Goal: Find approximate numerical solution of certain differential equations.

Specifically, we'll focus on \( f''(x) + q(x) f(x) = r(x) \), where \( q(x) \) and \( r(x) \) are known functions.

\( f(x) \) is unknown and will be approximated.

Example: \( f''(x) = 4(x) \) fits this format with \( q(x) = 0 \) and \( r(x) = 4 \).

To solve exactly,

\[
f'(x) = 4x + C_1, \quad C_1 \text{ is an arbitrary constant}
\]

\[
f''(x) = 2x^2 + C_1 x + C_2, \quad C_1, C_2 \text{ are arbitrary constants}
\]

This \( f(x) \) is the general solution of the DE (*)

To get a unique solution, we need to impose additional constraints:

Possibility 1: Specify the "initial values" of \( f(x) \) and \( f'(x) \).

\[
f(0) = 1, \quad f'(0) = 2.
\]

Problem: \( f''(x) = 4 \) Initial Value Problem (IVP).

\[
f(0) = 1
\]

\[
f'(0) = 2
\]

Solution: We already know that \( f(x) = 2x^2 + C_1 x + C_2 \).

To find \( C_1 \) and \( C_2 \) use initial values:

\[
f(0) = 1 \Rightarrow C_2 = 1
\]

\[
f'(0) = 2
\]
Possibility 2: Specify the "boundary values" of \( f \), i.e.,

\[
\begin{align*}
  f''(x) &= 4 \\
  f(0) &= 1 \\
  f(1) &= 6
\end{align*}
\]

Boundary value problem (BVP)

Solution: \( f(x) = 2x^2 + c_1 x + c_2 \)

\[
\begin{align*}
  f(0) &= 1 \Rightarrow c_2 = 1 \\
  f(1) &= 6 \Rightarrow 2 + c_1 + c_2 = 6 \\
  2 + c_1 &= 4 \Rightarrow c_1 = 2 \\
  f(x) &= 2x^2 + 3x + 1
\end{align*}
\]

\( f(x) \) solves the BVP.

**Problem:** Often, even simple-looking DEs are hard to solve exactly; OR there is no explicit solution!

**Example:** \( f''(x) + \cos(x) f(x) = x^2 \)

\( \Rightarrow \) no explicit solution!

**Remedy:** Find an approximate numerical solution by discretizing the BVP and turning it into a matrix equation!

**How?**

\[
f_j = f(x_j), \quad x_j = x_0 + j \cdot \Delta x \quad (N+1)-pt \text{ discretization of } f \text{ on the interval } [x_0, x_N].
\]
Define: \( F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix} \) discrete \((N+1)\) point approximation of \( f(x) \). Note \( F \in \mathbb{R}^{N+1} \).

We need:

1. Approximate the first derivative \( f'(x_0), f'(x_1), \ldots, f'(x_N) \)

That is: \( f'(x_j) \approx \frac{f_{j+1} - f_j}{\Delta x} \).

Then,
\[
F' = \frac{1}{\Delta x} \begin{bmatrix}
    f_1 - f_0 \\
    f_2 - f_1 \\
    \vdots \\
    f_{N+1} - f_N
\end{bmatrix} \approx \begin{bmatrix}
    f'(x_0) \\
    f'(x_1) \\
    \vdots \\
    f'(x_N)
\end{bmatrix}
\]

Observe:
\[
F' = \frac{1}{\Delta x} \begin{bmatrix}
    -1 & 1 & 0 & \cdots & 0 \\
    0 & -1 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}
\]
\( D_N : N \times (N+1) \) matrix.

In short: \( F' = \frac{1}{\Delta x} D_N \cdot F \).
Next:
\[ f'' = \frac{1}{\Delta x} \begin{bmatrix} F'_1 - F'_0 \\ F'_2 - F'_1 \\ \vdots \\ F'_{N-1} - F'_{N-2} \end{bmatrix} \approx \begin{bmatrix} f''(x_1) \\ f''(x_2) \\ \vdots \\ f''(x_{N-1}) \end{bmatrix} \]
\[ = \frac{1}{\Delta x} D_{N-1} F' \]

Second derivatives at the "interior points."

Let's combine:
\[ f'' = \frac{1}{(\Delta x)^2} D_{N-1} D_N F \]

Note that:
\[ D_{N-1} D_N = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 & -2 & 1 \end{bmatrix}_{(N-1)\times(N+1)} \]

Example: Solve the BVP approximately using an \((N+1)\) point discretization.
\[ \begin{cases} f''(x) = r(x) \\ f(0) = A \\ f(1) = B \end{cases} \]

Solution: Need:
(a) discretize the LHS of the equation
(b) discretize the RHS of the equation at the same points as the LHS
(c) incorporate the boundary values. \[ \cdots \]
We already know (a):
\[
\text{LHS } \rightarrow \quad F'' = \frac{1}{(4x)^2} D_{N-1}D_N F
\]

at \( x_1, x_2, ..., x_{N-1} \) (interior points).

(b) \[
R = \begin{bmatrix}
    r(x_1) \\
    \vdots \\
    r(x_{N-1})
\end{bmatrix}
\] (remember that \( r(x) \) is known)

Thus, at \( x_1, x_2, ..., x_{N-1} \) we have

\[
F'' = R \quad \leftarrow \text{discretized DE!}
\]

\[
(\Rightarrow) \quad \frac{1}{(4x)^2} D_{N-1}D_N F = R
\]

N-1 equations
N+1 unknowns
Need 2 more equations:

(c) \( f(0) = f_0 = A \) (*)

\( f(N) = f_N = B \) (**) 

\[
(*) \quad \Leftrightarrow \quad [1 \ 0 \ ... \ 0] \begin{bmatrix}
    f_0 \\
    f_1 \\
    \vdots \\
    f_{N-1} \\
    f_N
\end{bmatrix} = A
\]

(**) \( \Leftrightarrow \quad [0 \ 0 \ ... \ 0 \ 1] \begin{bmatrix}
    f_0 \\
    f_1 \\
    \vdots \\
    f_{N-1} \\
    f_N
\end{bmatrix} = B
\]

Then, the full discretization of the BVP is:

\[
\begin{bmatrix}
    1 & 0 & 0 & \cdots & 0 \\
    -D_{N-1}D_N & F = \begin{bmatrix}
        A \\
        B \\
        \vdots \\
        1
    \end{bmatrix}
\end{bmatrix}
\]

\( \Rightarrow \) The discretized solution \( F \) is given by the solution of \( LF = b \) only unknown.
Question: Is \( L \) invertible?

Answer: Yes! In fact, \( \det(L) = \pm N \).

Exercise: Prove this for \( N = 4 \).

Now: bring in \( q(x) \):

\[
\begin{align*}
f''(x) + q(x)f(x) &= r(x) \\
f(0) &= A; f(1) = B
\end{align*}
\]

Solve on \([0, 1]\)

Need to discretize \( q(x) \) \( f(x) \) as well!

\[
\begin{bmatrix}
q_0 f_0 \\
q_1 f_1 \\
\vdots \\
q_{N-1} f_{N-1}
\end{bmatrix}
\]

let \( q_i = q(x_i) \)

Then, \( A \), \( (N+1) \times (N+1) \)

\[ F \]

\[ L + \beta (dx)^2 Q \]

\[ F = \begin{bmatrix} A \\ \beta (dx)^2 R \\ B \end{bmatrix} \]

\[ b \]

MATLAB.