ALGORITHM (POWER METHOD)

Input: $A; \text{ random } x_0 \in \mathbb{D}; N$

Iterate: for $k = 1 : N$

\[ x_k = Ax_{k-1} \]

\[ x_k = \frac{x_k}{||x_k||_2} \]

end

Output: $\lambda = x_N$

$\lambda = \langle v, AV \rangle$

max # of iterations.

Note: We need that in

\[ x_0 = c_1 v_1 + \ldots + c_n v_n \]

$c_i \neq 0$ in order for

\[ \frac{A x_0}{||A x_0||} \]

But, if $x_0$ is chosen at

random, this will be true.

A modification: Given $s \in \mathbb{R}$, how can we find the closest

eigenvalue to $s$?

Proposition: Let $s \in \mathbb{R}$, and suppose $A$ is $n \times n$ with real

eigenvalues (as above). Then, one of the following holds:

(a) $s$ is an eigenvalue of $A$, i.e., $A - sI$ is not invertible,

or (b) The eigenvalues of $(A - sI)^{-1}$ are exactly $\frac{1}{\lambda - s}$ and the

corresponding eigenvectors are still $v_1, \ldots, v_n$.

Proof: $Av = \lambda v, \quad V \neq 0$

\[ A v - sI v = \lambda v - sv \]

\[ (A - sI)v = (A - s)v. \]

If $\lambda = s$, then (a) holds

otherwise $A - sI$ is invertible, and

\[ \frac{1}{\lambda - s} v = (A - sI)^{-1} v \]

i.e. $\frac{1}{\lambda - s}$ and $v$ are an eigenvalue-eigenvector pair.

for $(A - sI)^{-1}$. 
Eigenvalues of \((A - sI)^{-1}\) are \(\frac{1}{\lambda_i - s}, \ldots, \frac{1}{\lambda_n - s}\).

And the dominant eigenvalue is \(\frac{1}{\lambda_j - s}\) if and only if

\[|\lambda_j - s| < |\lambda_e - s| \text{ for all } e \neq j.\]

Thus, using the power method on \((A - sI)^{-1}\), we will return the eigenvector \(v_j\) that corresponds to \(\lambda_j\) closest to \(s\). To get this \(\lambda_j\), just compute \(\langle v_j, Av_j \rangle\).

**Important point:** When computing \((A - sI)^{x_k}\), we don't actually need to compute \((A - sI)^{-1}x_k\). But first

\[(A - sI)y = x_k.\]

In MATLAB: \(x_k = (A - s\cdot eye(n))\backslash x_k\).

Smallest eigenvalue in absolute value:

Recurrence relations (IV.4)

**Examples:**

\[a_n = a_{n-1} + 2 \quad a_0 = 5\]

5, 7, 9, ...

**Formula:**

\[a_n = a_0 + 2n\]

\[a_n = 5 + 2n\]
A more involved example (Fibonacci sequence)

0, 1, 1, 2, 3, 5, 8, 13

\[ F_{n+1} = F_n + F_{n-1}; \quad F_0 = 0, \quad F_1 = 1. \]  

Goal: Find a formula for \( F_n \).

How? Rewrite this relation as a matrix equation!

\[ \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ x & 0 \end{bmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \]

\[ V_n = A^2 V_0 \]

Then, \( V_1 = A V_0 \); \( V_2 = A^2 V_0 \); \ldots ; \( V_n = A^n V_0 \).

That is: \( \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ x & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

Noting that \( A \) is real symmetric (Hilbert–Hurwitz), we know that \( A \) is unitarily decomposable. Eigenvalues of \( A \): \( \lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2} \).
Corresp. eigenvectors: \( V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

Then: \( A = SDS^{-1} \) with
\[
S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\]
\[
S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}.
\]

Then, \( A^n = SD^nS^{-1} \)
\[
\Rightarrow \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = SD^nS^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1^n \\ \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \ldots = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix}
\]

\( \Rightarrow F_n = \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n). \)

A more general example:
\[
x_{n+1} = 3x_n + x_{n-1} + 2x_{n-2}
\]
\[
x_0 = a, \quad x_1 = b, \quad x_2 = c.
\]

\[
\begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{bmatrix} \quad \text{with} \quad V_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}.
\]

Then: diagonalize to get a formula for \( F_n = A^n V_0 \) and, thus, for \( x_n \).
Markov chains

- \( P_{ij} \): "transition probability" of getting from state \( j \) to state \( i \).
- \( 0 \leq P_{ij} \leq 1 \).
- \( \sum_{j=1}^{3} P_{ij} + P_{ij} + P_{ij} = 1 \) for \( j = 1, 2, 3 \) getting from \( j \) to \( 1 \).

Let \( x_{n,i} = \) probability of being at state \( i \) at time \( n \).

Assume we make \( n \) transitions.

Every day (e.g., a bird is flying between islands 1, 2, and 3, \( \times_{n,i} \) is the prob it is on island \( i \) on day \( n \)).

- \( x_{0,i} = \) prob of starting at state \( i \)
- \( x_{1,i} = \) prob of being at \( i \) after 1 step

\[
\begin{bmatrix}
x_{n,1} \\
x_{n,2} \\
x_{n,3}
\end{bmatrix}
\]

"distribution" after \( n \) steps.

Such vectors are called state vectors.

Note that:
- \( 0 \leq x_{n,i} \leq 1 \)
- \( \sum_{i=1}^{3} x_{n,i} = 1 \).
Question: Given \( x_n \), how do we find \( x_{n+1} \)?

\[
x_{n+1,i} = x_{n,1}p_{i1} + x_{n,2}p_{i2} + x_{n,3}p_{i3}
\]

OR

\[
x_{n+1} = \begin{bmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{bmatrix} x_n
\]

Then,

\[
x_{n+1} = P x_n
\]

\[
= P^2 x_{n-1} = \ldots = P^{n+1} x_0.
\]

So,

\[
x_n = P^n x_0
\]

a matrix recursion

initial state.

To investigate the limit as \( n \to \infty \) ("steady state" or "equilibrium," \ldots) if such a limit exists, we resort to eigenbasis as before, except that \( P \) has nice properties that will help!

Properties: \( P = [p_{ij}]_{1 \times n} \)

\[
\begin{align*}
& 0 \leq p_{ij} \leq 1 \\
& \sum_{i=1}^{k} p_{ij} = 1 \quad \forall j.
\end{align*}
\]

Such matrices are called stochastic.

Suppose \( P \) is stochastic and \( \lim_{n \to \infty} P^n x_0 \) exists and equals \( x \).

Then, \( P^n x_0 \approx x \) when \( n \) is large.
\( P^N_{x_0} \approx x \) \( \Rightarrow x \approx P^{N+1}_{x_0} = P(P^N_{x_0}) \approx P \)

That is, \( Px = x \).

i.e., If \( \lim_{n \to \infty} P^n_{x_0} = x \), then \( x \) is a stationary point,

i.e., \( Px = x \).

and \( \lambda = 1 \) is an eigenvalue of \( P \) with correspond. eigenvector \( x \).

**Some special facts about stochastic matrices**

1. If \( v = \left( \begin{array}{c} v_1 \\ \vdots \\ v_k \end{array} \right) \) is a state vector (i.e., \( \Sigma v_i = 1 \)) \( \geq v_i \geq 0 \)

Then so is \( Pv \) (proof: exercise)

2. \( P \) has an eigenvalue \( \lambda = 1 \)

**Proof:** \( P^T \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \) \( \Rightarrow \lambda = 1 \) is an eigenvalue of \( P^T \)

Since eigenvalues of any matrix are same as those of its transpose, \( \Rightarrow \lambda = 1 \) eigenvalue of \( P \).

3. All other eigenvalues of \( P \) satisfy \( |\lambda| < 1 \).

4. The eigenvector \( v_i \) (for \( \lambda = 1 \)) has non-negative (or non-positive) entries!

5. Eigenvectors correspond to \( |\lambda| < 1 \) have entries that add up to 0.

\( \Sigma P v_i = \left[ 1 \ldots 1 \right] v_i = \left( \Sigma v_i \right) \), but \( \left[ 1 \ldots 1 \right] P v_i = \left[ 1 \ldots 1 \right] \lambda v_i = \lambda \Sigma v_i \).
@When can we guarantee that \( \lambda_1 = 1 \) and \( |\lambda_j| < 1 \) for \( j \neq 1 \)?

If \( P \) or \( p_k \) for \( k \in \mathbb{N} \) has all positive entries, then

\[ \lambda_1 = 1 \] and \( |\lambda_j| < 1 \) for \( j \neq 1 \).

(no 0 entries!)

**Examples:**

\[ P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{stochastic, } P^2 = I \]

\[ \lambda_1 = 1, \lambda_2 = -1 \]

\( \lambda_1 = 1 \) is not true down eig.

\[ P = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \quad \text{stochastic} \]

not guaranteed to have \( \lambda_1 = 1 \) as down. eig.

But using **MATLAB**:

\[ \lambda_1 = 1; \quad v_1 = [0, 0.7071, 0.7071]^T \]

\[ \lambda_2 = 0.75; \quad v_2 = [0.8165, -0.4082, -0.4082]^T \]

\[ \lambda_3 = 0.5; \quad v_3 = [0, 0.7071, -0.7071]^T \]

\[ \Rightarrow \lim_{n \to \infty} P^n x_0 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \]

\( x_0 \) is a "state vector."

In general, \( \lim_{n \to \infty} P^n x_0 = \frac{1}{\| V_1 \|} V_1 \) so that entries are all nonnegative.