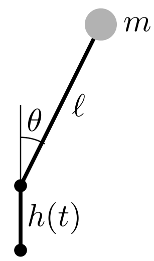
The Upside Down Pendulum

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of fixed length ℓ. Invert the pendulum and shake the pivot point vertically.



Suppose that at time t, the location of the pivot point is $x = 0$, $y = h(t)$. Denote by θ the angle between the rod and vertical. At time t, the position and velocity of the mass are

$$\begin{aligned} x(t) &= \ell \sin\theta(t) & \dot{x}(t) &= \ell\dot{\theta}(t)\cos\theta(t) \\ y(t) &= h(t) + \ell\cos\theta(t) & \dot{y}(t) &= \dot{h}(t) - \ell\dot{\theta}(t)\sin\theta(t) \end{aligned}$$

So the kinetic and potential energies of the mass are

$$\begin{aligned} T &= \tfrac{1}{2}m\big[\dot{x}(t)^2 + \dot{y}(t)^2\big] &&= \tfrac{1}{2}m\big[\ell^2\dot{\theta}(t)^2 + \dot{h}(t)^2 - 2\ell\dot{h}(t)\dot{\theta}(t)\sin\theta(t)\big] \\ U &= mgy(t) &&= mgh(t) + mg\ell\cos\theta(t) \end{aligned}$$

The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L}(\theta,\dot{\theta},t) = T - U &= \tfrac{1}{2}m\big[\ell^2\dot{\theta}^2 + \dot{h}(t)^2 - 2\ell\dot{h}(t)\dot{\theta}\sin\theta\big] - mgh(t) - mg\ell\cos\theta \\ &= m\big[\tfrac{1}{2}\ell^2\dot{\theta}^2 - \ell\dot{h}(t)\dot{\theta}\sin\theta - g\ell\cos\theta + \tfrac{1}{2}\dot{h}(t)^2 - gh(t)\big] \end{aligned}$$

and Lagrange's equation of motion is

$$\begin{aligned} 0 &= \tfrac{d}{dt}\big(\tfrac{\partial\mathcal{L}}{\partial\dot{\theta}}\big) - \tfrac{\partial\mathcal{L}}{\partial\theta} \\ &= \tfrac{d}{dt}m\big[\ell^2\dot{\theta} - \ell\dot{h}(t)\sin\theta\big] - m\big[-\ell\dot{h}(t)\dot{\theta}\cos\theta + g\ell\sin\theta\big] \\ &= m\big[\ell^2\ddot{\theta}(t) - \ell\ddot{h}(t)\sin\theta(t) - g\ell\sin\theta(t)\big] \end{aligned}$$

or

$$\tfrac{d^2\theta}{dt^2}(t) - \tfrac{1}{\ell}\big(g + \ddot{h}(t)\big)\sin\theta(t) = 0$$

In particular, if $h(t) = A\sin(\omega t)$, the equation of motion is

$$\tfrac{d^2\theta}{dt^2}(t) - \tfrac{1}{\ell}\big[g - A\omega^2\sin(\omega t)\big]\sin\theta(t) = 0$$

The horizontally driven case

For a horizontally driven pendulum, the corresponding equations are

$$\begin{aligned} x(t) &= h(t) + \ell\sin\theta(t) & \dot x(t) &= \dot w(t) + \ell\dot\theta(t)\cos\theta(t) \\ y(t) &= \ell\cos\theta(t) & \dot y(t) &= -\ell\dot\theta(t)\sin\theta(t) \end{aligned}$$

So the kinetic and potential energies and Lagrangian of the mass are

$$\begin{aligned} T &= \tfrac{1}{2}m\big[\dot x(t)^2 + \dot y(t)^2\big] &&= \tfrac{1}{2}m\big[\ell^2\dot\theta(t)^2 + \dot w(t)^2 + 2\ell\dot w(t)\dot\theta(t)\cos\theta(t)\big] \\ U &= mgy(t) &&= mg\ell\cos\theta(t) \\ \mathcal{L}(\theta,\dot\theta,t) &= T - U &&= m\big[\tfrac{1}{2}\ell^2\dot\theta^2 + \ell\dot w(t)\dot\theta\cos\theta - g\ell\cos\theta + \tfrac{1}{2}\dot w(t)^2\big] \end{aligned}$$

and Lagrange's equation of motion is

$$\begin{aligned} 0 &= \tfrac{d}{dt}\big(\tfrac{\partial\mathcal{L}}{\partial\dot\theta}\big) - \tfrac{\partial\mathcal{L}}{\partial\theta} \\ &= \tfrac{d}{dt}m\big[\ell^2\dot\theta + \ell\dot w(t)\cos\theta\big] - m\big[-\ell\dot w(t)\dot\theta\sin\theta + g\ell\sin\theta\big] \\ &= m\big[\ell^2\ddot\theta(t) + \ell\ddot w(t)\cos\theta(t) - g\ell\sin\theta(t)\big] \end{aligned}$$

or

$$\tfrac{d^2\theta}{dt^2}(t) - \tfrac{g}{\ell}\sin\theta(t) + \tfrac{1}{\ell}\ddot w(t)\cos\theta(t) = 0$$

In particular, if $w(t) = A\sin(\omega t)$, the equation of motion is

$$\tfrac{d^2\theta}{dt^2}(t) - \tfrac{1}{\ell}\big[g\sin\theta(t) + A\omega^2\sin(\omega t)\cos\theta(t)\big] = 0$$