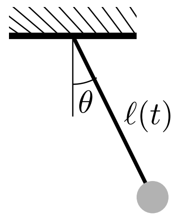
The Variable Length Pendulum

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of time–dependent length $\ell(t)$. Denote by θ the angle between the rod and



vertical. At time t, the position and velocity of the mass are

$$\begin{aligned} x(t) &= \ell(t)\sin\theta(t) & \dot{x}(t) &= \dot{\ell}(t)\sin\theta(t) + \ell(t)\dot{\theta}(t)\cos\theta(t) \\ y(t) &= -\ell(t)\cos\theta(t) & \dot{x}(t) &= -\dot{\ell}(t)\cos\theta(t) + \ell(t)\dot{\theta}(t)\sin\theta(t) \end{aligned}$$

So the kinetic and potential energies of the mass are

$$\begin{aligned} T &= \tfrac{1}{2}m\big[\dot{x}(t)^2 + \dot{y}(t)^2\big] &&= \tfrac{1}{2}m\big[\dot{\ell}(t)^2 + \ell(t)^2\dot{\theta}(t)^2\big] \\ U &= mgy(t) &&= -mg\ell(t)\cos\theta(t) \end{aligned}$$

The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L}(\theta,\dot{\theta},t) = T - U &= \tfrac{1}{2}m\big[\dot{\ell}(t)^2 + \ell(t)^2\dot{\theta}^2\big] + mg\ell(t)\cos\theta \\ &= m\big[\tfrac{1}{2}\ell(t)^2\dot{\theta}^2 + g\ell(t)\cos\theta + \tfrac{1}{2}\dot{\ell}(t)^2\big] \end{aligned}$$

and Lagrange's equation of motion is

$$\begin{aligned} 0 &= \tfrac{d}{dt}\big(\tfrac{\partial\mathcal{L}}{\partial\dot{\theta}}\big) - \tfrac{\partial\mathcal{L}}{\partial\theta} \\ &= \tfrac{d}{dt}m\big[\ell(t)^2\dot{\theta}\big] - m\big[-g\ell(t)\sin\theta\big] \\ &= m\big[\ell(t)^2\ddot{\theta}(t) + 2\ell(t)\dot{\ell}(t)\dot{\theta}(t) + g\ell(t)\sin\theta(t)\big] \end{aligned}$$

or

$$\tfrac{d^2\theta}{dt^2}(t) + 2\tfrac{\dot{\ell}(t)}{\ell(t)}\dot{\theta}(t) + \tfrac{g}{\ell(t)}\sin\theta(t) = 0$$

In particular, if $\ell(t) = L + A\cos(\omega t)$, the equation of motion is

$$\tfrac{d^2\theta}{dt^2}(t) - 2A\omega\tfrac{\sin(\omega t)}{L+A\cos(\omega t)}\dot{\theta}(t) + \tfrac{g}{L+A\cos(\omega t)}\sin\theta(t) = 0$$