\zeta \mathbb{1} - \hat{a}_k)^{-1}\) refers to the inverse of the \(|\hat{B}| \times |\hat{B}|\) matrix \([\zeta \delta_{\ell,\ell'} - \hat{a}_k(\ell, \ell')]_{\ell, \ell' \in \hat{B}}\).

**Proof:** Each matrix element of \(\zeta \mathbb{1} - \hat{a}_k\) is continuous on

\[
D = \{(\zeta, k) \in \mathbb{C}^2 \mid \zeta \in \mathbb{C} \setminus \mathcal{O}_C, |\text{Im } k| < m\}
\]

Furthermore \(\det(\zeta \mathbb{1} - \hat{a}_k)\) does not vanish on \(D\). Hence every matrix element of \((\zeta \mathbb{1} - \hat{a}_k)^{-1}\) is also continuous on \(D\) and in particular is bounded on compact subsets of \(D\). Set, for each \(\zeta \in \mathbb{C} \setminus \mathcal{O}_C\), and \(u, u' \in \mathbb{Z}_{\text{fin}}\),

\[
r_\zeta(u, u') = \sum_{\ell, \ell' \in \hat{B}} \int_{\hat{Z}_{\text{crs}}} e^{i\ell \cdot u - i\ell' \cdot u'} e^{\text{i}k \cdot (u - u')} \frac{d^{1+D_k}}{(2\pi)^{1+D}}
\]

By Lemma A.12, \(r_\zeta(u, u')\) obeys the conditions of Definition A.2 and

\[
\hat{r}_\zeta(\ell, \ell') = (\zeta \mathbb{1} - \hat{a}_k)^{-1}(\ell, \ell')
\]

By Lemma A.6, \(r_\zeta = (\zeta \mathbb{1} - a)^{-1}\), as operators on \(L^2(\mathbb{Z}_{\text{fin}})\). By Remark A.3.c, for each \(\zeta \in \mathbb{C} \setminus \mathcal{O}_C\), the periodization of \(r_\zeta(u, u')\) is \((\zeta \mathbb{1} - A)^{-1}\). In particular, \(\sigma(A) \subset \mathcal{O}_C\). By Lemma A.11.b,

\[
\|\left(\zeta \mathbb{1} - A\right)^{-1}\|_{m''} \leq \frac{C_{m' - m''}}{\text{vol}_{f}} \sup_{\|\text{Im } k\| = m'} \sum_{\ell, \ell' \in \hat{B}} \left|\left(\zeta \mathbb{1} - \hat{a}_k\right)^{-1}(\ell, \ell')\right|
\]

Then, by (A.12),

\[
\|f(A)\|_{m''} \leq \frac{1}{2\pi} |C| \sup_{\zeta \in C} |f(\zeta)| \sup_{\zeta \in C} \left\|\left(\zeta \mathbb{1} - A\right)^{-1}\right\|_{m''}
\]

\[
\leq \frac{C_{m' - m''}}{2\pi \text{vol}_{f}} |C| \sup_{\zeta \in C} |f(\zeta)| \sup_{\|\text{Im } k\| = m'} \sum_{\ell, \ell' \in \hat{B}} \left|\left(\zeta \mathbb{1} - \hat{a}_k\right)^{-1}(\ell, \ell')\right|
\]

\[
\leq \frac{C_{m' - m''}}{2\pi \text{vol}_{f}} |C| \sup_{\zeta \in C} |f(\zeta)| \sup_{\|\text{Im } k\| = m'} \sum_{\ell, \ell' \in \hat{B}} \left|\left(\zeta \mathbb{1} - \hat{a}_k\right)^{-1}(\ell, \ell')\right|
\]

\[\square\]
Scaling of Periodized Operators

Scaling plays an important in the construction of [parabolic-all.tex]. See, for example, [parabolic-all.tex, Definition I.3 and §III] and (I.3). For the abstract setting of this appendix, select scaling factors \( \sigma_T \) and \( \sigma_X \) and define the scaled lattices

\[ Z^{(s)}_{\text{fin}} = \frac{\sigma_T}{\sigma_X} \mathbb{Z} \times \frac{\sigma_X}{\sigma_X} \mathbb{Z}^D \]

\[ Z^{(s)}_{\text{crs}} = L_T \frac{\sigma_T}{\sigma_T} \mathbb{Z} \times L_X \frac{\sigma_X}{\sigma_X} \mathbb{Z}^D \]

\[ \hat{Z}^{(s)}_{\text{crs}} = \left( \mathbb{R} / \frac{2\pi \sigma_T}{\sigma_T} L_T \mathbb{Z} \right) \times \left( \mathbb{R}^D / \frac{2\pi \sigma_X}{\sigma_X} L_X \mathbb{Z}^D \right) \]

\[ B^{(s)} = \left( \frac{\sigma_T}{\sigma_T} \mathbb{Z} / L_T \frac{\sigma_T}{\sigma_T} \mathbb{Z} \right) \times \left( \frac{\sigma_X}{\sigma_X} \mathbb{Z}^D / L_X \frac{\sigma_X}{\sigma_X} \mathbb{Z}^D \right) \]

\[ \hat{B}^{(s)} = \left( \frac{2\pi \sigma_T}{L_T \epsilon_T} \mathbb{Z} / \frac{2\pi \sigma_X}{L_X \epsilon_X} \mathbb{Z} \right) \times \left( \frac{2\pi \sigma_X}{L_X \epsilon_X} \mathbb{Z}^D / \frac{2\pi \sigma_X}{\epsilon_X} \mathbb{Z}^D \right) \]

The map \( \mathbb{L}(\tau, x) = (\sigma_T \tau, \sigma_X x) \) gives bijections

\[ \mathbb{L} : Z^{(s)}_{\text{fin}} \rightarrow Z_{\text{fin}} \quad \mathbb{L} : Z^{(s)}_{\text{crs}} \rightarrow Z_{\text{crs}} \quad \mathbb{L} : B^{(s)} \rightarrow B \]

\( \mathbb{L} \) induces linear bijections \( \mathbb{L}_* : L^2(Z^{(s)}_{\text{fin}}) \rightarrow L^2(Z_{\text{fin}}) \) and \( \mathbb{L}_* : L^2(Z^{(s)}_{\text{crs}}) \rightarrow L^2(Z_{\text{crs}}) \) by \( \mathbb{L}_*(\alpha)(\mathbb{L}u) = \alpha(u) \). Observe that

\[ \langle \mathbb{L}_* \alpha, \mathbb{L}_* \beta \rangle_f = \sigma_T \sigma_X^D \langle \alpha, \beta \rangle_f \quad \langle \mathbb{L}_* \alpha, \mathbb{L}_* \beta \rangle_c = \sigma_T \sigma_X^D \langle \alpha, \beta \rangle_c \]

**Lemma A.14** Let \( a : L^2(Z^{(s)}_{\text{fin}}) \rightarrow L^2(Z_{\text{fin}}) \) have kernel \( a(u, u') \).

(a) The kernel of \( \mathbb{L}_*^{-1} a \mathbb{L}_* \) is

\[ a^{(s)}(v, v') = \sigma_T \sigma_X^D a(\mathbb{L}v, \mathbb{L}v') \]

(b) The Fourier transform of the kernel of \( \mathbb{L}_*^{-1} a \mathbb{L}_* \) is

\[ \hat{a}^{(s)}_k(\ell, \ell') = \hat{a} \mathbb{L}_*^{-1} k(\mathbb{L}^{-1} \ell, \mathbb{L}^{-1} \ell') \quad \text{for } k \in \mathbb{R} \times \mathbb{R}^D, \quad \ell, \ell' \in \hat{B}^{(s)} \]

(c) If \( m \geq \max \left\{ \frac{1}{\sigma_T}, \frac{1}{\sigma_X} \right\} m_s \), then \( \|a^{(s)}\|_{m_s} \leq \|a\|_m \).

**Proof:** (a) For \( \alpha \in L^2(Z^{(s)}_{\text{fin}}) \) and \( v \in Z^{(s)}_{\text{fin}} \),

\[ (\mathbb{L}_*^{-1} a \mathbb{L}_* \alpha)(v) = \text{vol}_f \sum_{u' \in Z_{\text{fin}}} a(\mathbb{L}v, u') (\mathbb{L}_* \alpha)(u') \]

\[ = \text{vol}_f \sum_{u' \in Z_{\text{fin}}} a(\mathbb{L}v, u') \alpha(\mathbb{L}_*^{-1} u') \]

\[ = \text{vol}_f^{(s)} \sum_{v' \in Z^{(s)}_{\text{fin}}} \sigma_T \sigma_X^D a(\mathbb{L}v, \mathbb{L}v') \alpha(v') \]

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(b) By (A.6) and part (a),
\[
\hat{a}_k^{(s)}(\ell, \ell') = \frac{\text{vol}_f}{|B|} \sum_{[v] \in B^{(s)}\atop v' \in \mathbb{Z}_{\text{fin}}} e^{-i\ell \cdot v} a(\mathbb{L}v, \mathbb{L}v') e^{i\ell' \cdot v'} e^{-ik \cdot (v - v')}
\]
\[
= \frac{\text{vol}_f}{|B|} \sum_{[u] \in B\atop u' \in \mathbb{Z}_{\text{fin}}} e^{-i(L^{-1} \ell \cdot u} a(u, u') e^{i(L^{-1} \ell' \cdot u')} e^{-i(L^{-1}k \cdot (u - u')}
\]
\[
= \hat{a}_{L^{-1}k}(\mathbb{L}^{-1} \ell, \mathbb{L}^{-1} \ell')
\]

(c) This part follows from the inequality
\[
\sup_{v \in \mathbb{Z}_{\text{fin}}^{(s)}} \text{vol}_f^{(s)} \sum_{v' \in \mathbb{Z}_{\text{fin}}^{(s)}} e^{m_s |v - v'|} |a^{(s)}(v, v')| = \sup_{u \in \mathbb{Z}_{\text{fin}}} \text{vol}^{(s)} \sum_{u' \in \mathbb{Z}_{\text{fin}}} e^{m_s |u - u'|} |a^{(s)}(u, u')|
\]
and the corresponding inequality with v summed over and v' supped over.

More generally,

**Lemma A.15** Let \( b : L^2(\mathbb{Z}_{\text{crs}}) \to L^2(\mathbb{Z}_{\text{fin}}) \) and \( c : L^2(\mathbb{Z}_{\text{fin}}) \to L^2(\mathbb{Z}_{\text{crs}}) \) have kernels \( b(u, x) \) and \( c(x, u) \) respectively.

(a) The kernels of \( \mathbb{L}_*^{-1} b \mathbb{L}_* \) and \( \mathbb{L}_*^{-1} c \mathbb{L}_* \) are
\[
b^{(s)}(v, x) = \sigma_T \sigma_X^0 b(\mathbb{L}v, \mathbb{L}x) \quad c^{(s)}(x, v) = \sigma_T \sigma_X^0 c(\mathbb{L}x, \mathbb{L}v)
\]

(b) The Fourier transform of the kernels of \( \mathbb{L}_*^{-1} b \mathbb{L}_* \) and \( \mathbb{L}_*^{-1} c \mathbb{L}_* \) are
\[
\hat{b}_k^{(s)}(\ell) = \hat{b}_{L^{-1}k}(\mathbb{L}^{-1} \ell) \quad \hat{c}_k^{(s)}(\ell') = \hat{c}_{L^{-1}k}(\mathbb{L}^{-1} \ell') \quad \text{for } k \in \mathbb{R} \times \mathbb{R}^0, \quad \ell, \ell' \in \hat{B}^{(s)}
\]

(c) If \( m \geq \max \{ \frac{1}{\sigma_T}, \frac{1}{\sigma_X} \} m_s \), then
\[
\|b^{(s)}\|_{m_s} \leq \|b\|_m \quad \|c^{(s)}\|_{m_s} \leq \|c\|_m
\]
Appendix B: Trigonometric Inequalities

Lemma B.1

(a) For \( x, y \) real with \( |x| \leq \frac{\pi}{2} \),
\[
| \sin(x + iy) | \geq \frac{\sqrt{2}}{\pi} |x + iy|
\]

(b) For \( x, y \) real with \( |y| \leq 1 \),
\[
| \frac{\sin(x + iy)}{|x + iy|} | \leq 2 \min \left\{ 1, \frac{1}{|x + iy|} \right\} \quad \left| \operatorname{Im} \frac{\sin(x + iy)}{x + iy} \right| \leq 2 |y| \min \left\{ |x|, \frac{2}{|x| + 1} \right\}
\]

(c) For \( 0 < \varepsilon \leq 1 \) and \( x, y \) real with \( |\varepsilon x| \leq \pi, |y| \leq 2 \)
\[
\left| \frac{\sin \frac{1}{2}(x + iy)}{\varepsilon \sin \frac{1}{2}(x + iy)} \right| \leq 4 \min \left\{ 1, \frac{2}{|x|} \right\} \quad \left| \operatorname{Im} \frac{\sin \frac{1}{2}(x + iy)}{\varepsilon \sin \frac{1}{2}(x + iy)} \right| \leq 6 |y| \min \left\{ |x|, \frac{8}{|x|} \right\}
\]

(d) For \( x \) real with \( |x| \leq \frac{\pi}{2} \),
\[
\frac{2}{x} \leq \frac{|\sin x|}{|x|} \leq 1
\]

(e) For any complex number \( z \) obeying \( |z| \leq 2 \),
\[
\left| \frac{\sin z}{z} - 1 \right| \leq \frac{1}{2} |z|^2
\]

Proof: By the standard trig identity
\[
\sin(x + iy) = \sin(x) \cos(iy) + \cos(x) \sin(iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)
\]

(a) For \( x, y \) real with \( |x| \leq \frac{\pi}{2} \)
\[
|\operatorname{Re} \sin(x + iy)| = |\sin(x) \cosh(y)| \geq |\sin(x)| \geq \frac{2}{\pi} |x|
\]
\[
|\operatorname{Re} \sin(x + iy)| = |\sin(x) \cosh(y)| \geq |\sin(x) \sinh(y)| \geq |\sin(x)| |y|
\]
\[
|\operatorname{Im} \sin(x + iy)| = |\cos(x) \sinh(y)| \geq |\cos(x)| |y|
\]

so that
\[
|\sin(x + iy)| \geq \max \left\{ \frac{2}{\pi} |x|, |y| \right\} \geq \frac{\sqrt{2}}{\pi} |x + iy|
\]

(b) For \( x, y \) real with \( |y| \leq 1 \),
\[
|\sin(x + iy)| = |\sin(x) \cosh(y) + i \cos(x) \sinh(y)| \leq \cosh(1) |\sin(x) + i \cos(x)| = \cosh(1)
\]
and, since \( |\sinh(y)| = \left| \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} \right| \leq |y| \sum_{n=0}^{\infty} \frac{|y|^{2n}}{(2n)!} = |y| \cosh(y) \leq \cosh(1)|y| \),

\[ |\sin(x + iy)| = |\sin(x) \cosh(y) + i \cos(x) \sinh(y)| \leq \cosh(1)|x| + i|y| = \cosh(1)|x + iy| \]

Thus

\[ \frac{|\sin(x+iy)|}{|x+iy|} \leq \cosh(1) \min \left\{ 1, \frac{1}{|x+iy|} \right\} \leq 2 \min \left\{ 1, \frac{1}{|x+iy|} \right\} \]

giving the first bound.

For the second bound

\[ \left| \text{Im} \frac{\sin(x+iy)}{x+iy} \right| = \left| \frac{x \text{Im} \sin(x+iy) - y \text{Re} \sin(x+iy)}{x^2 + y^2} \right| = \left| \frac{x \cos(x) \sinh(y) - y \sin(x) \cosh(y)}{x^2 + y^2} \right| \]

Using

\[ \cos x - 1 = -\int_0^x dt \sin t = -\int_0^x dt \int_0^t ds \cos s \]

\[ \sin x - x = \int_0^x dt [\cos t - 1] = -\int_0^x dt \int_0^t ds \int_0^s du \cos u \]

\[ \cosh y - 1 = \int_0^y dt \sinh t = \int_0^y dt \int_0^t ds \cosh s \]

\[ \sinh y - y = \int_0^y dt [\cosh t - 1] = \int_0^y dt \int_0^t ds \int_0^s du \cosh u \]

and \( \cosh(1) < 2 \), we have

\[ \cos x = 1 + \alpha(x) \frac{x^2}{2} \quad \sin x = x + \beta(x) \frac{x^3}{6} \quad \cosh y = 1 + \gamma(y) y^2 \quad \sinh y = y + \delta(y) \frac{y^3}{3} \]

with, for \(|y| \leq 1\), \(|\alpha(x)|, |\beta(x)|, |\gamma(y)|, |\delta(y)| \leq 1\). Consequently,

\[ \left| \text{Im} \frac{\sin(x+iy)}{x+iy} \right| \leq 2|xy| \]

Alternatively, using \(|\sin(x)| \leq |x|\), \(|\sinh(y)| \leq 2|y|\) and \(|\cosh(y)| \leq 2\),

\[ \left| \text{Im} \frac{\sin(x+iy)}{x+iy} \right| \leq \frac{4|xy|}{x^2 + y^2} \leq \frac{4|y|}{\sqrt{x^2 + y^2}} \]

(c) For \(0 < \varepsilon \leq 1\), \(x, y\) real and \(|\varepsilon x| \leq \pi\), \(|y| \leq 2\),

\[ \left| \frac{\sin \frac{1}{\varepsilon}(x + iy)}{\sin \frac{1}{\varepsilon}(x + iy)} - \frac{\sin \frac{1}{\varepsilon}(x + iy)}{\sin \frac{1}{\varepsilon}(x + iy)} \right| \leq \frac{\pi}{\sqrt{2}} \cosh(1) \min \left\{ 1, \frac{2}{|x + iy|} \right\} \leq 4 \min \left\{ 1, \frac{2}{|x|} \right\} \]
and
\[ \left| \text{Im} \left( \frac{x}{2} \sin \left( \frac{1}{2} (x + iy) \right) \right) \right| = \left| \text{Im} \left( \frac{x}{2} \sin \left( \frac{1}{2} x + iy \right) \right) \right| \leq \left| \frac{y}{2} \min \left\{ \frac{1}{2} |x|, \frac{4}{|x| + |y|} \right\} + 4 \min \left\{ 1, \frac{2}{|x|} \right\} \right| \cdot \left| \frac{x}{2} \sin \left( \frac{1}{2} x + iy \right) \right| \]
\[ \leq \frac{\sqrt{\pi}}{\sqrt{2}} |y| \min \left\{ \left( \frac{1}{2} + 2 \varepsilon \right) |x|, \frac{4 + 16}{|x| + |y|} \right\} \]
\[ \leq 6 |y| \min \left\{ |x|, \frac{8}{|x| + |y|} \right\} \]

(d) This follows from the observation that, for \( 0 \leq x \leq \frac{\pi}{2} \), \( \sin x \) is monotone decreasing. To see this, observe that
\[ \frac{d}{dx} \sin x = \frac{1}{x} \left( \cos x - \sin x \right) \]
and
\[ \cos x - \frac{\sin x}{x} = \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right) x^{2n} = \sum_{n=1}^{\infty} (-1)^n \frac{2n}{(2n+1)!} x^{2n} \]
This is an alternating series with the first term being negative and with the ratio between term \( n + 1 \) and term \( n \) having magnitude
\[ \frac{(2n+1)!}{2n} \frac{2n+2}{(2n+3)!} x^2 = \frac{x^2}{2n(2n+3)} \leq \frac{x^2}{10} \]
which is less than 1 for all \(|x| \leq \sqrt{10} = 3.162\).

(e) We have
\[ \left| \sin \frac{z}{z} - 1 \right| = \left| \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \right| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+3)!} \right| = \left| z \right|^2 \left| \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+3)!} \right| \]
As \( (2n+3)! = n! (n+1) \cdots (2n+3) = n! (n+1) \cdots (2n+3) \geq n! 3^n \)
we have
\[ \left| \sin \frac{z}{z} - 1 \right| \leq \frac{|z|^2}{3^n} \cdot |z|^{2/5} \]
When \(|z| \leq 2\)
\[ e^{|z|^2/5} \leq e \quad \Rightarrow \quad \left| \sin \frac{z}{z} - 1 \right| \leq \frac{1}{2} |z|^2 \]

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Appendix C: Lattice and Operator Summary

The following table gives, for most of the operators considered in this paper,

- the definition of the operator
- a reference to where in [parabolic-all.tex], the operator is introduced and
- the translation invariance properties of the operator.

A later table will specify where, in this paper, bounds on the operators are proven.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Definition</th>
<th>Tiwrt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0 = -e^{-h_0} \partial_0 + \left[ I - e^{-h_0} \right]$</td>
<td>$\mathcal{H}_0 \to \mathcal{H}_0$</td>
<td>§I.5 $\mathcal{X}_0$</td>
</tr>
<tr>
<td>$D_n = L_{s}^{n} D_0 L_{s}^{n}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Definition I.5.a $\mathcal{X}_n$</td>
</tr>
<tr>
<td>$\Omega_n = a \left( 1 + \sum_{j=1}^{n-1} \frac{1}{L^2} Q_j Q_j^* \right)^{-1}$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>Definition I.5.b $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$\Delta^{(0)} = D_0$</td>
<td>$\mathcal{H}_0 \to \mathcal{H}_0$</td>
<td>(I.14) $\mathcal{X}_0$</td>
</tr>
<tr>
<td>$\Delta^{(n)} = \left( I + \Omega_n Q_n D_{n}^{-1} Q_{n}^* \right)^{-1} \Omega_n$, $n \geq 1$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>(I.14) $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$C^{(n)} = \left( \frac{\mathcal{H}}{L^2} Q^* Q + \Delta^{(n)} \right)^{-1}$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>(I.15) $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$S_n^{-1} = D_n + Q_n^* \Omega_n Q_n$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Theorem I.15 $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$S_n(\mu)^{-1} = D_n + Q_n^* \Omega_n Q_n - \mu$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Theorem I.15 $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$\hat{S}_{n+1} = \left( \frac{L^2}{\mathcal{H}} I + Q \Omega^{-1} Q^* \right)^{-1}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>Lemma III.4.b $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$\hat{S}<em>{n+1}(\mu) = { D_n + \hat{Q}</em>{n+1}^* \hat{Q}<em>{n+1} \hat{Q}</em>{n+1} - \mu }^{-1}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>(G.27) $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_0^{(n)}$</td>
<td>Proposition G.13 $\mathcal{X}_0^{(n)}$</td>
</tr>
</tbody>
</table>

The references in the above table are to [parabolic-all.tex] and “Tiwrt” stands for “translation invariant with respect to”.

Here is a table giving where some other operators were defined in this paper.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Definition</th>
<th>Tiwrt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n Q_n^* = D_{n}^{-1} Q_n^* \left( I + \Omega_n Q_n D_{n}^{-1} Q_{n}^* \right)^{-1}$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_n$</td>
<td>Lemma V.6 $\mathcal{X}_0^{(n)}$</td>
</tr>
<tr>
<td>$\hat{S}<em>{n+1} = \hat{S}</em>{n+1}(0)$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_n$</td>
<td>after (VI.2) $\mathcal{X}_0^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_0^{(n)}$</td>
<td>(VI.4) $\mathcal{X}_0^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\phi,\nu}$</td>
<td>$\mathcal{H}_n \to \mathcal{H}_0^{(n)}$</td>
<td>Remark VI.2 $\mathcal{X}_0^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi^{(<em>)},\theta^{(</em>)}} = \frac{2}{L^2} C^{(n)(<em>)} Q^</em>$</td>
<td>$\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>before Remark VI.2 $\mathcal{X}_0^{(n+1)}$</td>
</tr>
<tr>
<td>$A_{\psi,\theta,\nu}$, $A_{\psi,\theta,\nu}(\mu)$, $\mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$</td>
<td>Remark VI.2 $\mathcal{X}_0^{(n+1)}$</td>
<td></td>
</tr>
</tbody>
</table>

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The lattices involved are

\[ X_n = \left( \varepsilon_n^2 \mathbb{Z} / \varepsilon_n^2 L_{tp} \mathbb{Z} \right) \times \left( \varepsilon_n \mathbb{Z}^3 / \varepsilon_n L_{sp} \mathbb{Z}^3 \right) \]

\[ \hat{X}_n = \left( \frac{2\pi}{\varepsilon_n L_{tp}} \mathbb{Z} / 2\pi \mathbb{Z} \right) \times \left( \frac{2\pi}{\varepsilon_n L_{sp}} \mathbb{Z}^3 / 2\pi \mathbb{Z}^3 \right) \]

\[ X_0^{(n)} = \left( \mathbb{Z} / \varepsilon_n^2 L_{tp} \mathbb{Z} \right) \times \left( \mathbb{Z}^3 / \varepsilon_n L_{sp} \mathbb{Z}^3 \right) \]

\[ \hat{X}_0^{(n)} = \left( \frac{2\pi}{\varepsilon_n L_{tp}} \mathbb{Z} / 2\pi \mathbb{Z} \right) \times \left( \frac{2\pi}{\varepsilon_n L_{sp}} \mathbb{Z}^3 / 2\pi \mathbb{Z}^3 \right) \]

\[ X_{-1}^{(n+1)} = \left( L^2 \mathbb{Z} / \varepsilon_n^2 L_{tp} \mathbb{Z} \right) \times \left( L \mathbb{Z}^3 / \varepsilon_n L_{sp} \mathbb{Z}^3 \right) \]

\[ \hat{X}_{-1}^{(n+1)} = \left( \frac{2\pi}{\varepsilon_n L_{tp}} \mathbb{Z} / 2\pi \mathbb{Z} \right) \times \left( \frac{2\pi}{\varepsilon_n L_{sp}} \mathbb{Z}^3 / 2\pi \mathbb{Z}^3 \right) \]

where \( \varepsilon_n = \frac{1}{L_n} \). The “single period” lattices are

\[ B_n = \left( \varepsilon_n^2 \mathbb{Z} / \mathbb{Z} \right) \times \left( \varepsilon_n \mathbb{Z}^3 / \mathbb{Z}^3 \right) \]

\[ \hat{B}_n = \left( 2\pi \mathbb{Z} / \varepsilon_n^2 \mathbb{Z} \right) \times \left( 2\pi \mathbb{Z}^3 / \varepsilon_n \mathbb{Z}^3 \right) \]

\[ B^+ = \left( \mathbb{Z} / L^2 \mathbb{Z} \right) \times \left( \mathbb{Z}^3 / L \mathbb{Z}^3 \right) \]

\[ \hat{B}^+ = \left( \frac{2\pi}{L} \mathbb{Z} / 2\pi \mathbb{Z} \right) \times \left( \frac{2\pi}{L} \mathbb{Z}^3 / 2\pi \mathbb{Z}^3 \right) \]

The following table specifies where, in this paper, bounds on the various operators are proven.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
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<td>( Q_n )</td>
<td>Lemma II.2.a</td>
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<tr>
<td>( \Omega_n )</td>
<td>Proposition II.4</td>
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<tr>
<td>( Q_{n,\nu}^{(\pm)} )</td>
<td>Lemma II.6</td>
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<tr>
<td>( D_n )</td>
<td>Lemma III.2</td>
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<tr>
<td>( \Delta_n^{(n)} )</td>
<td>Lemma IV.2</td>
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<td>( C_n^{(n)} )</td>
<td>Corollary IV.5</td>
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<tr>
<td>( D_n )</td>
<td>Corollary IV.5</td>
</tr>
<tr>
<td>( S_n(\mu), S_n )</td>
<td>Proposition V.1</td>
</tr>
<tr>
<td>( S_n^{(\pm)}(\mu), S_n^{(\pm)} )</td>
<td>Proposition V.1</td>
</tr>
<tr>
<td>( A_{\psi,\phi} )</td>
<td>Proposition VI.1</td>
</tr>
<tr>
<td>( A_{\psi,\phi,\nu} )</td>
<td>Proposition VI.1</td>
</tr>
<tr>
<td>( A_{\psi(\ast),\theta(\ast)} )</td>
<td>Proposition VI.1</td>
</tr>
<tr>
<td>( A_{\psi(\ast\ast),\theta(\ast\ast)}(\mu) )</td>
<td>Proposition VI.1</td>
</tr>
<tr>
<td>( \hat{S}_{n+1} )</td>
<td>Corollary VI.5</td>
</tr>
</tbody>
</table>
References


