Announcements

- Final exam review session: Thursday, Dec 5, 10:30am - 12pm in MATH 100
- Please fill out the teaching evaluations!
- If you are interested in undergraduate summer research opportunities, now is the time to start looking for a faculty, and to apply for a USRA or a Work Learn International undergraduate fellowship!

Stay tuned for:
- Extra office hours next week
- The problem set (just for exercise)
The power method

Given an \( n \times n \) matrix \( A \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and corresponding eigenvectors \( v_1, \ldots, v_n \) such that

- \( |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n| \)
- \( \{v_1, \ldots, v_n\} \) is an eigenbasis
- \( ||v_i||_2 = 1 \) for all \( j = 1, \ldots, n. \)
The power method

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- \(|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|\)
- \( \{v_1, \ldots, v_n\} \) is an eigenbasis
- \( \|v_i\|_2 = 1 \) for all \( j = 1, \ldots, n \).

Given \( x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \), where \( c_1 \neq 0 \), we have that

\[
A^k x_0 = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n
\]

\[
= \lambda_1^k \left( c_1 v_1 + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^k v_2 + \cdots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^k v_n \right)
\]

\[
= \lambda_1^k \left( c_1 v_1 + \varepsilon_k \right), \text{ where } \varepsilon_k \to 0 \text{ as } k \to \infty.
\]
The power method

Given an $n \times n$ matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$ such that

- $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$
- $\{v_1, \ldots, v_n\}$ is an eigenbasis
- $\|v_i\|_2 = 1$ for all $j = 1, \ldots, n$.

Given $x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, where $c_1 \neq 0$, we have that

$$A x_0 = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n$$

Thus,

$$A^k x_0 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \cdots + c_n \lambda_n^k v_n$$

$$= \lambda_1^k \left( c_1 v_1 + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^k v_2 + \cdots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^k v_n \right)$$

$$= \lambda_1^k (c_1 v_1 + \varepsilon_k), \text{ where } \varepsilon_k \to 0 \text{ as } k \to \infty.$$
The power method

Algorithm:

**Input:** Matrix $A$; random $x_0 \neq 0$; max number of iterations $N$.

**Iterate:** for $k = 1 : N$
\[
x_k = A x_{k-1}; \quad x_k = \frac{x_k}{\|x_k\|};
\]
end

**Output:** $v_1 = x_N$, $\lambda_1 = \langle v_1, Av_1 \rangle$. 
Can we find any other eigenvectors/eigenvalues?

Given $s \in \mathbb{R}$, how can we find the closest eigenvalues to $s$?

**Proposition:** Let $s \in \mathbb{R}$, and suppose that $A$ is $n \times n$ with real eigenvalues, and corresponding eigenbasis $\{v_1, \ldots, v_n\}$. Then, one of the following holds:

(a) $s$ is an eigenvalue of $A$, i.e. $A - sl$ is not invertible; OR

(b) the eigenvalues of $(A - sl)^{-1}$ are exactly $\frac{1}{\lambda_1 - s}, \ldots, \frac{1}{\lambda_n - s}$, and the corresponding eigenvectors are still $v_1, \ldots, v_n$.

Proof: on board.
Can we find any other eigenvectors/eigenvalues?

Given \( s \in \mathbb{R} \), how can we find the closest eigenvalues to \( s \)?

**Proposition:** Let \( s \in \mathbb{R} \), and suppose that \( A \) is \( n \times n \) with real eigenvalues, and corresponding eigenbasis \( \{v_1, \ldots, v_n\} \). Then, one of the following holds:

(a) \( s \) is an eigenvalue of \( A \), i.e. \( A - sI \) is not invertible; OR

(b) the eigenvalues of \( (A - sI)^{-1} \) are exactly \( \frac{1}{\lambda_1 - s}, \ldots, \frac{1}{\lambda_n - s} \), and the corresponding eigenvectors are still \( v_1, \ldots, v_n \).

Proof: on board.

**Consequence:** the eigenvalues of \( (A - \lambda I)^{-1} \) are \( \frac{1}{\lambda_1 - s}, \ldots, \frac{1}{\lambda_n - s} \), therefore, the dominant eigenvalue is \( \frac{1}{\lambda_j - s} \) if and only if

\[ \frac{1}{|\lambda_j - s|} > \frac{1}{|\lambda_i - s|}, \quad \forall i \neq j \]

\[ \Leftrightarrow |\lambda_j - s| < |\lambda_i - s|, \quad \forall i \neq j, \]

i.e. \( \lambda_j \) is closest to \( s \)!
The closest eigenvalue to $s \in \mathbb{R}$.

Given $s \in \mathbb{R}$, we can use the power method on $(A - sI)^{-1}$ to get the eigenvector $v_j$ that corresponds to the $\lambda_j$ closest to $s$ (if such a $\lambda_j$ is unique). To get $\lambda_j$, just compute $\langle v_j, Av_j \rangle$. 

How do we find the smallest $|\lambda_j|$?

Choose $s = 0$, and perform the power method on $A^{-1}$!
The closest eigenvalue to $s \in \mathbb{R}$.

Given $s \in \mathbb{R}$, we can use the power method on $(A - sl)^{-1}$ to get the eigenvector $v_j$ that corresponds to the $\lambda_j$ closest to $s$ (if such a $\lambda_j$ is unique). To get $\lambda_j$, just compute $\langle v_j, Av_j \rangle$.

power iterations: $x_{k+1} = (A - sl)^{-1}x_k$,
in MATLAB: $x = (A - s \ast \text{eye}(n))\backslash x$. 
The closest eigenvalue to $s \in \mathbb{R}$.

Given $s \in \mathbb{R}$, we can use the power method on $(A - sl)^{-1}$ to get the eigenvector $v_j$ that corresponds to the $\lambda_j$ closest to $s$ (if such a $\lambda_j$ is unique). To get $\lambda_j$, just compute $\langle v_j, Av_j \rangle$.

Power iterations: $x_{k+1} = (A - sl)^{-1}x_k$,

in MATLAB: $x = (A - s \times \text{eye}(n)) \backslash x$.

How do we find the smallest $|\lambda_j|$?
The closest eigenvalue to \( s \in \mathbb{R} \).

Given \( s \in \mathbb{R} \), we can use the power method on \((A - sl)^{-1}\) to get the eigenvector \( v_j \) that corresponds to the \( \lambda_j \) closest to \( s \) (if such a \( \lambda_j \) is unique). To get \( \lambda_j \), just compute \( \langle v_j, Av_j \rangle \).

power iterations: \( x_{k+1} = (A - sl)^{-1}x_k \),

in MATLAB: \( x = (A - s * \text{eye}(n)) \backslash x \).

How do we find the smallest \( |\lambda_j| \)? Choose \( s = 0 \), and perform the power method on \( A^{-1} \)!
Recurrence relations

A recursive definition of a sequence of numbers $a_0, a_1, a_2, \ldots$:

$$a_n = a_{n-1} + 2; \quad a_0 = 5.$$

The sequence is

$$5, 7, 9, \ldots.$$
Recurrence relations

A recursive definition of a sequence of numbers $a_0, a_1, a_2, \ldots$:

$$a_n = a_{n-1} + 2; \quad a_0 = 5.$$  

The sequence is

$$5, 7, 9, \ldots$$

But we can also find a formula for this sequence so we can quickly evaluate $a_n$ for any $n$:

$$a_n = a_0 + 2n = 5 + 2n.$$
The Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, ....

Here

\[ F_{n+1} = F_n + F_{n-1}; \quad F_0 = 0, F_1 = 1. \]

How can we find a formula for \( F_n \)?
The Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, ....

Here

\[ F_{n+1} = F_n + F_{n-1}; \quad F_0 = 0, F_1 = 1. \]

How can we find a formula for \( F_n \)?

Rewrite relation as a matrix equation:

\[
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix}
=
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
F_n \\
F_{n-1}
\end{bmatrix},
\begin{bmatrix}
F_1 \\
F_0
\end{bmatrix}
=
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Thus,

\[
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix}
=
A^n
\begin{bmatrix}
F_1 \\
F_0
\end{bmatrix},
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix}
=
A^n
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Let \( \nu_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \), and \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). Then,

\[
\nu_n = A\nu_{n-1}, \quad \nu_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
The Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, . . .

Here

\[ F_{n+1} = F_n + F_{n-1}; \quad F_0 = 0, \ F_1 = 1. \]

How can we find a formula for \( F_n \)?

Rewrite relation as a matrix equation:

\[ F_{n+1} = F_n + F_{n-1} \]

\[ F_n = F_n. \]

Thus,

\[
\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}, \quad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Let \( \mathbf{v}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \), and \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). Then,

\[ \mathbf{v}_n = A\mathbf{v}_{n-1}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

Thus, \( \mathbf{v}_n = A\mathbf{v}_{n-1} = A^2\mathbf{v}_{n-2} = \cdots = A^n\mathbf{v}_0. \)
The Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, ….

Here

\[ F_{n+1} = F_n + F_{n-1}; \quad F_0 = 0, F_1 = 1. \]

How can we find a formula for \( F_n \)?
Rewrite relation as a matrix equation:

\[
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
F_n \\
F_{n-1}
\end{bmatrix}, \quad \begin{bmatrix}
F_1 \\
F_0
\end{bmatrix} =
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Thus,

\[
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix} = A^n \begin{bmatrix}
F_1 \\
F_0
\end{bmatrix} = A^n \begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Let \( v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \), and \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). Then,

\[ v_n = Av_{n-1}, \quad v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

Thus, \( v_n = A^n v_0 = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

\[ \ldots F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \]
A more general example

\[ x_{n+1} = 3x_n + x_{n-1} + 2x_{n-2}, \quad x_0 = a, x_1 = b, x_2 = c. \]

In matrix form:

\[
\begin{bmatrix}
  x_{n+1} \\
  x_n \\
  x_{n-1}
\end{bmatrix}
= \begin{bmatrix}
  3 & 1 & 2 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  x_n \\
  x_{n-1} \\
  x_{n-2}
\end{bmatrix}; \quad v_0 = \begin{bmatrix}
  c \\
  b \\
  a
\end{bmatrix}.
\]

\[ v_{n-1} = Av_{n-2}, \quad v_0 = \begin{bmatrix}
  c \\
  b \\
  a
\end{bmatrix}. \]

Then,

\[ v_n = A^n v_0. \]

Diagonalize \( A \) to get a formula for \( v_n \), and, thus, for \( x_n \).
Special facts about stochastic matrices.

Let $P$ be an $n \times n$ stochastic matrix, i.e., $P_{ij} \geq 0$, and $\sum_{i=1}^{n} P_{ij} = 1$ for all $j$. Then,

1. If $v = [v_1, \ldots, v_n]^T$ is a state vector, i.e., $\sum v_i = 1$ and $0 \leq v_i \leq 1$, then $Pv$ is also a state vector! (Proof: exercise).

2. $\lambda = 1$ is an eigenvalue of $P$. (Proof: on board.)

3. All other eigenvalues of $P$ satisfy $|\lambda_j| \leq 1$.

4. If $v$ is an eigenvector with eigenvalue 1, i.e., $Pv = v$, then, $v$ has all nonnegative (or nonpositive) entries.

5. Eigenvectors corresponding to $|\lambda_j| < 1$ have entries that add up to 0.

6. When can we guarantee that $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j \neq 1$?
   
   If $P$ or $P^k$ for some $k \in \mathbb{N}$ has all positive entries, then $\lambda_1 = 1, |\lambda_j| < 1$ for all $j \neq 1$. 