Assignment 7: Due Wednesday, March 18 at start of class

Problems to be handed in

Problem 1

Determine, with justification and using Taylor series if needed, which of the following functions are $o(h)$ as $h \to 0$, and which are not:
(1) $h^{1/3}$, (2) $h^{1/3} + h^{3/2}$, (3) $1 + he^{-h}$, (4) $\frac{\sin h}{h} - \cos(h)$.

Problem 2

Bob waits for buses 14 or 99 to take him home. Both independently arrive according to a Poisson process of rate 1 bus per 5 min for the 99, and 1 per 10 min for the 14. The probability that a bus arrives full (so Bob cannot take it) is 0.5 for the 99 and 0.2 for the 14.
1. In average, how much time does Bob have to wait before getting in a bus? What is the probability that this bus is a 99?
2. What is the probability that Bob sees 3 full buses before one that is not full?
3. Leaving from the bus station, the 99 takes 15 min in average to take Bob home, while the 14 takes 20 (the two times are independent). In average, how much time does it take for Bob to get back home (from the time he arrives at the bus station)?

Problem 3

Tired of waiting, Bob decides to walk from the bus station to a parking lot located 10 min away, where shared cars are available. Due to the high demand, Bob can see shared cars on his phone appearing according to a Poisson process, at a rate of 1 car per 2 min. Independently, there is a probability p that Bob manages to book a car when it appears.
1. What is the probability that Bob books a car before he arrives at the parking lot?
2. What is the probability that a non-full bus arrives while Bob walks to the parking lot?
3a. Once at the parking lot and if he hasn't been able to book a car yet, Bob can take the next one with probability one, and it then takes him 10 min in average to get home (driving time and waiting times are independent). How much time does Bob take in average to get back home?
3b. Given that there was exactly one non-full 99 bus that arrived while Bob was walking, and no bus 14, and assuming the driving times (by bus and car) are deterministic (with same times as given in Problem 2), what is the probability that Bob makes it home faster than if he had decided to wait?

Problem 4

1. Let $\{N(t), t \geq 0\}$ be a λ-rate Poisson process. Given that $N(t) = n$ ($n \in \mathbb{N}$), find the conditional probability of $N(u) = k$, for $0 \leq u \leq t$ and $k = 0, 1, (\ldots), n$.
2a. Given that $N(t) = 2$, find the conditional (marginal) probability density functions of the arrival times S_1 and S_2.
2b (bonus). In general, given that $N(t) = n$, find the marginal pdf's of the arrival times $S_1, \ldots, S_n$.

Problem 5

Cars get parked in a lot (with infinite capacity) according to a λ-rate Poisson process, and independently stay parked for a random duration. The parking time duration of a car follows a common distribution X, with cdf $F(x) = P(X \leq x)$. Let $N(t)$ be the number of cars parked at time t.
1a. What is the distribution of $N(t)$?
1b. Assuming a car arrival rate of 1 per minute, and X (in min) $\sim Gamma(3, 1)$ (as defined in class), what is the expected number of cars parked after 1 hour?
2. In the long run ($t \to \infty$), what is the expected number of cars, as a function of λ and the moments of X?

These provide additional practice but are not to be handed in. Textbook Chapter 5 examples 5.15-19, and exercices 43,44, 47-51,58-60, 64, 66.