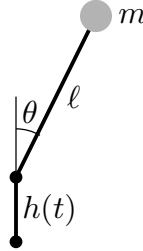


The Upside Down Pendulum

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of fixed length ℓ . Invert the pendulum and shake the pivot point vertically.



Suppose that at time t , the location of the pivot point is $x = 0$, $y = h(t)$. Denote by θ the angle between the rod and vertical. At time t , the position and velocity of the mass are

$$\begin{aligned} x(t) &= \ell \sin \theta(t) & \dot{x}(t) &= \ell \dot{\theta}(t) \cos \theta(t) \\ y(t) &= h(t) + \ell \cos \theta(t) & \dot{y}(t) &= \dot{h}(t) - \ell \dot{\theta}(t) \sin \theta(t) \end{aligned}$$

So the kinetic and potential energies of the mass are

$$\begin{aligned} T &= \frac{1}{2}m[\dot{x}(t)^2 + \dot{y}(t)^2] = \frac{1}{2}m[\ell^2\dot{\theta}(t)^2 + \dot{h}(t)^2 - 2\ell\dot{h}(t)\dot{\theta}(t)\sin\theta(t)] \\ U &= mgy(t) = mgh(t) + mg\ell \cos \theta(t) \end{aligned}$$

The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L}(\theta, \dot{\theta}, t) &= T - U = \frac{1}{2}m[\ell^2\dot{\theta}^2 + \dot{h}(t)^2 - 2\ell\dot{h}(t)\dot{\theta}\sin\theta] - mgh(t) - mg\ell \cos \theta \\ &= m\left[\frac{1}{2}\ell^2\dot{\theta}^2 - \ell\dot{h}(t)\dot{\theta}\sin\theta - g\ell \cos \theta + \frac{1}{2}\dot{h}(t)^2 - gh(t)\right] \end{aligned}$$

and Lagrange's equation of motion is

$$\begin{aligned} 0 &= \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} \\ &= \frac{d}{dt} m[\ell^2\dot{\theta} - \ell\dot{h}(t)\sin\theta] - m[-\ell\dot{h}(t)\dot{\theta}\cos\theta + g\ell \sin\theta] \\ &= m[\ell^2\ddot{\theta}(t) - \ell\ddot{h}(t)\sin\theta(t) - g\ell \sin\theta(t)] \end{aligned}$$

or

$$\frac{d^2\theta}{dt^2}(t) - \frac{1}{\ell}(g + \ddot{h}(t)) \sin\theta(t) = 0$$

In particular, if $h(t) = A \sin(\omega t)$, the equation of motion is

$$\frac{d^2\theta}{dt^2}(t) - \frac{1}{\ell}[g - A\omega^2 \sin(\omega t)] \sin\theta(t) = 0$$

The horizontally driven case

For a horizontally driven pendulum, the corresponding equations are

$$\begin{aligned}x(t) &= h(t) + \ell \sin \theta(t) & \dot{x}(t) &= \dot{w}(t) + \ell \dot{\theta}(t) \cos \theta(t) \\y(t) &= \ell \cos \theta(t) & \dot{y}(t) &= -\ell \dot{\theta}(t) \sin \theta(t)\end{aligned}$$

So the kinetic and potential energies and Lagrangian of the mass are

$$\begin{aligned}T &= \frac{1}{2}m[\dot{x}(t)^2 + \dot{y}(t)^2] = \frac{1}{2}m[\ell^2\dot{\theta}(t)^2 + \dot{w}(t)^2 + 2\ell\dot{w}(t)\dot{\theta}(t)\cos\theta(t)] \\U &= mgy(t) = mg\ell\cos\theta(t) \\ \mathcal{L}(\theta, \dot{\theta}, t) &= T - U = m\left[\frac{1}{2}\ell^2\dot{\theta}^2 + \ell\dot{w}(t)\dot{\theta}\cos\theta - g\ell\cos\theta + \frac{1}{2}\dot{w}(t)^2\right]\end{aligned}$$

and Lagrange's equation of motion is

$$\begin{aligned}0 &= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} \\ &= \frac{d}{dt}m[\ell^2\dot{\theta} + \ell\dot{w}(t)\cos\theta] - m[-\ell\dot{w}(t)\dot{\theta}\sin\theta + g\ell\sin\theta] \\ &= m[\ell^2\ddot{\theta}(t) + \ell\ddot{w}(t)\cos\theta(t) - g\ell\sin\theta(t)]\end{aligned}$$

or

$$\frac{d^2\theta}{dt^2}(t) - \frac{g}{\ell}\sin\theta(t) + \frac{1}{\ell}\ddot{w}(t)\cos\theta(t) = 0$$

In particular, if $w(t) = A\sin(\omega t)$, the equation of motion is

$$\frac{d^2\theta}{dt^2}(t) - \frac{1}{\ell}[g\sin\theta(t) + A\omega^2\sin(\omega t)\cos\theta(t)] = 0$$