Math 256. Sample final exam

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE \( xy' + y^2 = 4 \) with \( y(1) = 0 \) has the solution,

(a) \( 2 - 2x^4 \)  
(b) \( (x^4 - 1)/(1 + x) \)  
(c) \( 2(x^4 - 1)/(1 + x^4) \)  
(d) \( e^{x-1} - 1 \)  
(e) None of the above.

2. The system

\[
y' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 2 & -1 \end{pmatrix} y
\]

has the general solution,

(a) \( u_1 + u_2 e^t + u_3 e^{-t} \)  
(b) \( u_1 e^t + u_2 e^{2t} + u_3 e^{-2t} \)  
(c) \( u_1 + u_2 e^{2t} + u_3 e^{3t} \)  
(d) \( u_1 e^t + u_2 e^{-4t} + u_3 \)  
(e) None of the above,

for three constant vectors \( u_1 \), \( u_2 \) and \( u_3 \).

3. The inverse Laplace transform of

\[
\tilde{y}(s) = -\frac{8}{s(s^2 - 4)}
\]

(a) \( y(t) = 2t + e^t - e^{-t} \)  
(b) \( y(t) = 2t - \cos 2t \)  
(c) \( y(t) = 2 - e^{2t} - e^{-2t} \)  
(d) \( y(t) = 2 + \cos 2t \)  
(e) None of the above.

4. Which of the following is a solution to the PDE \( u_{tt} = u_{xx} \):

(a) \( u = \sin(x - t) \)  
(b) \( u = \cos x \sin 4t \)  
(c) \( u = \sin x e^{-t} \)  
(d) \( u = e^t \sin x \)  
(e) None of the above.

5. A cool question
Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. (12 points) The charge in a photon collector satisfies

\[ q'' + 2q' + 10q = 0, \quad q(0) = 0, \quad q'(0) = A, \]

where \( A \) is the energy of an incident photon. Find \( q(t) \). The collector is coupled to a detector whose signal \( d(t) \) satisfies

\[ d' + d = q, \quad d(0) = 0. \]

For the detector to register the photon, \( d(\pi) \) must exceed a threshold of 1. What incident energy \( A \) will trigger the detector?

2. (12 points) Write the ODEs

\[ x' = 3x + y + e^t, \quad y' = 3y + x, \]

as a 2 \times 2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if \( x(0) = y(0) = 0 \).

3. (12 points) From the definition of the Laplace transform and the properties of the delta function (plus an integration by parts) show that

\[ \mathcal{L} \{ t\delta'(t - c) \} = (cs - 1)e^{-cs}, \quad c > 0, \]

where \( \delta'(t) = \frac{d\delta}{dt} \). Hence solve the ODE

\[ y'' + 2y' + 5y = t\delta'(t - 3), \quad y(0) = y'(0) = 0. \]

4. (16 points) (a) Find the Fourier series of the function

\[ f(x) = -x \text{ for } 0 < x < \pi, \quad f(x) = -f(-x) \text{ for } -\pi < x < 0, \quad \& \quad f(x) = f(x + 2\pi). \]

(b) Find the \( U(x) \) that satisfies \( U_{xx} = 0 \) with \( U(0) = 0 \) and \( U(\pi) = \pi \).

(c) Using the method of separation of variables, solve

\[ u_t = u_{xx}, \quad u(0, t) = 0, \quad u(\pi, t) = \pi, \quad u(x, 0) = 0. \]
Math 256. Another one

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Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE \( y' \tan x + y = 4 \) with \( y(\pi/2) = 2 \) has the solution,

(a) \( 4 - \frac{2}{\sin x} \)  
(b) \( 2 \sin x \)  
(c) \( (x + 2) \sin x + \cos x \)  
(d) \( \frac{4x}{\pi} - \cos x \)  
(e) None of the above.

2. The system

\[
\begin{pmatrix}
-3 & 0 & 1 \\
0 & 1 & 0 \\
-5 & 0 & 3
\end{pmatrix} \mathbf{y}
\]

has the general solution,

(a) \( \mathbf{u}_1 + \mathbf{u}_2 e^t + \mathbf{u}_3 e^{-t} \)  
(b) \( \mathbf{u}_1 e^t + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{-2t} \)  
(c) \( \mathbf{u}_1 + \mathbf{u}_2 e^{2t} + \mathbf{u}_3 e^{3t} \)  
(d) \( \mathbf{u}_1 e^t + \mathbf{u}_2 e^{-4t} + \mathbf{u}_3 \)  
(e) None of the above,

for three constant vectors \( \mathbf{u}_1, \mathbf{u}_2 \) and \( \mathbf{u}_3 \).

3. The inverse Laplace transform of

\[ \tilde{y}(s) = \frac{18}{s(s^2 - 9)} \]

is

(a) \( y(t) = 2t + e^{3t} - e^{-3t} \),  
(b) \( y(t) = 2t - \cos 3t \),  
(c) \( y(t) = e^{3t} + e^{-3t} - 2 \),  
(d) \( y(t) = 2 + \cos 3t \),  
(e) None of the above.

4. Which of the following is a solution to the PDE \( u_{tt} = -u_{xxxx} \):

(a) \( u = \sin(x - t) \)  
(b) \( u = \cos x + \sin t \)  
(c) \( u = \sin x e^{-t} \)  
(d) \( u = e^t \sin x \)  
(e) None of the above.

5. A cool question
Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. (12 points) Solve
   \[ q'' + 4q' + 40q = 0, \quad d' + 2d = q, \quad f' = q(f + f^{-1}), \quad q(0) = d(0) = f(0) = 0, \quad q'(0) = 1. \]

2. (12 points) Write the ODEs
   \[ x' = x + 2y, \quad y' = 4x - y + 4e^{-t}, \]
as a 2 \times 2 system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if \( x(0) = y(0) = 0. \)

3. (12 points) Using the Laplace transform, solve the ODE
   \[ \ddot{y} + 4\dot{y} + 5y = (3t^3 - 2)\delta(t - 1), \quad y(0) = 0 \quad \dot{y}(0) = 1, \]
where \( \delta(t) \) denotes Dirac’s delta function.

4. (16 points) (a) Find the Fourier series of the function
   \[ f(x) = \pi \sin^2 \left( \frac{x}{2} \right) - x \text{ for } 0 < x < \pi, \quad f(x) = -f(-x) \text{ for } -\pi < x < 0, \quad \& \quad f(x) = f(x + 2\pi). \]
   Hint: the helpful trig identities might prove handy.
   (b) Find the \( U(x) \) that satisfies \( U_{xx} = 0 \) with \( U(0) = 0 \) and \( U(\pi) = \pi. \)
   (c) Using the method of separation of variables, solve
   \[ u_t = u_{xx}, \quad u(0, t) = 0, \quad u(\pi, t) = \pi, \quad u(x, 0) = \pi \sin^2 \left( \frac{x}{2} \right). \]
Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx.$$

Helpful trig identities:

- $\sin 0 = \sin \pi = 0$, $\sin(\pi/2) = 1 = -\sin(3\pi/2)$,
- $\cos 0 = -\cos \pi = 1$, $\cos(\pi/2) = \cos(3\pi/2) = 0$,
- $\sin(-A) = -\sin A$, $\cos(-A) = \cos A$, $\sin^2 A + \cos^2 A = 1$,
- $\sin(2A) = 2 \sin A \cos A$, $\sin(A + B) = \sin A \cos B + \cos A \sin B$,
- $\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A$, $\cos(A + B) = \cos A \cos B - \sin A \sin B$,
- $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$.

Useful Laplace Transforms:

- $f(t) \to \bar{f}(s)$
- $1 \to 1/s$
- $t^n, \quad n = 0, 1, 2, \ldots \to n!/s^{n+1}$
- $e^{at} \to 1/(s - a)$
- $\sin at \to a/(s^2 + a^2)$
- $\cos at \to s/(s^2 + a^2)$
- $t \sin at \to 2as/(s^2 + a^2)^2$
- $t \cos at \to (s^2 - a^2)/(s^2 + a^2)^2$
- $y'(t) \to s\bar{y}(s) - y(0)$
- $y''(t) \to s^2\bar{y}(s) - y'(0) - sy(0)$
- $e^{at}f(t) \to \bar{f}(s - a)$
- $f(t - a)H(t - a) \to e^{-as}\bar{f}(s)$

$$\int_{-\infty}^{\infty} \delta(x - a) f(x) \, dx = f(a)$$