Graphs & Networks

Ex. social network

Ex. resistor network

A graph is a set of vertices (nodes) and edges.

We can describe directed graphs (uniquely) via their incidence matrix $A$. $D$

Ex: Consider

Incidence matrix of this graph
Our next job: Understand \( N(D) \), \( N(D^T) \), \( R(D) \), \( R(D^T) \).

Example 1: \( N(D) \): Find the set of all \( \mathbb{R}^4 \) (in general \( \mathbb{R}^\# \text{vertices} \)) \( \mathbf{v} \) such that:

\[
D \mathbf{v} = 0.
\]

\( v_j \): "Voltage" at node \( j \).

\( \Rightarrow \) \[
D \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 0
\]

\( \Rightarrow \) \[
\begin{bmatrix}
v_2 - v_1 \\
v_4 - v_2 \\
v_3 - v_2 \\
v_3 - v_4
\end{bmatrix} = 0
\]

\( \Rightarrow \) \[
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
\]

\( \Rightarrow \) \[
\text{Solve } v_1 = s_1; \quad v_2 = s_2; \quad v_3 = s_3
\]

\( \Rightarrow \) \[
N(D) = \text{span}\{ [1] \}
\]
Note: If $v \in N(D)$, then the voltages at any nodes that are connected (eventually) have to be identical.

So, for any connected graph, $N(D) = \text{span} \{ [v_1], [v_2], \ldots, [v_n] \}$ is 1-dimensional.

Ex:

\[
D = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

$V_1 = V_2 = V_3 = 5$
$V_4 = V_5 = 6$

$N(D) = \text{span} \{ [5], [0], [0], [1], [0] \}$

2. $R(D) = \{ DV : v \in \mathbb{R}^n \}$

Ex:

$V_j$: voltage at vertex $j$.

In our original example

\[
Dv = \begin{bmatrix}
v_2 - v_1 \\
v_4 - v_2 \\
\vdots \\
v_3 - v_4 \\
\end{bmatrix}
\]

Note: $\dim(R(D)) = 4 - \dim(N(D))$.

When the graph is connected, $\dim(N(D)) = 1$

$\Rightarrow \dim(R(D)) = 4 - 1 = 4$ (number of vertices - 1)
Let's investigate $D^T$. Using our original example:

$$D^T = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
-1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & -1
\end{bmatrix}$$

Consider:

$$D^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$X_i$: Current along edge $i$.

$Y_i$: Current accumulating at vertex # $i$.

Then: $N(D^T) = \{ z : D^Tz = 0 \}$.

$\text{4 no accumulation of current at any node!}$

In our example, $N(D^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

$$D^T \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \text{col}\#1 + \text{col}\#2 - \text{col}\#3 \quad \text{(of } D^T)$$

Edge 1 Edge 2 - Edge 3

Corresponds to loop 1

$$D^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = -\text{col}\#2 + \text{col}\#4 - \text{col}\#5$$

corresp. to loop 2
Digression:

\[
Ax = \begin{bmatrix}
a_1 & a_2 & \cdots & a_n \\
1 & 1 & \cdots & 1 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n
\]

Let's talk about \( R(D^T) \) later.

Resistor networks (circuits)

Ohm's law:

\[
\begin{align*}
& \text{I: current from 1 to 2} \\
& \text{R: resistance (in } \Omega \text{)} \\
& \text{V_i: voltage at node i.}
\end{align*}
\]

\[
V_2 - V_1 = \mathbf{j} \cdot R_1
\]

\[
V_4 - V_2 = \mathbf{j}_2 \cdot R_2
\]

As before: \( V_i \): voltage at vertex i. Let \( \mathbf{j} \) be the vector of currents, i.e., \( j_i \) is the current on edge i (across \( R_i \)).

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For example:

\[
V_2 - V_1 = \mathbf{j}_1 \cdot R_1
\]

\[
V_4 - V_2 = \mathbf{j}_2 \cdot R_2
\]
Using new notation:

\[
D \mathbf{v} = \begin{bmatrix}
\dot{j}_1 R_1 \\
\dot{j}_2 R_2 \\
\vdots \\
\dot{j}_5 R_5
\end{bmatrix} = \begin{bmatrix}
R_1 & R_2 & 0 \\
0 & R_2 & \cdots \\
\vdots & \ddots & \ddots \\
0 & \cdots & R_5
\end{bmatrix}
\begin{bmatrix}
\dot{j}_1 \\
\dot{j}_2 \\
\vdots \\
\dot{j}_5
\end{bmatrix}
\]

\[R \text{ (matrix)}\]

So,

\[D \mathbf{v} = R \dot{\mathbf{j}} \Rightarrow R^{-1} D \mathbf{v} = \dot{\mathbf{j}}\]

where \( R^{-1} = \begin{bmatrix}
\frac{1}{R_1} & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{1}{R_5} & 0 & 0
\end{bmatrix} \)